Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 743–752
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23006.2452

Zero-free regions for bicomplex polynomials and related bicomplex entire functions

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(Communicated by Ali Jabbari)

Abstract

In this paper, we find the zero free regions of bicomplex polynomial and related bicomplex entire functions with certain restrictions on the coefficients. Our results generalize many results already known in the literature.

Keywords: Bicomplex polynomials, Bicomplex entire functions, Zero free region
2020 MSC: 30D20, 30C10, 30C15, 30D10, 30G35

1 Introduction

A bicomplex number is a number of the form $z = z_1 + jz_2$, where z_1, z_2 are complex numbers. The set of bicomplex numbers is denoted by

$$\mathbb{C}_2 = \{z = z_1 + jz_2, z_1, z_2 \in \mathbb{C}_1\},$$

i and j are commuting imaginary units, that is, $ij = ji, i^2 = j^2 = -1$, and $\mathbb{C}_1$ is the set of complex numbers with imaginary unit i. Thus bicomplex numbers are "complex numbers with complex coefficients" (see [5],[6]).

It is easy to see that $(\mathbb{C}_2, +, .)$ is a commutative ring with zero as additive identity and unity as multiplicative identity but $\mathbb{C}_2$ is not a field due to presence of zero divisors, namely the set

$$\mathrm{O} = \{z_1 + jz_2 \in \mathbb{C}_2 : z_1^2 + z_2^2 = 0\} = \{a(1 \pm ij) : a \in \mathbb{C}_1\}.$$

If $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, $x_1, x_2, x_3, x_4 \in \mathbb{R}$, then $z = z_1 + jz_2 = x_1 + ix_2 + jx_3 + jix_4$. So, $\mathbb{C}_2$ can be viewed as a real vector space which is isomorphic to $\mathbb{R}^4$ via the map $x_1 + ix_2 + jx_3 + ijx_4 \to (x_1, x_2, x_3, x_4)$.

There are four idempotent elements in $\mathbb{C}_2$ namely $0, 1, \frac{1+ij}{2}, \frac{1-ij}{2}$. Two of these idempotent elements namely $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$, play an important role since every element in $\mathbb{C}_2$ has a unique representation as a linear combination of them and $e_1 + e_2 = 1$, $e_1 - e_2 = ij$, $e_1e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$. Any number $z = z_1 + jz_2 \in \mathbb{C}_2$ can be written uniquely as $z = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2$. This representation is known as idempotent representation of $z \in \mathbb{C}_2$.

The norm function $\|.\| : \mathbb{C}_2 \to \mathbb{R}^+$ ($\mathbb{R}^+$ denotes the set of all non negative real numbers) is defined as follows:

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Received: March 2021 *Accepted:* August 2021

If $z = z_1 + jz_2 = \phi_1 e_1 + \phi_2 e_2 \in \mathbb{C}_2$, then

$$\|z\| = \{|z_1|^2 + |z_2|^2\}^{\frac{1}{2}} = \left\{\frac{|\phi_1|^2 + |\phi_2|^2}{2}\right\}^{\frac{1}{2}}.$$

The complex spaces $\mathcal{A}_1,\ \mathcal{A}_2$ defined as:
$\mathcal{A}_1$=$\{z_1 - iz_2 : z_1\ and\ z_2\ in\ \mathbb{C}_1\}$, $\mathcal{A}_2$=$\{z_1 + iz_2 : z_1\ and\ z_2\ in\ \mathbb{C}_1\}$, are known as auxiliary complex spaces.
Since each element in $\mathbb{C}_1$ can be expressed in the form $z_1 - iz_2$ and $z_1 + iz_2$ (in many ways), the elements in $\mathcal{A}_1$ and $\mathcal{A}_2$ are the same as the elements in $\mathbb{C}_1$. A set X in $\mathbb{C}_2$ is called a cartesian set if and only if there exist sets X_1 in $\mathcal{A}_1$ and X_2 in $\mathcal{A}_2$ such that

$$X = \{z_1 + jz_2 \in \mathbb{C}_2 : z_1 + jz_2 = \phi_1 e_1 + \phi_2 e_2, (\phi_1, \phi_2) \in X_1 \times X_2\}.$$

If X satisfies the above condition, then X is called cartesian set determined by X_1 and X_2. An open discus $D(a; r_1, r_2)$ with centre $a = a_1e_1 + a_2e_2$ with radii $r_1 > 0, r_2 > 0$ is defined as

$$\begin{aligned} D(a; r_1, r_2) =& B_1(a_1, r_1) \times B_2(a_2, r_2) \\ =& \{\phi_1 e_1 + \phi_2 e_2 \in \mathbb{C}_2 : |\phi_1 - a_1| < r_1, |\phi_2 - a_2| < r_2\} \end{aligned}$$

where $B_1(a_1, r_1)$ is an open ball with centre $a_1 \in \mathbb{C}_1$ and $r_1 > 0$.
A closed discus $D(a; r_1, r_2)$ with centre $a = a_1e_1 + a_2e_2$ with radii $r_1 > 0, r_2 > 0$ is defined as

$$\begin{aligned} D(a; r_1, r_2) =& \overline{B}_1(a_1, r_1) \times \overline{B}_2(a_2, r_2) \\ =& \{\phi_1 e_1 + \phi_2 e_2 \in \mathbb{C}_2 : |\phi_1 - a_1| \le r_1, |\phi_2 - a_2| \le r_2\} \end{aligned}$$

where $\overline{B}_1(a_1, r_1)$ is a closed ball with centre $a_1 \in \mathbb{C}_1$ and $r_1 > 0$.

Let $f(z) = \sum\limits_{k=0}^{n} a_k z^k$ be a bicomplex polynomial (see [6]) of degree n of bicomplex variable z. Let us write $z = z_1 + jz_2$ in its $\mathbb{C}_2$-idempotent representation $z = \beta_1 e_1 + \beta_2 e_2$ with $\beta_1 = z_1 - iz_2$ and $\beta_2 = z_1 + jz_2$. We write also the complex coefficients as $a_k = \gamma_k e_1 + \delta_k e_2,\ k = 0, 1, ..., n$. Then $z^k = \beta_1^k e_1 + \beta_2^k e_2$ and we rewrite our polynomial as

$$f(z) = \sum_{k=0}^{n} \big(\gamma_k \beta_1^k\big) e_1 + \sum_{k=0}^{n} \big(\delta_k \beta_2^k\big) e_2 =: \phi(\beta_1) e_1 + \phi(\beta_2) e_2.$$

Recently, Sandip et al [4] proved the following results on the zeros of bicomplex entire functions.

Theorem 1.1. Let $f(z) = \sum\limits_{j=0}^{\infty} a_j z^j$ be a bicomplex entire function with real positive coefficients and for some $k \le 1, t > 0$, and $\lambda \ge 1$

$$ka_0 \le ta_1 \le t^2 a_2 \le ... \le t^\lambda a_\lambda \ge t^{\lambda+1} a_{\lambda+1} \ge ...$$

then $f(z)$ does not vanish in the discs $D(0; r_0, r_0)$ where $r_0 = \frac{ta_0}{(1-2k)a_0 + 2t^\lambda a_\lambda}$.

Remark. If $k = 1$, then Theorem 1.1 can be regarded as a bicomplex version of Theorem 1 of [1].

Theorem 1.2. Let $f(z) = \sum\limits_{j=0}^{\infty} a_j z^j$ be a bicomplex entire function with complex coefficients such that $a_0 \ne 0$ and for some $t > 0$

$$|a_0| \ge t|a_1| \ge t^2 |a_2| \ge ...$$

Then no zero of $f(z)$ lies in the discus $D(0; r_0, r_0)$, where

$$r_0 = \frac{t|a_0|}{(|a_0| + ||a_0| - a_0|) + 2\sum\limits_{j=1}^{\infty} ||a_j| - a_j| t^j}.$$

Remark. Theorem 1.2 can be regarded as bicomplex version of Theorem B of [2].

In this paper we first find zero free regions for bicomplex polynomials and then extend them to bicomplex entire functions. Our results generalize the few results proved by Sandip et al.[4]. In fact, we prove the following results:

2 Main Results

Theorem 2.1. Let $f(z) = \sum_{j=0}^{n} a_j z^j$ be a bicomplex polynomial with real positive coefficients and for some $k \leq 1$, $t > 0$, $\rho \geq 1$ and $\lambda \geq 1$,

$$ka_0 \leq ta_1 \leq t^2 a_2 \leq ... \leq t^\lambda \rho a_\lambda \geq t^{\lambda+1} a_{\lambda+1} \geq ... \geq t^n a_n.$$

Then $f(z)$ does not vanish in the discus $D(0; r_0, r_0)$ where

$$r_0 = \frac{ta_0}{(1-2k)a_0 + (2\rho t^\lambda a_\lambda + 2(\rho - 1)a_\lambda t^\lambda) + (|a_n| - a_n)t^{n+1}}.$$

Example 1.Let $f(z) = 5z^4 + 4z^3 + z^2 + 2z + 1$.
Here $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 4, a_4 = 5$. So, it follows that all the coefficients of $f(z)$ are positive real numbers and for $k = \frac{3}{5}$, $t = 1$, $\lambda = 2$ and $\rho = 5$. Here $r_0 \approx 0.056$. Hence by Theorem 2.1, we obtain $f(z)$ does not vanish in $D(0.056, 0.056)$.

Theorem 2.2. Let $f(z) = \sum_{j=0}^{n} a_j z^j$ be a bicomplex polynomial with complex coefficients such that $a_0 \neq 0$ and for some $t > 0$,

$$k|a_0| \geq t|a_1| \geq t^2|a_2| \geq ... \geq t^n|a_n|.$$

Then $f(z)$ does not vanish in the discus $D(0; r_0, r_0)$ where

$$r_0 = \frac{t|a_0|}{(|a_n|t^{n+1} + (2k-1)|a_0| + \big||a_0| - a_0\big|) + 2\sum_{j=1}^{n}\big||a_j| - a_j\big|t^j}.$$

Example 2.Let $f(z) = (1+i)z^4 + (2+i)z^3 + (1+i)z^2 + 1$.
Here $|a_0| = 1, |a_1| = 0, |a_2| = \sqrt{2}, |a_3| = \sqrt{5}, |a_4| = \sqrt{2}$. So, it follows that all the coefficients of $f(z)$ are complex numbers and for $k = 3$, $t = 1$, the condition of Theorem 2.2 are satisfied. Here $r_0 \approx 0.0742$. Hence by Theorem 2.2, we find that $f(z)$ does not vanish in $D(0.0742, 0.0742)$.

The following Theorem 2.3 and Theorem 2.4 generalize the results proved by Sandip et al.[4] for bicomplex entire functions, which are the bicomplex versions of the results already proved by Aziz and Mohammad.

Theorem 2.3. . Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a bicomplex entire function with real positive coefficients and for some $k \leq 1$, $t > 0$, $\rho \geq 1$ and $\lambda \geq 1$

$$ka_0 \leq ta_1 \leq t^2 a_2 \leq ... \leq t^\lambda \rho a_\lambda \geq t^{\lambda+1} a_{\lambda+1} \geq ... \quad .$$

Then $f(z)$ does not vanish in the discus $D(0; r_0, r_0)$ where

$$r_0 = \frac{ta_0}{(1-2k)a_0 + (2\rho t^\lambda a_\lambda + 2(\rho - 1)a_\lambda t^\lambda)}.$$

Example 3. Let us consider $f(z) = e^z + 2 + \frac{z}{2} + \frac{z^2}{3}$. Then $f(z) = 3 + \frac{3z}{2} + \frac{5z^2}{6} + \frac{z^3}{3!} + \frac{z^4}{4!} + ...$.
Here $a_0 = 3, a_1 = \frac{3}{2}, a_2 = \frac{5}{6}, a_j = \frac{1}{j!}, \quad j = 3, 4, ...$.
So, it follows that all the coefficients are positive real numbers, Theorem 1.1 is not applicable there but for $k = \frac{1}{6}, t = 1$ and $\lambda = 2, \rho = \frac{11}{5}$,

$$ka_0 \leq ta_1 \leq t^2 a_2 \leq ... \leq t^\lambda \rho a_\lambda \geq t^{\lambda+1} a_{\lambda+1} \geq ...$$

Here $r_0 \approx 0.391$. Hence by Theorem 2.3, we obtain $f(z) = e^z + 2 + \frac{z}{2} + \frac{z^2}{3}$ does not vanish in $D(0; 0.391, 0.391)$.
Remark If $\rho = 1$, then theorem 2.3 reduces to theorem 1.1.

Theorem 2.4. Let $f(z)=\sum_{j=0}^{\infty}a_jz^j$ be a bicomplex entire function with complex coefficients such that $a_0\neq 0$ and for some $t>0$,

$$k|a_0|\geq t|a_1|\geq t^2|a_2|\geq ...$$

Then no zero of $f(z)$ lie in the discus $D(0;r_0,r_0)$ where

$$r_0=\frac{t|a_0|}{((2k-1)|a_0|+\big||a_0|-a_0\big|)+2\sum_{j=1}^{\infty}\big||a_j|-a_j\big|t^j}.$$

Example 4 .Let $f(z)=(1+i)+(2+3i)z+(3-i)z^2+z^3$. Here, $a_0=1+i, a_1=2+3i, a_2=3-i, a_3=1, a_j=0, j=4,5,....$ So it follows that all the coefficients are complex numbers, Theorem 1.2 is not applicable there but for $k=4, t=1$, the condition of Theorem 2.4 are satisfied. Now, $r_0\approx 0.0713$. Hence by Theorem 2.4, the polynomial $P(z)$ has no zero in $D(0;0.0713,0.0713)$.

Remark If $k=1$, then Theorem 2.4 reduces to Theorem 1.2.

3 Lemmas

In this section, we present the following Lemmas [7] which will be needed in the forefront.

Lemma 3.1. [7] Let $X=X_1+X_2:=\{\phi_1e_1+\phi_2:\phi_1\in X_1,\phi_2\in X_2\}$ be a domain in $\mathbb{C}_2$. A bicomplex function $F=G_1e_1+G_2e_2:X\to\mathbb{C}_2$ is holomorphic if and only if both the component function G_1 and G_2 are holomorphic in X_1 and X_2 respectively.

Lemma 3.2. [7] Let F be a bicomplex holomorphic function defined in a domain $X=X_1e_1+X_2e_2:=\{\phi_1e_1+\phi_2:\phi_1\in X_1,\phi_2\in X_2\}$ such that $F(z)=G_1(\phi_1)e_1+G_2(\phi_2)e_2$, for all $z=\phi_1e_1+\phi_2e_2\in X$. Then $F(z)$ has zero on X if and only if $G_1(\phi_1)$ and $G_2(\phi_2)$ both have zero at ϕ_1 in X_1 and at ϕ_2 in X_2 respectively.

Lemma 3.3. [3]If $F(z)$ is holomorphic in $|z|\leq R$ in $\mathbb{C}_1$, $F(0)=0$ and $|F(z)|\leq M$ for $|z|=R$, then

$$|g(z)|\leq\frac{M|z|}{R}.$$

The above Lemma is termed as Schwarz's Lemma [3] in $\mathbb{C}_1$.

4 Proof of the Theorems

Proof of Theorem 2.1. Since $a_j=a_je_1+a_je_2$ and $z=\phi_1e_1+\phi_2e_2$,then $f(z)$ can be expressed as

$$\begin{aligned}f(z)&=\sum_{j=0}^{n}(a_je_1+a_je_2)(\phi_1e_1+\phi_2e_2)^j\\&=\sum_{j=0}^{n}(a_je_1+a_je_2)(\phi_1^je_1+\phi_2^je_2)\\&=\sum_{j=0}^{n}a_j\phi_1^je_1+\sum_{j=0}^{n}a_j\phi_2^je_2\\&=f_1(\phi_1)e_1+f_2(\phi_2)e_2\end{aligned}$$

where

$$f_1(\phi_1)=\sum_{j=0}^{n}a_j\phi_1^j\quad\text{and}\quad f_2(\phi_2)=\sum_{j=0}^{n}a_j\phi_2^j.$$

Since $f(z)$ is holomorphic in any closed discus $\overline{D}(0;t,t)\subset\mathbb{C}_2, 0<t<\infty$, by Lemma 3.1, $f_1(\phi_1)$ and $f_2(\phi_2)$ both are holomorhic respectively in $X_1=\{\phi_1\in A_1:|\phi_1|\leq t\}\subset\mathbb{C}_1$ and $X_2=\{\phi_2\in A_2:|\phi_2|\leq t\}\subset\mathbb{C}_1$.

Now, let us consider

$$
\begin{aligned}
F(\phi_1) =&(\phi_1 - t)f_1(\phi_1)\\
=&(\phi_1 - t)(a_0 + a_1\phi_1 + a_2\phi_1^2 + ... + a_\lambda\phi_1^\lambda + +a_{\lambda+1}\phi_1^{\lambda+1} + ... + a_n\phi^n)\\
=&-ta_0 + (a_0 - ta_1)\phi_1 + (a_1 - ta_2)\phi_1^2 + ... + (a_{\lambda-1} - ta_\lambda)\phi_1^\lambda + (a_\lambda - ta_{\lambda+1})\phi_1^{\lambda+1} + ...\\
&+ (a_{n-1} - ta_n)\phi^n - a_n\phi_1^{n+1}\\
=&-ta_0 + (a_0 - ka_0 + ka_0 - ta_1)\phi_1 + (a_1 - ta_2)\phi_1^2 + ... + (a_{\lambda-1} - \rho a_\lambda t + \rho a_\lambda t + ta_\lambda)\phi_1^\lambda\\
&+ (a_\lambda - \rho a_\lambda + \rho a_\lambda - ta_{\lambda+1})\phi_1^{\lambda+1} + ... + (a_{n-1} - ta_n)\phi_1^n - a_n\phi_1^{n+1}\\
=&-ta_0 + (1 - k)a_0\phi_1 + R(\phi_1)
\end{aligned}
$$

where

$$
\begin{aligned}
R(\phi_1) =&(ka_0 - ta_1)\phi_1 + (a_1 - ta_2)\phi_1^2 + ... + (a_{\lambda-1} - \rho a_\lambda t + \rho a_\lambda t - ta_\lambda)\phi_1^\lambda\\
&+ (a_\lambda - \rho a_\lambda + \rho a_\lambda - ta_{\lambda+1})\phi_1^{\lambda+1} + ... + (a_{n-1} - ta_n)\phi_1^n - a_n\phi_1^{n+1}.
\end{aligned}
$$

Also, for $|\phi_1| = t$,

$$
\begin{aligned}
|R(\phi_1)| \le&|ka_0 - ta_1||\phi_1| + |a_1 - ta_2||\phi_1|^2 + ... + (\rho - 1)a_\lambda t|\phi_1|^\lambda + |a_{\lambda-1} - \rho a_\lambda t||\phi_1|^\lambda\\
&+ (\rho - 1)a_\lambda|\phi_1|^{\lambda+1} + |\rho a_\lambda - ta_{\lambda+1}||\phi_1|^{\lambda+1} + ... + |a_{n-1} - ta_n||\phi_1|^n + |a_n||\phi_1|^{n+1}\\
=&(ta_1 - ka_0)t + (ta_2 - a_1)t^2 + ... + (\rho - 1)a_\lambda t^{\lambda+1} + (\rho a_\lambda t - a_{\lambda-1})t^\lambda + (\rho - 1)a_\lambda t^{\lambda+1}\\
&+ (\rho a_\lambda - ta_{\lambda+1})t^{\lambda+1} + ... + (a_{n-1} - ta_n)t^n - a_n t^{n+1}\\
=&(|a_n| - a_n)t^{n+1} + 2\rho t^{\lambda+1}a_\lambda + 2(\rho - 1)a_\lambda t^{\lambda+1} - ka_0 t.
\end{aligned}
$$

Now, $R(\phi_1)$ is holomorphic in $|\phi_1| \le t$.
Also, $R(0) = 0$ and $R(\phi_1) \le \big((|a_n| - a_n)t^n + 2\rho t^\lambda a_\lambda + 2(\rho - 1)a_\lambda t^\lambda - ka_0\big)t$ for $|\phi_1| = t$. Therefore, by Lemma 3.3, we get

$$
\begin{aligned}
|R(\phi_1)| \le&\frac{((|a_n| - a_n)t^n + 2\rho t^\lambda a_\lambda + 2(\rho - 1)a_\lambda t^\lambda - ka_0\big)t|\phi_1|}{t}\\
=&((|a_n| - a_n)t^n + 2\rho t^\lambda a_\lambda + 2(\rho - 1)a_\lambda t^\lambda - ka_0\big)|\phi_1|.
\end{aligned}
$$

For $|\phi_1| < t$, we see that

$$
\begin{aligned}
|F(\phi_1)| \ge&|-ta_0 + (1 - k)a_0\phi_1| - |R(\phi_1)|\\
\ge&ta_0 - (1 - k)a_0|\phi_1| - ((|a_n| - a_n)t^n + 2\rho t^\lambda a_\lambda + 2(\rho - 1)a_\lambda t^\lambda - ka_0\big)|\phi_1|\\
=&ta_0 - \{(1 - 2k)a_0 + ((|a_n| - a_n)t^n + 2\rho t^\lambda a_\lambda + 2(\rho - 1)a_\lambda t^\lambda\}|\phi_1|\\
>&0 \ if \ |\phi_1| < \frac{ta_0}{(1 - 2k)a_0 + ((|a_n| - a_n)t^n + 2\rho t^\lambda a_\lambda + 2(\rho - 1)a_\lambda t^\lambda}.
\end{aligned}
$$

Therefore, for $|\phi_1| < t$
$|f_1(\phi_1)| > 0$ if $|\phi_1| < r_0$, where $r_0 = \frac{ta_0}{(1-2k)a_0+((|a_n|-a_n)t^n+2\rho t^\lambda a_\lambda+2(\rho-1)a_\lambda t^\lambda)}$.

Similarly for $|\phi_2| < t$, $|f_2(\phi_2)| > 0 \ if \ |\phi_2| < r_0$.
Thus both $f_1(\phi_1)$ and $f_2(\phi_2)$ have no zeros in $X_1' = \{\phi_1 \in X_1 : |\phi_1| < r_0\}$ and $X_2' = \{\phi_2 \in X_2 : |\phi_2| < r_0\}$.

Consequently, by Lemma 3.2 $f(z) = f_1(\phi_1)e_1 + f_2(\phi_2)e_2$ has no zero in $X_1'e_1 + X_2'e_2 = D(0, r_0, r_0)$.

Proof of Theorem 2.2 : Since $f(z)$ can be expressed as

$$
\begin{aligned}
f(z) =& \sum_{j=0}^{n} a_j\phi_1^j e_1 + \sum_{j=0}^{n} a_j\phi_2^j e_2\\
=& f_1(\phi_1)e_1 + f_2(\phi_2)e_2.
\end{aligned}
$$

Since $f(z)$ is holomorphic in any closed discus $\overline{D}(0;t,t) \subset \mathbb{C}_2, 0 < t < \infty$, by Lemma 2.1, $f_1(\phi_1)$ and $f_2(\phi_2)$ both are holomorhic respectively in $X_1 = \{\phi_1 \in A_1 : |\phi_1| \leq t\} \subset \mathbb{C}_1$ and $X_2 = \{\phi_2 \in A_2 : |\phi_2| \leq t\} \subset \mathbb{C}_1$.
Now, let us consider

$$\begin{aligned}
F(\phi_1) =&(\phi_1 - t)f_1(\phi_1),\\
=&(\phi_1 - t)(a_0 + a_1\phi_1 + a_2\phi_1^2 + ... + a_n\phi_1^n)\\
=&-ta_0 + (a_0 - ta_1)\phi_1 + (a_1 - ta_2)\phi_1^2 + ... + (a_{n-1} - ta_n)\phi_1^n + a_n\phi_1^{n+1}\\
=&-ta_0 + a_n\phi_1^{n+1} + R(\phi_1), where\ R(\phi_1) = \sum_{j=1}^{n}(a_{j-1} - ta_j)\phi_1^j.
\end{aligned}$$

For $|\phi_1| = t$,

$$\begin{aligned}
|R(\phi_1)| =&|\sum_{j=1}^{n}(a_{j-1} - ta_j)\phi_1^j|\\
=&|\sum_{j=1}^{n}\{(|a_{j-1}| - t|a_j|) + (a_{j-1} - |a_{j-1}|) + t(|a_j| - a_j)\}\phi_1^j|\\
\leq&\sum_{j=1}^{n}\big||a_{j-1}| - t|a_j|\big|t^j + \sum_{j=1}^{n}\big||a_{j-1}| - a_{j-1}\big|t^j + \sum_{j=1}^{n}\big||a_j| - a_j\big|t^{j+1}\\
=&\big||a_0| - k|a_0| + k|a_0| - t|a_1|\big|t + \sum_{j=2}^{n}\big(|a_{j-1}| - t|a_j|\big)t^j + \sum_{j=1}^{n}\big||a_{j-1}| - a_{j-1}\big|t^j\\
&+ \sum_{j=1}^{n}\big||a_j| - a_j\big|t^{j+1}\\
=&(k-1)|a_0|t + (k|a_0| - t|a_1|)t + \sum_{j=2}^{n}\big(|a_{j-1}| - t|a_j|\big)t^j + \sum_{j=1}^{n}\big||a_{j-1}| - a_{j-1}\big|t^j\\
&+ \sum_{j=1}^{n}\big||a_j| - a_j\big|t^{j+1}\\
=&t\big((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{n}\big||a_j| - a_j\big|t^{j+1}.
\end{aligned}$$

Since $R(\phi_1)$ is holomorphic in $|\phi_1| \leq t$. Also, $R(0) = 0$ and

$$R(\phi_1) \leq t\big((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{n}\big||a_j| - a_j\big|t^{j+1} \quad \text{for} \quad |\phi_1| = t,$$

by Lemma 3.3

$$\begin{aligned}
R(\phi_1) \leq&\frac{t\big((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{n}\big||a_j| - a_j\big|t^{j+1}}{t}\\
=&\big((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{n}\big||a_j| - a_j\big|t^j
\end{aligned}$$

Therefore, for $|\phi_1| < t$,

$$\begin{aligned}|F(\phi_1)| \geq & t|a_0| - \left(|a_n|t^{n+1} + R(\phi_1)\right) \\ \geq & t|a_0| - (|a_n|t^{n+1} + (2k-1)|a_0| + \left||a_0| - a_0\right|) + 2\sum_{j=1}^{n}\left||a_j| - a_j\right|t^j \\ > & 0, \text{ if } |\phi_1| < \frac{t|a_0|}{|a_n|t^{n+1} + ((2k-1)|a_0| + \left||a_0| - a_0\right|) + 2\sum_{j=1}^{n}\left||a_j| - a_j\right|t^j}.\end{aligned}$$

Hence for $|\phi_1| < t$,

$$|f_1(\phi_1)| > 0 \; if \; |\phi_1| < r_0 \; where \; r_0 = \frac{t|a_0|}{(|a_n|t^{n+1} + (2k-1)|a_0| + \left||a_0| - a_0\right|) + 2\sum_{j=1}^{n}\left||a_j| - a_j\right|t^j}.$$

Similarly for $|\phi_2| < t$,$|f_2(\phi_2)| > 0 \; if \; |\phi_2| < r_0$. Thus both $f_1(\phi_1)$ and $f_2(\phi_2)$ have no zeros in $X_1' = \{\phi_1 \in X_1 : |\phi_1| < r_0\}$ and $X_2' = \{\phi_2 \in X_2 : |\phi_2| < r_0\}$.

Consequently, by Lemma 3.2 $f(z) = f_1(\phi_1)e_1 + f_2(\phi_2)e_2$ has no zero in $X_1'e_1 + X_2'e_2 = D(0; r_0, r_0)$.

Proof of Theorem 2.3. Since $a_j = a_je_1 + a_je_2$ and $z = \phi_1e_1 + \phi_2e_2$, $f(z)$ can be expressed as

$$\begin{aligned}f(z) &= \sum_{j=0}^{\infty}(a_je_1 + a_je_2)(\phi_1e_1 + \phi_2e_2)^j \\ &= \sum_{j=0}^{\infty}(a_je_1 + a_je_2)(\phi_1^je_1 + \phi_2^je_2) \\ &= \sum_{j=0}^{\infty}a_j\phi_1^je_1 + \sum_{j=0}^{\infty}a_j\phi_2^je_2 \\ &= f_1(\phi_1)e_1 + f_2(\phi_2)e_2\end{aligned}$$

where

$$f_1(\phi_1) = \sum_{j=0}^{\infty}a_j\phi_1^j \quad \text{and} \quad f_2(\phi_2) = \sum_{j=0}^{\infty}a_j\phi_2^j.$$

Since $f(z)$ is holomorphic in any closed discus $\overline{D}(0; t, t) \subset \mathbb{C}_2, 0 < t < \infty$, by Lemma 3.1, $f_1(\phi_1)$ and $f_2(\phi_2)$ both are holomorphic respectively in $X_1 = \{\phi_1 \in A_1 : |\phi_1| \leq t\} \subset \mathbb{C}_1$ and $X_2 = \{\phi_1 \in A_2 : |\phi_2| \leq t\} \subset \mathbb{C}_1$.

Clearly $\lim\limits_{\alpha \to \infty} a_jt^j = 0$. Now, let us consider

$$\begin{aligned}F(\phi_1) =& (\phi_1 - t)f_1(\phi_1), \\ =& (\phi_1 - t)(a_0 + a_1\phi_1 + a_2\phi_1^2 + ... + a_\lambda\phi_1^\lambda + +a_{\lambda+1}\phi_1^{\lambda+1} + ...) \\ =& -ta_0 + (a_0 - ta_1)\phi_1 + (a_1 - ta_2)\phi_1^2 + ... + (a_{\lambda-1} - ta_\lambda)\phi_1^\lambda + (a_\lambda - ta_{\lambda+1})\phi_1^{\lambda+1} + ... \\ =& -ta_0 + (a_0 - ka_0 + ka_0 - ta_1)\phi_1 + (a_1 - ta_2)\phi_1^2 + ... + (a_{\lambda-1} - \rho a_\lambda t + \rho a_\lambda t + ta_\lambda)\phi_1^\lambda \\ &+ (a_\lambda - \rho a_\lambda + \rho a_\lambda - ta_{\lambda+1})\phi_1^{\lambda+1} + ... \\ =& -ta_0 + (1-k)a_0\phi_1 + R(\phi_1)\end{aligned}$$

where

$$\begin{aligned}R(\phi_1) =& (ka_0 - ta_1)\phi_1 + (a_1 - ta_2)\phi_1^2 + ... + (a_{\lambda-1} - \rho a_\lambda t + \rho a_\lambda t - ta_\lambda)\phi_1^\lambda + \\ &+ (a_\lambda - \rho a_\lambda + \rho a_\lambda - ta_{\lambda+1})\phi_1^{\lambda+1} + ... \;.\end{aligned}$$

Also, for $|\phi_1| = t$,

$$\begin{aligned}|R(\phi_1)| \leq & |ka_0 - ta_1||\phi_1| + |a_1 - ta_2||\phi_1|^2 + ... + (\rho-1)a_\lambda t|\phi_1|^\lambda + |a_{\lambda-1} - \rho a_\lambda t||\phi_1|^\lambda \\ & + (\rho-1)a_\lambda|\phi_1|^{\lambda+1} + |\rho a_\lambda - ta_{\lambda+1}||\phi_1|^{\lambda+1} + ... \\ =& (ta_1 - ka_0)t + (ta_2 - a_1)t^2 + ... + (\rho-1)a_\lambda t^{\lambda+1} + (\rho a_\lambda t - a_{\lambda-1})t^\lambda + (\rho-1)a_\lambda t^{\lambda+1} \\ & + (\rho a_\lambda - ta_{\lambda+1})t^{\lambda+1} + ... \\ =& 2\rho t^{\lambda+1}a_\lambda + 2(\rho-1)a_\lambda t^{\lambda+1} - ka_0 t.\end{aligned}$$

Now, $R(\phi_1)$ is holomorphic in $|\phi_1| \leq t$. Also, $R(0) = 0$ and $R(\phi_1) \leq \big(2\rho t^\lambda a_\lambda + 2(\rho-1)a_\lambda t^\lambda - ka_0\big)t$ for $|\phi_1| = t$. Therefore, by Lemma 3.3, we get

$$\begin{aligned}|R(\phi_1)| \leq & \frac{(2\rho t^\lambda a_\lambda + 2(\rho-1)a_\lambda t^\lambda - ka_0)t|\phi_1|}{t} \\ =& \big(2\rho t^\lambda a_\lambda + 2(\rho-1)a_\lambda t^\lambda - ka_0\big)|\phi_1|.\end{aligned}$$

For $|\phi_1| < t$, we see that

$$\begin{aligned}|F(\phi_1)| \geq & | - ta_0 + (1-k)a_0\phi_1| - |R(\phi_1)| \\ \geq & ta_0 - (1-k)a_0|\phi_1| - \big(2\rho t^\lambda a_\lambda + 2(\rho-1)a_\lambda t^\lambda - ka_0\big)|\phi_1| \\ =& ta_0 - \{(1-2k)a_0 + (2\rho t^\lambda a_\lambda + 2(\rho-1)a_\lambda t^\lambda\} \\ >& 0 \; if \; |\phi_1| < \frac{ta_0}{(1-2k)a_0 + (2\rho t^\lambda a_\lambda + 2(\rho-1)a_\lambda t^\lambda)}.\end{aligned}$$

Therefore, for $|\phi_1| < t$, $|f_1(\phi_1)| > 0$ if $|\phi_1| < r_0$, where $r_0 = \frac{ta_0}{(1-2k)a_0 + (2\rho t^\lambda a_\lambda + 2(\rho-1)a_\lambda t^\lambda}$. Similarly, $|f_2(\phi_2)| > 0$ if $|\phi_2| < r_0$.

Thus both $f_1(\phi_1)$ and $f_2(\phi_2)$ have no zeros in $X_1' = \{\phi_1 \in X_1 : |\phi_1| < r_0\}$ and $X_2' = \{\phi_2 \in X_2 : |\phi_2| < r_0\}$. Consequently, by Lemma 3.2 $f(z) = f_1(\phi_1)e_1 + f_2(\phi_2)e_2$ has no zero in $X_1'e_1 + X_2'e_2 = D(0; r_0, r_0)$.

Proof of Theorem 2.4: $f(z)$ can be expressed as

$$\begin{aligned}f(z) =& \sum_{j=0}^{\infty} a_j\phi_1^j e_1 + \sum_{j=0}^{\infty} a_j\phi_2^j e_2 \\ =& f_1(\phi_1)e_1 + f_2(\phi_2)e_2.\end{aligned}$$

Since $f(z)$ is holomorphic in any closed discus $\overline{D}(0; t, t) \subset \mathbb{C}_2, 0 < t < \infty$, by Lemma 3.1, $f_1(\phi_1)$ and $f_2(\phi_2)$ both are holomorphic respectively in $X_1 = \{\phi_1 \in A_1 : |\phi_1| \leq t\} \subset \mathbb{C}_1$ and $X_2 = \{\phi_2 \in A_2 : |\phi_2| \leq t\} \subset \mathbb{C}_1$.

Also, $\lim_{\alpha\to\infty} a_j t^j = 0$. Now, let us consider

$$\begin{aligned}F(\phi_1) =& (\phi_1 - t)f_1(\phi_1), \\ =& (\phi_1 - t)(a_0 + a_1\phi_1 + a_2\phi_1^2 + ...) \\ =& -ta_0 + (a_0 - ta_1)\phi_1 + (a_1 - ta_2)\phi_1^2 + ... \\ =& -ta_0 + R(\phi_1), where\ R(\phi_1) = \sum_{j=1}^{\infty}(a_{j-1} - ta_j)\phi_1^j.\end{aligned}$$

For $|\phi_1| = t,$,

$$\begin{aligned}
|R(\phi_1)| =& |\sum_{j=1}^{\infty}(a_{j-1} - ta_j)\phi_1^j| \\
=& |\sum_{j=1}^{\infty}\{(|a_{j-1}| - t|a_j|) + (a_{j-1} - |a_{j-1}|) + t(|a_j| - a_j)\}\phi_1^j| \\
\leq& \sum_{j=1}^{\infty}\big||a_{j-1}| - t|a_j|\big|t^j + \sum_{j=1}^{\infty}\big||a_{j-1}| - a_{j-1}\big|t^j + \sum_{j=1}^{\infty}\big||a_j| - a_j\big|t^{j+1} \\
=& \big||a_0| - k|a_0| + k|a_0| - t|a_1|\big|t + \sum_{j=2}^{\infty}\big(|a_{j-1}| - t|a_j|\big)t^j + \sum_{j=1}^{\infty}\big||a_{j-1}| - a_{j-1}\big|t^j + \sum_{j=1}^{\infty}\big||a_j| - a_j\big|t^{j+1} \\
=& (k-1)|a_0|t + (k|a_0| - t|a_1|)t + \sum_{j=2}^{\infty}\big(|a_{j-1}| - t|a_j|\big)t^j + \sum_{j=1}^{\infty}\big||a_{j-1}| - a_{j-1}\big|t^j + \sum_{j=1}^{\infty}\big||a_j| - a_j\big|t^{j+1} \\
=& t\big((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{\infty}\big||a_j| - a_j\big|t^{j+1}.
\end{aligned}$$

Since $R(\phi_1)$ is holomorphic in $|\phi_1| \leq t$. Also, $R(0) = 0$ and

$$R(\phi_1) \leq t\big((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{\infty}\big||a_j| - a_j\big|t^{j+1} \quad \text{for} \quad |\phi_1| = t,$$

by Lemma 3.3, we have

$$\begin{aligned}
R(\phi_1) \leq& \frac{t\big((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{\infty}\big||a_j| - a_j\big|t^{j+1}}{t} \\
=& \big((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{\infty}\big||a_j| - a_j\big|t^j
\end{aligned}$$

Therefore, for $|\phi_1| < t$,

$$\begin{aligned}
|F(\phi_1)| \geq& t|a_0| - R(\phi_1) \\
\geq& t|a_0| - ((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{\infty}\big||a_j| - a_j\big|t^j \\
>& 0, \text{ if } \ |\phi_1| < \frac{t|a_0|}{((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{\infty}\big||a_j| - a_j\big|t^j}.
\end{aligned}$$

Hence for $|\phi_1| < t$,

$$|f_1(\phi_1)| > 0 \ if \ |\phi_1| < r_0 \ where \ r_0 = \frac{t|a_0|}{((2k-1)|a_0| + \big||a_0| - a_0\big|\big) + 2\sum_{j=1}^{\infty}\big||a_j| - a_j\big|t^j}.$$

Similarly for $|\phi_2| < t$, $|f_2(\phi_2)| > 0 \ if \ |\phi_2| < r_0$. Thus both $f_1(\phi_1)$ and $f_2(\phi_2)$ have no zeros in $X_1' = \{\phi_1 \in X_1 : |\phi_1| < r_0\}$ and $X_2' = \{\phi_2 \in X_2 : |\phi_2| < r_0\}$. Consequently, by Lemma 3.2 $f(z) = f_1(\phi_1)e_1 + f_2(\phi_2)e_2$ has no zero in $X_1'e_1 + X_2'e_2 = D(0; r_0, r_0)$.

References

[1] A. Aziz and Q.G. Mohammad, *On the zeros of a certain class of polynomials and related analytic functions*, J. Math. Anal. Appl. **75** (1980), 495–502.

[2] A. Aziz and Q.G. Mohammad, *Zero-free regions for polynomials and some generalizations of Eneström-Kakeya theorem*, Can. Math. Bull. **27** (1984), no. 3, 265–272.

[3] J.B. Conway, *Function of one complex variable*, 2nd edition, Springer-Verlag, 1978.

[4] S.K. Datta, T. Molla, M. Sk and J. Saha, *A note on the bicomplex version of Eneström-Kakeya theorem*, J.Xi'an Univ. Agricul. Technol. **XII** (2020), 593–603.

[5] M.E. Luna-Elizarrarás, M. Shapiro, D.C. Struppa and A. Vajiac: *Bicomplex numbers and their elementary functions*, Cubo **14** (2012), no. 2, 61–80.

[6] M.E. Luna-Elizarrarás, M. Shapiro, D.C. Struppa and A. Vajiac, *Bicomplex holomorphic functions: The algebra, geometry and analysis of bicomplex numbers*, 1st edition, Birkhauser, 2015.

[7] G.B. Price, *An introduction to multicomplex spaces and functions*, Marcel Dekker, New York, 2nd Edition, 1991.