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The preimage of $A_\infty(Q_0)$ for the local Hardy-Littlewood maximal operator

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Abstract

We describe here all those weight functions u such that $Mu \in A_\infty(Q)$ for M the local Hardy-Littlewood maximal operator restricted to a cube $Q \subset \mathbb{R}^n$. In a recent paper it is shown that for the maximal operator in $\mathbb{R}^n$, $Mu \in A_\infty$ implies that $Mu \in A_1$; here we see that the same is true for the local M but this imposes a stronger condition for weights in Q, that is, for M restricted to a finite cube $Mu \in A_\infty$ if and only if $u \in A_\infty$. This differs from the case in $\mathbb{R}^n$ where there are weights u not belonging to A_∞ such that Mu is in A_∞. As an application we get a new shorter proof of a result of I. Wik. We also give a characterization for those weights in terms the K-functional of Peetre.

Keywords: Maximal Operators, A_∞ classes, Weigths, Rearrangements
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1 Introduction

The goal of this work is to characterize the weights u on a cube $Q_0 \subset \mathbb{R}^n$(Let's point out that along this work the cubes have their sides parallel the coordinate axes, and a weight is a positive measurable function in a cube). such that Mu are in $A_\infty(Q_0) = \bigcup_{p=1}^{\infty} A_p(Q_0)$ where $A_p(Q_0)$ are the Muckenhoupt classes of weights for M the local maximal operators of Hardy-Littlewood associated with a fixed cube Q_0, that is:

$$Mf(x) = \sup_{x \in Q \subset Q_0} \frac{1}{|Q|} \int_Q |f(z)|\, dz$$

To our knowledge, there are no previous works characterizing the weights in the preimage of A_∞ for the local maximal operator M.

We will show that as in the case for M in the whole $\mathbb{R}^n$, if $Mu \in A_\infty(Q_0)$ then, $Mu \in A_1(Q_0)$ (see 5 for $\mathbb{R}^n$). Then, following a result from 2 we have that weights u satisfying that $Mu \in A_\infty(Q_0)$ (and a fortiori $Mu \in A_1(Q_0)$) can be characterized by means of an inequality for Peetre's $K-functional$. Thus, this inequality ensures that u must satisfy a reverse Hölder condition, RH_p, for some $p > 1$ but this implies that u itself belongs to $A_\infty(Q_0)$. This contrasts with the $\mathbb{R}^n$ case, where there are weights not belonging to A_∞ but such that $Mu \in A_\infty$ -for instance weak-A_∞ weights-.

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As an application we give a new proof of an interesting result: In 12 I. Wik proved that if $u \in A_p(Q_0)$ then $u^* \in A_p([0,|Q_0|])$, being u^* the non-increasing rearrangement of u. Although it is not mentioned in 12, an immediate consequence of this is the fact that equimeasurable A_∞ weight functions supported on finite cubes included in $\mathbb{R}^n$, even for different n, belong to the same A_p classes. The proof we give here is much shorter; the original requires several previous lemmas, including one on coverings subsumed here in the Herz-Stein equivalence.

So the main results of this work are the following:

Proposition 1.1. Let $Q_0 \subset \mathbb{R}^n$ and u a weight $Mu \in A_1(Q_0)$ if and only $Mu \in A_\infty(Q_0)$.

Theorem 1.2. A weight u satisfies $Mu \in A_\infty(Q_0)$ if and only if for some $C>0$, $s>1$ and for any $Q \subset Q_0$ and

$$\left(\frac{1}{t}K\left(t,u^s,L^1,L^\infty\right)\right)^{\frac{1}{s}} \leq C.\frac{1}{t}K\left(t,u,L^1,L^\infty\right) \tag{1.1}$$

for $0<t<|Q|$, where L^1 and L^∞ means $L^1(Q)$ and $L^\infty(Q)$.

And finally, because it will be seen that the condition

$$\left(\frac{1}{t}K\left(t,u^s,L^1,L^\infty\right)\right)^{\frac{1}{s}} \leq C.\frac{1}{t}K\left(t,u,L^1,L^\infty\right) \tag{1.2}$$

for $0<t<|Q|$ implies that $u \in A_\infty(Q_0)$, we have:

Theorem 1.3. Let u a weight on a cube Q_0, the following statements are equivalent:

i) $u \in A_\infty(Q_0)$

ii) $u \in \bigcup_{r>1} RH_r(Q_0)$

iii) $(Mu^s)^{\frac{1}{s}}(x) \leq C.Mu(x)$ for some $s>1$, $C>0$ and $a.e.\ x \in Q_0$

iv) $Mu \in A_1(Q_0)$

v) $Mu \in A_\infty(Q_0)$

vi) $\exists C>0, s>1 : \left(\frac{1}{t}K\left(t,u^s,L^1,L^\infty\right)\right)^{\frac{1}{s}} \leq C.\frac{1}{t}K\left(t,u,L^1,L^\infty\right)$ for $0<t<|Q|$, $\forall Q \subset Q_0$.

Theorem 1.4. (I. Wik) Let $u \in A_p(Q_0)$ for a finite cube $Q_0 \subset \mathbb{R}^n$. Then $u^* \in A_p([0,|Q_0|])$ for u^* the non-increasing rearrangement of u.

2 Definitions, lemmas and some of the proofs

The definition of $A_p(Q_0)$ and $RH_p(Q_0)$ classes is analogous to the definition of A_p and RH_p classes in $\mathbb{R}^n$, but requiring that the cubes were included in Q_0. To lighten the notation, from now on, if there is no ambiguity we will write A_p, A_∞, RH_p instead of $A_p(Q_0)$, $A_\infty(Q_0)$ and $RH_p(Q_0)$.

A weight w is a non-negative locally integrable function. A weight $w \in A_p$ class for $1<p<\infty$ if and only if

$$[w]_{A_p} := \sup_{Q \subset Q_0}\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\right)^{p-1} < +\infty$$

A weight $w \in A_1$ if and only if

$$Mw(x) \leq Cw(x)\ \ a.e.\ x \in Q_0$$

and $[w]_{A_1}$ is the minimal constant C such that this inequality occurs.

Of course we will denote $A_\infty = \bigcup_{p<\infty} A_p$ and in this section A_p classes and $[A_p]$ constants refers to the ones for the local maximal Hardy-Littlewood operator for Q_0.

We also define the reverse Hölder classes: $w \in RH_r$ for $r > 1$ if and only if w fullfills a reverse Hölder inequality with exponent r for each $Q \subset Q_0$: $\left(\frac{1}{|Q|}\int_Q w^r\right)^{\frac{1}{r}} \leq C.\frac{1}{|Q|}\int_Q w$ with C independent from Q. It is well known (cf 6) that $A_\infty = \bigcup_{r>1} RH_r$.

As we have mentioned in the introduction we can reduce the problem of describing the weights whose image is in A_∞ to those whose images are in A_1; actually $Mu \in A_\infty \iff Mu \in A_1$ it is true both for the local and for the usual maximal operator of Hardy-Littlewood (see 5 for the usual, non-local, case); the proof for the non-local operator still works if we show that $(Mu)^\delta$ is in A_1 for $0 \leq \delta < 1$ and M the local maximal operator respect to Q_0. For the usual maximal operator of Hardy-Littlewood this is the first statement of the characterization of Coifman and Rochberg for the A_1 weights. The analogous result is true for the local operator, moreover we have:

Lemma 2.1. Let Q_0 any domain in $\mathbb{R}^n$ and $Mf(x) = \sup_{x\in Q\subset Q_0} \frac{1}{|Q|}\int_Q |f(z)|\,dz$

(1) Let $f \in L^1(Q_0)$ be such that $Mf(x) < \infty$ a.e. and $0 \leq \delta < 1$, then $w(x) = (Mf(x))^\delta$ is in A_1. Also the A_1 constant depends only on δ.

(2) Conversely, if $w \in A_1$ then there are $f \in L^1(Q_0)$ and $k(x)$ with k and k^{-1} both belonging to L^∞ such that $w(x) = k(x)(Mf(x))^\delta$.

The result is probably part of the folklore of the subject but we didn't see the result explicity written so we give the proof:

Proof . For the first statement we can rely in the corresponding result for $\mathbb{R}^n$. Then, let's write $M_{\mathbb{R}^n}$ for the usual Hardy-Littlewood operator in $\mathbb{R}^n$ and we keep M for the local maximal operator respect to Q_0. Also we extend f being null outside Q_0, and thus, if $x \in Q_0$, then $M_{\mathbb{R}^n}(f)(x) \leq M(f\cdot\chi_{Q_0})(x) = Mf(x)$. Therefore

$$M\left((Mf)^\delta\right)(x) \leq M_{\mathbb{R}^n}\left((Mf)^\delta\right)(x) = M_{\mathbb{R}^n}\left((Mf\cdot\chi_{Q_0})^\delta\right)(x)$$

$$\leq C\cdot(Mf\cdot\chi_{Q_0})^\delta(x) = C\cdot(Mf(x))^\delta$$

where the second inequality follows from the corresponding theorem for the usual case.

And then $(Mf(x))^\delta \in A_1$. The dependence of the constant only on δ is inherited for the usual case.

For the proof of (2) we can observe that for any cube $Q_1 \subset Q_0$ the local operator respect to Q_1: $M_{Q_1}f(x) = \sup_{x\in Q\subset Q_1} \frac{1}{|Q|}\int_Q |f(z)|\,dz$ satisfies $M_{Q_1}w \leq Mw \leq cw$ and then $M_{Q_1}(M_{Q_1}w) \leq cM_{Q_1}w$ and this condition for local M implies a reverse Hölder inequality for w in Q_1 (see for instance 2 sections 3 and 4 for two different proofs) with r and C independent from the cube Q_1:

$$\left(\frac{1}{|Q_1|}\int_{Q_1} w^r\right)^{\frac{1}{r}} \leq C.\frac{1}{|Q_1|}\int_{Q_1} w$$

and then Hölder inequality:

$$\frac{1}{|Q_1|}\int_{Q_1} w \leq \left(\frac{1}{|Q_1|}\int_{Q_1} w^r\right)^{\frac{1}{r}} \leq C.\frac{1}{|Q_1|}\int_{Q_1} w$$

and taking suprema for $Q_1 \subset Q$ and using that $a.e.$ in Q is $w \leq Mw \leq cw$ we have:

$$w \leq Mw \leq (M(w^r))^{\frac{1}{r}} \leq C.Mw \leq cCw$$

and then for $f = w^r$ and $\delta = \frac{1}{r} \in (0,1)$ we have for $a.e.$ $x \in Q_0$:

$$1 \leq \frac{(M(f))^\delta(x)}{w(x)} \leq cC$$

so if $k(x) = \frac{w(x)}{(M(f))^\delta(x)}$ we have that $k \in L^\infty(Q_0)$ and $k^{-1} \in L^\infty(Q_0)$ and $w(x) = k(x)(Mf(x))^\delta$ as we wished to prove. □

As we mentioned before, using this local version of the theorem, we can obtain for the local operator M the proposition that follows below. The analogous result for $M_{\mathbb{R}^n}$ can be found in 5. The proof for local M goes in the same way once it was established the statement of the previous lemma with the Coifman-Rochberg characterization for $A_1(Q_0)$; but we include it here for completeness:

Proposition 2.2. If u is any weight, $Mu \in A_\infty \iff Mu \in A_1$

Proof . The implication $Mu \in A_1 \implies Mu \in A_\infty$ is obvious because $A_1 \subset A_\infty$. So we need only to prove that $Mu \in A_\infty \implies Mu \in A_1$.

If $Mu \in A_\infty = \bigcup_{p<\infty} A_p$, then $Mu \in A_p$ for some $p \geq 1$. If $p = 1$ the result is ready. Let $p > 1$. Because of the latter lemma we have that $(Mu)^\delta \in A_1$ for any δ with $0 \leq \delta < 1$ and any u locally integrable. We will see that if $Mu \in A_p$ actually we can extend δ to be 1, that is: $Mu \in A_1$.

We will use the following result: For a measure space (Ω, μ) with measure $\mu(\Omega) = 1$ and $\left(\int_\Omega |f|^r \, d\mu\right)^{\frac{1}{r}} < \infty$ for some $r > 0$, we have that

$$\lim_{r \to 0^+} \left(\int_\Omega |f|^r \, d\mu\right)^{\frac{1}{r}} = \exp\left(\int_\Omega \log(|f|) \, d\mu\right)$$

(see, for instance, 11, ej 5 d) Chap 3).

Let's remark that using that $\mu(\Omega) = 1$ and Hölder Inequality we obtain $\left(\int_\Omega |f|^{r_1} \, d\mu\right)^{\frac{1}{r_1}} \geq \left(\int_\Omega |f|^{r_2} \, d\mu\right)^{\frac{1}{r_2}}$ if $r_1 \geq r_2$. So for $r > 0$ we have that

$$\left(\int_\Omega |f|^r \, d\mu\right)^{\frac{1}{r}} \geq \exp\left(\int_\Omega \log(|f|) \, d\mu\right) = \lim_{r \to 0^+} \left(\int_\Omega |f|^r \, d\mu\right)^{\frac{1}{r}}$$

.

Now for $q > p$,and using that

$$\sup_Q \frac{Mu(Q)}{|Q|} \left(\frac{1}{|Q|} \int_Q Mu(x)^{-\frac{1}{q-1}} \, dx\right)^{q-1} = [Mu]_{A_q} \leq [Mu]_{A_p}$$

, we obtain that for any cube $Q \subset Q_0$:

$$\frac{Mu(Q)}{|Q|} \left(\frac{1}{|Q|} \int_Q Mu(x)^{-\frac{1}{q-1}} \, dx\right)^{q-1} \leq [Mu]_{A_p} < \infty$$

.

If q tends to infinity then $\frac{1}{q-1}$ tends to 0^+, so taking $r = \frac{1}{q-1}$ and applying the mentioned result for $f = Mu^{-1}$, $\Omega = Q$ and $d\mu = \frac{dx}{|Q|}$, we have

$$\lim_{q \to +\infty} \left(\frac{1}{|Q|} \int_Q Mu(x)^{-\frac{1}{q-1}} \, dx\right)^{q-1} = \exp\left(\int_Q \log\left(Mu(x)^{-1}\right) dx\right)$$

$$= \exp\left(\int_Q -\log(Mu(x)) \, dx\right) = \frac{1}{\exp\left(\int_Q \log(Mu(x)) \, dx\right)}$$

.

Taking limit in $\frac{Mu(Q)}{|Q|} \left(\frac{1}{|Q|} \int_Q Mu(x)^{-\frac{1}{q-1}} \, dx\right)^{q-1} \leq [Mu]_{A_p}$ we have that

$$\frac{Mu(Q)}{|Q|} \frac{1}{\exp\left(\int_Q \log(Mu(x)) \, dx\right)} \leq [Mu]_{A_p}$$

, so

$$\frac{Mu(Q)}{|Q|} \leq [Mu]_{A_p} \cdot \exp\left(\int_Q \log(Mu(x)) \, dx\right)$$

.

Also, applying the observation for $f = Mu$ we get for any $r > 0$ that

$$\left(\frac{1}{|Q|}\int_Q (Mu)^r\,dx\right)^{\frac{1}{r}} \geq \exp\left(\int_Q \log\left(Mu\,(x)\right)dx\right)$$

.

Thus

$$\frac{Mu(Q)}{|Q|} \leq [Mu]_{A_p}\cdot\exp\left(\int_Q \log\left(Mu\,(x)\right)dx\right) \leq [Mu]_{A_p}\left(\frac{1}{|Q|}\int_Q |Mu|^r\,dx\right)^{\frac{1}{r}}$$

, and then

$$\frac{Mu(Q)}{|Q|} \leq [Mu]_{A_p}\left(\frac{1}{|Q|}\int_Q |Mu|^r\,dx\right)^{\frac{1}{r}}$$

.

We take $r = \delta$ with $0 \leq \delta < 1$ and we use that $(Mu)^r = (Mu)^\delta \in A_1$; then

$$\frac{1}{|Q|}\int_Q |Mu|^r\,dx \leq [(Mu)^r]_{A_1}\cdot(Mu\,(x))^r$$

a.e for every $x \in Q \subset Q_0$.

So we have a.e for $x \in Q$

$$\begin{aligned}\frac{Mu(Q)}{|Q|} &\leq [Mu]_{A_p}\left(\frac{1}{|Q|}\int_Q |Mu|^r\,dx\right)^{\frac{1}{r}}\\ &\leq [Mu]_{A_p}\cdot\left([(Mu)^r]_{A_1}\cdot(Mu\,(x))^r\right)^{\frac{1}{r}}\\ &= [Mu]_{A_p}\cdot\left([(Mu)^r]_{A_1}\right)^{\frac{1}{r}}\cdot(Mu\,(x))\end{aligned}$$

.

Taking $C = [Mu]_{A_p}\cdot\left([(Mu)^r]_{A_1}\right)^{\frac{1}{r}}$ independent of Q, for every Q we obtain that

$$\frac{Mu(Q)}{|Q|} \leq C\cdot Mu\,(x)$$

a.e for $x \in Q$.

Then for almost every $x \in Q_0$ we have that

$$M\,(Mu)\,(x) = \sup_{Q_0\supset Q\ni x}\frac{Mu(Q)}{|Q|} \leq C\cdot Mu\,(x)$$

, that is

$$M\,(Mu)\,(x) \leq C\cdot Mu\,(x)$$

and then we obtain that $Mu \in A_1\,(Q_0)$. $\square$

A result similar to the next lemma for the operator $M_{\mathbb{R}^n}$ is due to Neugebauer (see 8). For the local maximal operator M is essentially proven along 2; for completeness we isolate here their argument:

Lemma 2.3. Let M the local maximal operator of Hardy-Littlewood associated with Q_0. For a weight u it holds that $Mu \in A_1$ if and only if there exists $s > 1$ and $C_0 > 0$ such that $(Mu^s)^{\frac{1}{s}}\,(x) \leq C_0.Mu\,(x)$

Proof . The non trivial implication: If $Mu \in A_1$ then $(Mu^s)^{\frac{1}{s}}\,(x) \leq C_0.Mu\,(x)$ was already mentioned in the second part of the previous theorem, coming from the reverse Hölder inequalities.

The other implication is consequence of the first part of the theorem: If $(Mu^s)^{\frac{1}{s}}\,(x) \leq C_0.Mu\,(x)$, we name $u^s = f$ and $\delta = \frac{1}{s}$ and because $(Mu^s)^{\frac{1}{s}} = (Mf)^\delta \in A_1$ we have that $a.e\ x \in Q_0$

$$M\,(Mu)\,(x) \leq M\left((Mu^s)^{\frac{1}{s}}\right)(x) \leq$$

$$[(Mu^s)^{\frac{1}{s}}]_{A_1}(Mu^s)^{\frac{1}{s}}(x) \leq [(Mu^s)^{\frac{1}{s}}]_{A_1}C_0.Mu(x)$$

so for $C = [(Mu^s)^{\frac{1}{s}}]_{A_1}C_0$ we have $M(Mu)(x) \leq CMu(x)$ $a.e$ $x \in Q_0$. That is $Mu \in A_1$. □

Now, putting together the last lemma and the proposition we have the corresponding criterion for the local M operator:

Criterion 2.4. Let u a weight function in Q_0, $Mu \in A_\infty$ if and only if there exists $s > 1$ and $C_0 > 0$ such that $(Mu^s)^{\frac{1}{s}}(x) \leq C_0.Mu(x)$.

Binding the former arguments for M the maximal operator of Hardy-Littlewood associated with Q_0 and the corresponding A_p classes we have already shown the following implications of the statement of Theorem 3:

$$i) \Leftrightarrow ii) \Rightarrow iii) \Leftrightarrow iv) \Leftrightarrow v)$$

The first equivalence $i) \Leftrightarrow ii)$ is known; the second implication: $ii) \Rightarrow iii)$ is obvious taking suprema; the equivalence $iii) \Leftrightarrow iv)$ is the Criterion mentioned above; and the non trivial implication of $iv) \Leftrightarrow v)$ follows from Proposition 1.

To end the proof of Theorem 3 we can observe that $ii)$ gives $iii)$ for any $Q \subset Q_0$ with the same RH_r constante ant then we follow an argument from 2 (cf. 2 section 3) to see that for the local M operator if $(Mu^s)(x) \leq C.(Mu(x))^s$ for some $s > 1$, $C > 0$, then there is $C > 0$: for every $Q \subset Q_0$ and $0 < t < |Q|$ it occurs that

$$\left(\frac{1}{t}K\left(t, u^s, L^1, L^\infty\right)\right)^{\frac{1}{s}} \leq C.\frac{1}{t}K\left(t, u, L^1, L^\infty\right)$$

; that is $iii) \Rightarrow vi)$, and on the other hand that $vi)$ can be easily rewritten to obtain $ii)$ and then $vi) \Rightarrow ii)$ closing the chain of deductions and proving Theorem 3. In the last section we give some definitions related to Peetre's $K - functional$ and we sketch the proofs from 2 of the remaining implications of Theorem 3.

LetÂ´s remark again the difference with the global case, where the pointwise condition $(Mu^s)^{\frac{1}{s}}(x) \leq C_0.Mu(x)$ is strictly weaker than belonging to $\bigcup_{r>1} RH_r$ as we can see taking any non-doubling $weak - A_\infty$ weight u. That is, if u satisfies $\left(\frac{1}{|Q|}\int_Q u^s\right)^{\frac{1}{s}} \leq C.\frac{1}{|2Q|}\int_{2Q} u$ for every $Q \subset \mathbb{R}^n$ for some $C > 0$ and $s > 1$ and then $(Mu^s)^{\frac{1}{s}} \leq C.Mu$ and thus $Mu \in A_\infty$ but being u non-doubling $u \notin A_p$ for any $p < \infty$, so $u \notin A_\infty$ and $u \notin RH_r$ for any $r > 1$.

3 Some more definitions and the missing implications

The first proof from 2 of the fact that $u \in \bigcup_{r>1} RH_r$ whenever $Mu \in A_1$ is based on interpolation theory, the K functionals and Holmstedt formula. We begin introducing some necessary definitions and recalling some known results:

A compatible couple of Banach spaces is a pair of two Banach spaces A_0 and A_1 that are continously embedded in certain Hausdorff topological vector space Z. Clearly $A_0 \cap A_1$ and $A_0 + A_1$ with the norms $\|x\|_{A_0 \cap A_1} = \max\left(\|x\|_{A_0}, \|x\|_{A_1}\right)$ and $\|x\|_{A_0+A_1} = \inf\left(\|x_0\|_{A_0} + \|x_1\|_{A_1} : x_i \in A_i\right)$ are also subspaces of Z and the obvious injections of $A_0 \cap A_1$ in A_i and of A_i in $A_0 + A_1$ are continuous.

For $(A_{0,}A_1)$ a compatible couple of Banach spaces, $A_0 \supset A_1$, for $f \in A_0$ and $t > 0$ the K-functional is defined by

$$K(t, f, A_0, A_1) = \inf_{f=f_0+f_1, f_i \in A_i}\{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\}$$

If (X, μ) is a totally σ-finite measure space and $A_0 = L^1$, $A_1 = L^\infty$, respectively the μ-integrable and μ-essentially bounded real functions, they are continously embedded in the space Z of real μ-measurable functions. It is well known (see 4) that for any $f \in L^1 + L^\infty$.

$$K\left(t, f, L^1, L^\infty\right) = \int_0^t f^*(z)\,dz = t\left(\frac{1}{t}\int_0^t f^*(z)\,dz\right) = tf^{**}(t)$$

where $f^* = f^*_\mu$ denotes the non-increasing rearrangement of f respect to μ and $f^{**}(t) = \frac{1}{t}\int_0^t f^*(z)\,dz$ the action of the Hardy operator on f^*.

Another well known result (see 3) for $A_0 = L^p$ and $A_1 = L^\infty$ with $1 < p < \infty$ is that

$$K\left(t, f, L^p, L^\infty\right) \approx \left(\int_0^{t^p} f^*(z)^p\, dz\right)^{\frac{1}{p}}$$

Now if u satisfies $iii)$, that is $\left(Mu^s\right)(x) \leq C.\left(Mu\left(x\right)\right)^s$ for some $s > 1$, $C > 0$ and taking rearrangements in the last inequality one has

$$\left(\left(Mu^s\right)^*(t)\right)^{\frac{1}{s}} \leq C.\left(Mu\right)^*(t)$$

for $0 < t < |Q_0|$. Applying the equivalence, due to Herz, Stein (cf. Bennett-Sharpley 4, theorem 3.8, see also 1), and valid for every locally integrable f :

$$\left(Mf\right)^*(t) \approx f^{**}(t)$$

to the inequality $\left(Mu^s\right)^*(t) \leq C.\left(\left(Mu\right)^*(t)\right)^s$ one obtains

$$\left(\frac{1}{t}\int_0^t u^*(z)^s\, dz\right)^{\frac{1}{s}} \leq C.\frac{1}{t}\int_0^t u^*(z)\, dz$$

that in terms of the mentioned equivalences for the K-functionals is written like this:

$$\frac{1}{t^{\frac{1}{s}}}K\left(t^{\frac{1}{s}}, u, L^s, L^\infty\right) \leq C.\frac{1}{t}K\left(t, u, L^1, L^\infty\right)$$

or equivalently, using that $K\left(t^{\frac{1}{s}}, u, L^s, L^\infty\right) \approx \left(K\left(t, u^s, L^1, L^\infty\right)\right)^{\frac{1}{s}}$ we can also write

$$\left(\frac{1}{t}K\left(t, u^s, L^1, L^\infty\right)\right)^{\frac{1}{s}} \leq C.\frac{1}{t}K\left(t, u, L^1, L^\infty\right)$$

for $0 < t < |Q_0|$. So we have obtain that $iii) \Rightarrow vi)$.

If one translates back the last inequality obtaining $\left(\frac{1}{t}\int_0^t u^*(z)^s\, dz\right)^{\frac{1}{s}} \leq C.\frac{1}{t}\int_0^t u^*(z)\, dz$, then for $t = |Q_0|$ it results

$$\left(\frac{1}{|Q_0|}\int_0^{|Q_0|} u^*(z)^s\, dz\right)^{\frac{1}{s}} \leq C.\frac{1}{|Q_0|}\int_0^{|Q_0|} u^*(z)\, dz$$

and then

$$\left(\frac{1}{|Q_0|}\int_{Q_0} u(x)^s\, dx\right)^{\frac{1}{s}} \leq C.\frac{1}{|Q_0|}\int_{Q_0} u(x)\, dx$$

In 2 is observed that the argument can be localized for any $Q \subset Q_0$ because $M\left(Mu\right) \approx M_{L(\log L)}$ (see 10) where

$$M_{L(\log L)} = \sup_{Q \subset Q_0, x \in Q} \|u\|_{L(\log L)\left(Q, \frac{dx}{|Q|}\right)}$$

with $\|u\|_{L(\log L)\left(Q, \frac{dx}{|Q|}\right)}$ the Luxemburg norm respect to the Young function $\Phi(t) = t(1 + \log^+ t)$, being $\log^+ t = \max(\log t, 0)$.

That is

$$\|u\|_{L(\log L)\left(Q, \frac{dx}{|Q|}\right)} = \inf\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{u(x)}{\lambda}\right) dx \leq 1\}$$

Clearly for any $Q \subset Q_0$ and $M_{L(\log L)}(x) \approx M\left(Mu\right)(x) \leq C.Mu(x)$ a.e. $x \in Q_0$ we have an analogous inequality with the same constant a.e. $x \in Q$ and then for the restriction of u to any Q the argument of 2 gives a similar reverse Hölder inequality with the same constant C and exponent s:

$$\left(\frac{1}{|Q|}\int_Q u^s\right)^{\frac{1}{s}} \leq C.\frac{1}{|Q|}\int_Q u$$

from where one can recover that $\left(Mu^s\right)^{\frac{1}{s}}(x) \leq C.Mu(x)$ a.e. $x \in Q_0$. Then we have that $vi) \Rightarrow ii)$ for the statement of Theorem 3. Now putting together the results of this sections with the implications proven in the previous section we end the proof of the theorem.

4 An application

Now, let's give our proof of Wik's Theorem.

Proof . From our theorem 3 for Q_0 we have that

$$u \in A_\infty \iff Mu \in A_\infty \iff Mu \in A_1$$

Thus if we take $(Mu)^* : [0, |Q_0|] \longrightarrow \mathbb{R}^+$ the non-increasing rearrangement of Mu in Q_0, and using that for non-increasing positive functions the Hardy-Littlewood operator M and the Hardy operator P with $Pf(t) = \frac{1}{t}\int_0^t |f(s)|\, ds$ are the same one, for $t \in [0, |Q_0|]$, the Herz-Stein equivalence $(Mf)^*(t) \approx f^{**}(t)$, and the above mentioned fact that if $u \in A_\infty$ then $Mu \in A_1$ we have that for some $c > 0$:

$$M\left((Mu)^*\right)(t) = P\left((Mu)^*\right)(t) = (Mu)^{**}(t)$$

$$\approx (M(Mu))^*(t) \leq (cMu)^*)t = c\,(Mu)^*(t)$$

Thus $(Mu)^* \in A_1$, but from Herz-Stein again and using that for u^* decreasing M is the same as P we get: $(Mu)^* \approx u^{**} = P(u^*) = M(u^*)$, so $M(u^*) \in A_1$. Now using again our theorem 3 for $[0, |Q_0|]$ considered as cube of $\mathbb{R}$ and

for the weight u^* we have that $u^* \in A_\infty([0, |Q_0|])$. So far we have that $u^* \in A_q$ for some $q > 1$, but we we can't ensure yet that $q = p$ as in Wik's result. To obtain this we continue as follows: Because $u \in A_p$ we have that $\sigma = u^{1-p'} \in A_{p'}$ and then aplying he above argument to σ we get that $\sigma^* \in A_\infty([0, |Q_0|])$, but $\sigma^* = \left(u^{1-p'}\right)^* = (u^*)^{1-p'}$. And then we get that $u^* \in A_\infty$ and $(u^*)^{1-p'} \in A_\infty$, and it is a well known result (see for instance 9, theorem 2.17, chapter IV) that $w \in A_p \iff w \in A_\infty$ and $w^{1-p'} \in A_\infty$ for any weight w; so we have, at last, that $u^* \in A_p$. $\square$

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