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# On $A_\lambda$-almost null and $A_\lambda$-almost convergent Orlicz sequence spaces

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**Abstract**

The idea of almost convergent sequence was introduced by G. G. Lorentz [8]. In this paper, some new generalized sequence spaces on $A_\lambda$-almost null and $A_\lambda$-almost convergent sequences by Orlicz function are introduced and extended them to the paranormed sequence spaces. Some inclusion relation has also been established between the new spaces. In addition, the $\alpha$-, $\beta$- and $\gamma$-duals of these spaces, and the characterization of $(A_\lambda(f)(\Delta, M, q) : \nu)$ and $(\nu : A_\lambda(f)(\Delta, M, q))$ of infinite matrices are also given.

Keywords: Almost convergence, paranormed spaces, $\alpha$-, $\beta$- and $\gamma$-duals, matrix transformation
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## 1 Introduction

Throughout this paper, the set of all complex sequences will be denoted by $\omega$. The notations $\ell_\infty$, $c$ and $c_0$ and $\ell_p$ $(1 \leq p < \infty)$ are used for the sequences spaces of all bounded, convergent, null and absolutely $p$-summable sequences, respectively.

Furthermore, we denote $\ell_1$, $bs$ and $cs$ for sequence spaces of all absolutely convergent series, bounded series, and convergent series respectively.

A $K$-space on any sequence space $E$ with linear topology is defined as a map $q_r : E \to \mathbb{C}$, where $\mathbb{C}$ denote the set of complex field and $\mathbb{N} = \{0, 1, 2, ...\}$ such that $q_r(x) = x_r$ is continuous $\forall\ r \in \mathbb{N}$. A complete linear metric $K$-space is called an $FK$-space. An $FK$-space is called a $BK$-space whose topology is normable. The spaces $\ell_\infty$, $c$ and $c_0$ are $BK$-spaces endowed with the sup-norm as $\|x\|_\infty = \sup_{r\in\mathbb{N}} |x_r|$. (for details cf. [6, 9]).

The shift operator $D$ on $w$ is defined by $(Dx)_n = x_{n+1}\ \ \forall\ n \in \mathbb{N}$ . A Banach limit $F$ is defined as a non negative linear functional on $\ell_\infty$ such that $F(Dx) = F(x)$ and $F(e) = 1$. A sequence $(x_r)$ is said to be almost convergent to the generalized limit $\xi$ if all Banach limits of $x$ are equal to $\xi$, and is denoted by $f - \lim x_r = \xi$. Lorentz [8] proved the criterion for almost convergence, that is, $f - \lim x_r = \xi$ iff $\lim_{k\to\infty} \sum_{r=0}^{k} \frac{x_{n+r}}{k+1} = \xi$ uniformly in $n$.

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The notations $f_0$ and $f$ are used to define the spaces of all almost null & almost convergent sequences as follows:

$$f_0 = \left\{ x = (x_r) \in w : \lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{k} x_{n+r} = 0 \text{ uniformly in } n \right\}$$

$$f = \left\{ x = (x_r) \in w : \lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{k} x_{n+r} = l \text{ uniformly in } n \right\}$$

**Definition 1.1 (Matrix Transformation).** [4] Let $U$ and $V$ be two given sequence spaces and $A = (a_{nr})$ is an infinite matrix of real entries. Then the function $A : U \to V$ defined a matrix transformation between two sequence spaces, $U$ and $V$ as

$$(Ax)_n = \sum_r a_{nr} x_r \tag{1.1}$$

provided the series on the right hand side of (1.1) is convergent for each $n \in \mathbb{N}$. We call $Ax$ as $A$-transformation of sequence $x$. By $(U : V)$, we shall denote the collection of all infinite matrices from $U$ into $V$.

Here, and in what follows, the summation without limits runs from 0 to $\infty$.

Further the notion

$$U_A = \{x = (x_r) \in w : Ax \in U\}. \tag{1.2}$$

is called the matrix domain $A$ in a sequence space $U$, which itself is a sequence space.

In this work we continue to study the spaces of difference sequence and also the concept of all almost null & all almost convergent sequences by using the matrix transformations $A_\lambda$, where $\lambda = (\lambda_r)$ is strictly increasing sequence.

For two arbitrary sequence spaces $U$ and $V$. Define the set

$$S(U,\ V) = \{a = (a_r) \in w : xa = (x_r a_r) \in V \quad \forall\ x \in U\}$$

is called the multiplier space of the spaces $U$ and $V$. The $\alpha-$, $\beta-$ and $\gamma-$duals of subset $U \subset \omega$, are defined as

$$U^\alpha = S(U,\ l_1), \quad U^\beta = S(U, cs), \quad U^\gamma = S(U, bs)$$

## 2 $A_\lambda$-almost null and $A_\lambda$-almost convergent sequence spaces

Throughout this paper, let $\lambda = (\lambda_r)$ is a strictly increasing sequence of positive reals tending to infinity, that is, $0 < \lambda_0 < \lambda_1 < \lambda_2 < ......$ and $\lim_{r\to\infty} \lambda_r = \infty$.
We introduce the matrix $A_\lambda = \{a_{nr}(\lambda)\}$ by

$$a_{nr}(\lambda) = \begin{cases} \frac{\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}}{\lambda_n - \lambda_{n-1}}, & 0 \le r \le n \\ 0, & r > n \end{cases} \tag{2.1}$$

for all $r,\ n \in \mathbb{N}$.
Take $(\Delta x_r) \in w$ and $n \ge 1$. Then (2.1) gives by a short calculation that

$$\begin{aligned} \Delta x_n - (A_\lambda \Delta x)_n &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{r=0}^{n} (\lambda_r - 2\lambda_{r-1} + \lambda_{r-2})(\Delta x_n - \Delta x_r) \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{r=0}^{n-1} (\lambda_r - 2\lambda_{r-1} + \lambda_{r-2})(\Delta x_n - \Delta x_r) \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{r=0}^{n-1} (\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}) \sum_{i=r+1}^{n} (\Delta x_i - \Delta x_{i-1}) \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{i=1}^{n} (\Delta x_i - \Delta x_{i-1}) \sum_{r=0}^{i-1} (\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}) \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{i=1}^{n} (\lambda_{i-1} - \lambda_{i-2})(\Delta x_i - \Delta x_{i-1}) \end{aligned}$$

Thus for every $(\Delta x_r) \in w$, we have

$$\Delta x_n - (A_\lambda \Delta x)_n = (\Lambda \Delta x)_n \quad \forall\, n \in \mathbb{N} \tag{2.2}$$

where $\Lambda \Delta x = (\Lambda \Delta x)_n$ is as follows

$$(\Lambda \Delta x)_n = \begin{cases} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{r=1}^{n} (\lambda_{r-1} - \lambda_{r-2})(\Delta x_r - \Delta x_{r-1}), & n \geq 1 \\ 0, & n = 0 \end{cases}$$

Moreover we have the following result:

**Theorem 2.1.** Suppose $f\text{-}\lim \Delta x_n = \xi$ for $\Delta x_n \in w$ and $\xi \in \mathbb{C}$. Then $f\text{-}\lim (A_\lambda \Delta x)_n = \xi$ holds if and only if $\Lambda \Delta x \in f_0$.

**Proof .** First consider that $f\text{-}\lim \Delta x_n = f\text{-}\lim (A_\lambda \Delta x)_n = \xi$. Then, from equation (2.2), the equality

$$\frac{1}{k+1} \sum_{r=0}^{k} [\Delta x_{n+r} - (A_\lambda \Delta x)_{n+r}] = \frac{1}{k+1} \sum_{r=0}^{k} (\Lambda \Delta x)_{n+r} \tag{2.3}$$

holds for all $n, r \in \mathbb{N}$. Thus by passing $k \to \infty$ uniformly in $n$, the left hand side of equation (2.3) approach to zero which yields that

$$\lim_{k \to \infty} \frac{1}{k+1} \sum_{r=0}^{k} (\Lambda \Delta x)_{n+r} = 0, \text{ uniformly in } n.$$

Therefore, $\Lambda \Delta x \in f_0$.

On the contrary suppose that $\Lambda \Delta x \in f_0$ and take $f\text{-}\lim_{n \to \infty} \Delta x_n = \xi$.

By taking limit in the equality (2.3), we obtain

$$\lim_{k \to \infty} \frac{1}{k+1} \sum_{r=0}^{k} [\Delta x_{n+r} - (A_\lambda \Delta x)_{n+r}] = 0$$

This yields that

$$\lim_{k \to \infty} \frac{1}{k+1} \sum_{r=0}^{k} \Delta x_{n+r} = \lim_{k \to \infty} \frac{1}{k+1} \sum_{r=0}^{k} (A_\lambda \Delta x)_{n+r} = \xi.$$

Hence the result □

We approach on the construction of new sequence spaces $A_\lambda(f_0)(\Delta, \mathcal{M})$ and $A_\lambda(f)(\Delta, \mathcal{M})$ of all $A_\lambda$-almost null & $A_\lambda$-almost convergent sequences by means of difference sequence and Orlicz function $\mathcal{M}$, [3]. In [7], Lindenstrauss and Tzafriri introduced a sequence space $l_{\mathcal{M}}$ which consist of an Orlicz function $\mathcal{M}$ as follows,

$$\ell_{\mathcal{M}} = \left\{ x \in w : \sum_{r=1}^{\infty} \mathcal{M}\left(\frac{|x_r|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}$$

The Orlicz sequence space $\ell_{\mathcal{M}}$ under the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{r=1}^{\infty} \mathcal{M}\left(\frac{|x_r|}{\rho}\right) \leq 1 \right\}$$

is a Banach space which is called an Orlicz sequence space.

We define the spaces $A_\lambda(f_0)(\Delta, \mathcal{M})$ and $A_\lambda(f)(\Delta, \mathcal{M})$, as the set of all $A_\lambda$-almost null & $A_\lambda$-almost convergent sequences of complex numbers, respectively as follows:

$$A_\lambda(f_0)(\Delta, \mathcal{M}) = \left\{ (x_r) \in w : \lim_{k \to \infty} \frac{1}{k+1} \sum_{r=0}^{k} \mathcal{M}\left( \frac{|A_\lambda \Delta x|_{n+r}}{\rho} \right) = 0, \text{ uniformly in } n, \rho > 0 \right\}$$

$$A_\lambda(f)(\Delta, \mathcal{M}) = \left\{ (x_r) \in w : \lim_{k \to \infty} \frac{1}{k+1} \sum_{r=0}^{k} \mathcal{M}\left( \frac{|A_\lambda \Delta x|_{n+r}}{\rho} \right) = l, \text{ uniformly in } n, \rho > 0 \right\}.$$

With notation of (1.2) the spaces $A_\lambda(f_0)(\Delta,\mathcal{M})$ and $A_\lambda(f)(\Delta,\mathcal{M})$ can redefine as the matrix domain of triangle $A_\lambda$ in the spaces $f_0$ and $f$, respectively.

Throughout the text, $y=(y_r)=(\Delta y_r)$ will be used as the $A_\lambda$-transform of a sequence $x=(x_r)=(\Delta x_r)$, that is,

$$(y_r)=(A_\lambda x)_n=\frac{1}{\lambda_n-\lambda_{n-1}}\sum_{r=0}^{n}\mathcal{M}\left(\frac{\lambda_r-2\lambda_{r-1}+\lambda_{r-2}}{\rho}\Delta x_r\right) \tag{2.4}$$

for all $n\in\mathbb{N}$.

**Theorem 2.2.** The sequence space $A_\lambda(f)(\Delta,\mathcal{M})$ is a Banach- spaces under the norm defined as

$$\|x\|_{A_\lambda(f)(\Delta,\ \mathcal{M})}=\|A_\lambda x\|_{f(\Delta,\ \mathcal{M})}=\inf\left\{\rho>0:\sup_{k,n\in\mathbb{N}}\tau_{nk}\mathcal{M}\left(\frac{|A_\lambda\Delta x|_{n+r}}{\rho}\right)\le 1\right\}.$$

where $\tau_{nk}\mathcal{M}\left(\dfrac{A_\lambda\Delta x}{\rho}\right)=\dfrac{1}{k+1}\sum_{r=0}^{k}\mathcal{M}\left(\frac{(A_\lambda\Delta x)_{n+r}}{\rho}\right)$.

**Proof .** Let $\{x^s\}$ be any Cauchy sequence in $A_\lambda(f)(\Delta,\mathcal{M})$, where $x_k^s=(x_1^s,x_2^s,x_3^s,....)\in A_\lambda(f)(\Delta,\mathcal{M})$ for all $s\in\mathbb{N}$. Let $\delta$ be fixed and $q>0$, then $\exists$ a positive integer $n_0$ such that

$$\|x^s-x^t\|_{A_\lambda(f)(\Delta,\ \mathcal{M})}<\frac{\epsilon}{\delta q},\forall\ s,t\ge n_0$$

Thus by using norm definition, we have

$$\sup_{n,r\in\mathbb{N}}\sum_{r=0}^{k}\mathcal{M}\left(\frac{|A_\lambda(\Delta x^s-\Delta x^t)|_{n+r}}{\|x^s-x^t\|_{A_\lambda(f)(\Delta,\ \mathcal{M})}(k+1)}\right)\le 1,\forall\ s,t\ge n_0 \text{ and } k\in\mathbb{N}$$

$$\sum_{r=0}^{k}\mathcal{M}\left(\frac{|A_\lambda(\Delta x^s-\Delta x^t)|_{n+r}}{\|x^s-x^t\|_{A_\lambda(f)(\Delta,\ \mathcal{M})}(k+1)}\right)\le 1,\forall\ n,r\in\mathbb{N} \text{ and } s,t\ge n_0,\ k\in\mathbb{N}$$

Choose $q>0$ with $\mathcal{M}\left(\frac{\delta q}{2}\right)\ge 1$ so that

$$\sum_{r=0}^{k}\mathcal{M}\left(\frac{|A_\lambda(\Delta x^s-\Delta x^t)|_{n+r}}{\|x^s-x^t\|_{A_\lambda(f)(\Delta,\ \mathcal{M})}(k+1)}\right)\le\mathcal{M}\left(\frac{q\delta}{2}\right)$$

Since $\mathcal{M}$ is non-decreasing and $x_i^s$ is convergent in $\mathbb{R}$ for each $i\in\mathbb{N}$. Let $\lim_{s\to\infty}x_i^s=x_i$ for each $i\in\mathbb{N}$. Using the continuity of Orlicz function $\mathcal{M}$ and modulus, it yields that $(\Delta x^s-\Delta x)\in A_\lambda(f)(\Delta,\mathcal{M})$, it follows that $x\in A_\lambda(f)(\Delta,\mathcal{M})$. □

**Remark 2.3.** Note that the absolute property on the sequence spaces $A_\lambda(f)(\Delta,\mathcal{M})$ and $A_\lambda(f_0)(\Delta,\mathcal{M})$ is not true, i.e, $\|x\|_{A_\lambda(f)(\Delta,\mathcal{M})}\ne\|\,|x|\,\|_{A_\lambda(f)(\Delta,\mathcal{M})}$ for at least one sequence in each of these spaces, and this means that the spaces $A_\lambda(f)(\Delta,\mathcal{M})$ and $A_\lambda(f_0)(\Delta,\mathcal{M})$ are $BK$-spaces of non-absolute type.

Next, we discuss the following Theorem showing the isomorphism between the sequence spaces $A_\lambda(f)(\Delta,\mathcal{M})$, $A_\lambda(f_0)(\Delta,\mathcal{M})$ and $f$, $f_0$ respectively.

**Theorem 2.4.** The sequence space $A_\lambda(f)(\Delta,\mathcal{M})$ and $A_\lambda(f_0)(\Delta,\mathcal{M})$ of non-absolute type are linearly norm isomorphic to the spaces $f$ and $f_0$ respectively.

**Proof .** We establish the result $A_\lambda(f)(\Delta,\mathcal{M})\cong f$. The fact $A_\lambda(f_0)(\Delta,\mathcal{M})\cong f_0$ can be proved in the similar lines.

We show the existence of a linear bijection between the space $A_\lambda(f)(\Delta,\mathcal{M})$ and $f$. The mapping $P$ from $A_\lambda(f)(\Delta,\mathcal{M})$ to $f$ by $x\to y=Px=A_\lambda(f)(\Delta,\mathcal{M})x$ is linear (by equation (2.4)). Further, $Px=\theta$ implies $x=\theta$.

Let $y=(y_r)\in f$ and the sequence $x=(x_r)$ defined as

$$\Delta x_r = \sum_{j=r-1}^{r} (-1)^{r-j} \frac{1}{\mathcal{M}} \left( \frac{\lambda_j - \lambda_{j-1}}{\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}} \rho \Delta y_r \right) \quad \forall\ r \in \mathbb{N} \tag{2.5}$$

Then,

$$\begin{aligned}
\sum_{r=0}^{n+j} \frac{\lambda_r - 2\lambda_{r-1} + \lambda_{k-2}}{\lambda_{n+j} - \lambda_{n+j-1}} \Delta x_r &= \sum_{r=0}^{n+j} \sum_{j=r-1}^{r} (-1)^{r-j} \frac{1}{\mathcal{M}} \left( \frac{\lambda_j - \lambda_{j-1}}{\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}} \rho \Delta y_r \right) \\
&= \sum_{r=0}^{n+j} \frac{1}{\mathcal{M}} \left( \frac{(\lambda_r - \lambda_{k-1}) \Delta y_r - (\lambda_{r-1} - \lambda_{r-2}) \Delta y_{r-1}}{\lambda_{n+j} - \lambda_{n+j-1}} \rho \right) \\
&= \frac{1}{\mathcal{M}} \rho \Delta y_{n+j}
\end{aligned}$$

Since, $\mathcal{M}$ is continuous and for some $\rho > 0$, then

$$\lim_{k \to \infty} \frac{1}{k+1} \sum_{j=0}^{k} \sum_{r=0}^{n+j} \mathcal{M} \left( \frac{\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}}{\rho(\lambda_{n+j} - \lambda_{n+j-1})} \right) \Delta x_r = \lim_{k \to \infty} \frac{1}{k+1} \sum_{j=0}^{k} \Delta y_{n+j}$$

and

$$\lim_{k \to \infty} \frac{1}{k+1} \sum_{j=0}^{k} \mathcal{M} \left( \frac{|A_\lambda \Delta x_r|_{n+j}}{\rho} \right) = f - lim\, y_k = l \text{ uniformly in n.}$$

This shows that $x \in A_\lambda(f)(\Delta, \mathcal{M})$ and consequently $P$ is surjective. Hence, $P$ is a linear bijection. Also, by Theorem (2.2), $P$ preserves the norm and then $A_\lambda(f)(\Delta\mathcal{M}) \approx f$. □

**Theorem 2.5.** As the Orlicz function $\mathcal{M}$ which satisfy $\Delta_2$-condition. Then
(a) $A_\lambda(f)(\Delta) \subset A_\lambda(f)(\Delta, \mathcal{M})$
(b) $A_\lambda(f_0)(\Delta) \subset A_\lambda(f_0)(\Delta, \mathcal{M})$

**Proof .** (a) Let $x \in A_\lambda(f)(\Delta)$. Then $\exists$ some $C > 0$ such that $|A_\lambda \Delta x|_{n+r} \leq C, \quad \forall\ n, r$. Thus, for some $\rho > 0$

$$\mathcal{M} \left( \frac{|A_\lambda \Delta x|_{n+r}}{\rho} \right) \leq \mathcal{M} \left( \frac{C}{\rho} \right) \leq K.\ell\mathcal{M}(C), \text{ by } \Delta_2 - \text{condition.}$$

Hence

$$\sup_{n,k \in \mathbb{N}} \tau_{nk} \mathcal{M} \left( \frac{|A_\lambda \Delta x|_{n+r}}{\rho} \right) < \infty$$

This proves that $A_\lambda(f)(\Delta) \subset A_\lambda(f)(\Delta, \mathcal{M})$. Proof of $(b)$ follows similarly. □

**Theorem 2.6.** The inclusion $A_\lambda(f_0)(\Delta, \mathcal{M}) \subset A_\lambda(f)(\Delta, \mathcal{M})$ strictly holds.

**Proof .** Take $x = (x_r) \in A_\lambda(f_0)(\Delta, \mathcal{M})$. Then $A_\lambda \Delta x \in f_0(\mathcal{M})$. Since $f_0 \subset f$, we have $A_\lambda \Delta x \in f(\mathcal{M})$, and hence $x \in A_\lambda(f)(\Delta, \mathcal{M})$. Therefore the inclusion $A_\lambda(f_0)(\Delta, \mathcal{M}) \subset A_\lambda(f)(\Delta, \mathcal{M})$ is strict.
Next take the sequence $x \in A_\lambda(f)(\Delta, \mathcal{M})$ as defined by $x = (\Delta x_r) = 1 \ \ \forall r \in \mathbb{N}$. Then

$$\lim_{k \to \infty} \frac{1}{k+1} \sum_{j=0}^{k} \mathcal{M} \left( \frac{|A_\lambda \Delta x|_{n+r}}{\rho} \right) = 1 \neq 0.$$

Thus $x \in A_\lambda(f)(\Delta, \mathcal{M})$ but not in $A_\lambda(f_0)(\Delta, \mathcal{M})$. Hence, the inclusion $A_\lambda(f_0)(\Delta, \mathcal{M}) \subset A_\lambda(f)(\Delta, \mathcal{M})$ is strict. □

**Theorem 2.7.** The inclusions $A_\lambda(c)(\Delta, \mathcal{M}) \subset A_\lambda(f)(\Delta, \mathcal{M}) \subset A_\lambda(l_\infty)(\Delta, \mathcal{M})$ strictly hold.

**Proof .** Consider the sequence $x \in A_\lambda(c)(\Delta, \mathcal{M})$, then $A_\lambda \Delta x \in c(\mathcal{M})$. Since $c \subset f$, we have $A_\lambda \Delta x \in f(\mathcal{M})$, that is, $x \in A_\lambda(f)(\Delta, \mathcal{M})$. Therefore, $A_\lambda(c)(\Delta, \mathcal{M}) \subset A_\lambda(f)(\Delta, \mathcal{M})$. Now, take $y = (y_r) \in A_\lambda(f)(\Delta, \mathcal{M})$. Then $A_\lambda \Delta y \in f(\mathcal{M})$ and $f \subset l_\infty$, we obtain $A_\lambda \Delta y \in l_\infty(\mathcal{M})$. Hence $A_\lambda(f)(\Delta, \mathcal{M}) \subset A_\lambda(l_\infty)(\Delta, \mathcal{M})$ holds. □

## 3 Kothe-duals of the space $A_\lambda(f)(\Delta,\mathcal{M})$

In this section the Kothe duals ($\alpha$-, $\beta$- and $\gamma$-duals) of the space $A_\lambda(f)(\Delta,\mathcal{M})$ have been dteremined and studied thoroughly.

**Lemma 3.1.** [12] $A=(a_{nr})\in(f:\ell_1)$ if and only if

$$\sup_{K,N\in\mathcal{F}}\left|\sum_{n\in N}\sum_{r\in K}a_{nr}\right|<\infty \tag{3.1}$$

**Theorem 3.2.** The $\alpha$-dual of $A_\lambda(f)(\Delta,\mathcal{M})$ is the set $a_1(\lambda)$, where

$$a_1(\lambda)=\left\{a=(a_r)\in w:\sum_{r=0}^{k}\frac{1}{\mathcal{M}}\left(\frac{\lambda_r-\lambda_{r-1}}{\lambda_r-2\lambda_{r-1}+\lambda_{r-2}}\right)\rho|\Delta a_r|<\infty\right\} \tag{3.2}$$

**Proof .** Define the matrix $B=(b_{nr})$ with the aid of a sequence $a=(a_r)$ as follows

$$b_{nr}=\begin{cases}(-1)^{n-r}\frac{1}{\mathcal{M}}\left(\frac{\lambda_r-\lambda_{r-1}}{\lambda_r-2\lambda_{r-1}+\lambda_{r-2}}\right)\rho\Delta a_r, & n-1\le r\le n\\ 0, & 0\le r\le n-1 \ or \ r>n\end{cases} \tag{3.3}$$

Then $x=(x_n)\in A_\lambda(f)(\Delta,\mathcal{M})$, we have

$$a_rx_r=a_r\sum_{j=r-1}^{r}(-1)^{r-j}\frac{1}{\mathcal{M}}\left(\frac{\lambda_j-\lambda_{j-1}}{\lambda_r-2\lambda_{r-1}+\lambda_{r-2}}\right)\rho y_r=(By)_r,\ \forall r\in\mathbb{N}. \tag{3.4}$$

Therefore, $ax=(a_rx_r)\in\ell_1$ whenever $x\in A_\lambda(f)(\Delta,\mathcal{M})$ iff $By\in\ell_1$ whenever $y\in A_\lambda(f)(\Delta,\mathcal{M})$. This yields that $a\in\{A_\lambda(f)\}^\alpha$ iff $B\in(f:\ell_1)$. By Lemma (3.1) this is possible iff

$$\sup_{K,N\in\mathcal{F}}\left|\sum_{n\in N}\sum_{r\in K}b_{nr}\right|<\infty \tag{3.5}$$

It follows that equation (3.5) holds iff $\sum_r\frac{1}{\mathcal{M}}\left(\frac{\lambda_r-\lambda_{r-1}}{\lambda_r-2\lambda_{r-1}+\lambda_{r-2}}\right)\rho|a_r|<\infty$ which gives that $\{A_\lambda(f)(\Delta,\mathcal{M})\}^\alpha=a_1(\lambda)$. □

**Lemma 3.3.** $A=(a_{nr})\in(f:l_\infty)$ iff

$$\sup_{n\in\mathbb{N}}\sum_r|a_{nr}|<\infty. \tag{3.6}$$

**Theorem 3.4.** The $\gamma$-dual of the space $A_\lambda(f)(\Delta,\mathcal{M})$ is the set $d_1\cap d_2$, where

$$d_1=\left\{a=(a_r)\in w:\sup_{n\in\mathbb{N}}\sum_{r=0}^{n-1}\left|\frac{1}{\mathcal{M}}\Delta\left(\frac{\rho a_r}{\lambda_r-2\lambda_{r-1}+\lambda_{r-2}}\right)(\lambda_r-\lambda_{r-1})\right|<\infty\right\}$$

$$d_2=\left\{a=(a_r)\in w:\frac{1}{\mathcal{M}}\left(\frac{\lambda_n-\lambda_{n-1}}{\lambda_n-2\lambda_{n-1}+\lambda_{n-2}}\rho a_n\right)\in\ell_\infty\right\}$$

**Proof .** Take $a=(a_r)\in w$ and considering the equality obtained with (2.5) between the sequences $x=(x_r)$ and $y=(y_r)$ that

$$\begin{aligned}\sum_{r=0}^{n}a_rx_r &= \sum_{r=0}^{n}a_r\left[\sum_{j=r-1}^{r}(-1)^{r-j}\frac{1}{\mathcal{M}}\left(\frac{\lambda_j-\lambda_{j-1}}{\lambda_r-2\lambda_{r-1}+\lambda_{r-2}}\right)\rho y_j\right]\\ &= \sum_{r=0}^{n-1}\frac{1}{\mathcal{M}}\Delta\left(\frac{a_r}{\lambda_r-2\lambda_{r-1}+\lambda_{r-2}}\right)(\lambda_r-\lambda_{r-1})\rho y_r+\frac{1}{\mathcal{M}}\left(\frac{\rho a_n(\lambda_n-\lambda_{n-1})}{\lambda_n-2\lambda_{n-1}+\lambda_{n-2}}\right)y_n\\ &= (By)n,\ \forall n\in\mathbb{N},\end{aligned} \tag{3.7}$$

where, $B = (b_{nr})$ is defined as

$$b_{nr} = \begin{cases} \frac{1}{\mathcal{M}}\Delta\left(\frac{a_r}{\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}}\right)\rho(\lambda_r - \lambda_{r-1}, & 0 \le r \le n-1 \\ \frac{1}{\mathcal{M}}\left(\frac{\rho a_n(\lambda_n - \lambda_{n-1})}{\lambda_n - 2\lambda_{n-1} - \lambda_{n-2}}\right), & r = n \\ 0, & r > n \end{cases} \tag{3.8}$$

$\forall\, r, n \in \mathbb{N}$. Thus from (3.7), $ax = (a_r x_r) \in bs$ whenever $x = (x_r) \in A_\lambda(f)(\Delta, \mathcal{M})$ iff $By \in \ell_\infty$ whenever $y \in f$. Hence by Lemma (3.3) that $\{A_\lambda(f)(\Delta, \mathcal{M})\}^\gamma = d_1 \cap d_2$. $\square$

**Lemma 3.5.** [11] $A = (a_{nr}) \in (f : c)$ iff equation (3.6) holds and there are $\beta_r, \beta \in \mathbb{C}$ such that

$$\lim_{n\to\infty} a_{nr} = \beta_r \ for\ all\ r \in \mathbb{N} \tag{3.9}$$

$$\lim_{n\to\infty} \sum_r a_{nr} = \beta \tag{3.10}$$

and

$$\lim_{n\to\infty} \sum_r |\Delta(a_{nr} - \beta_r)| = 0. \tag{3.11}$$

**Theorem 3.6.** Define the sets $d_3, d_4$ and $d_5$ as follows:

$$d_3 = \left\{a = (a_r) \in w : \frac{1}{\mathcal{M}}\left(\frac{\rho a_r}{\lambda_r - 2\lambda_{r-1} + \lambda_{r-1}}(\lambda_r - \lambda_{r-1})\right) \in c\right\},$$

$$d_4 = \left\{a = (a_r) \in w : \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{\mathcal{M}}\Delta\left(\frac{\rho a_r}{\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}}\right)(\lambda_r - \lambda_{r-1})\ exists\right\},$$

$$d_5 = \left\{a = (a_r) \in w : \left\{\left|\Delta'\left[\frac{1}{\mathcal{M}}\Delta\left(\frac{\rho a_r}{\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}}\right)(\lambda_r - \lambda_{r-1})\right]\right|\right\} \in cs\right\}.$$

Then, $\{A_\lambda(f)(\Delta, \mathcal{M})\}^\beta = \cap_{i=1}^5 d_i$.

**Proof .** Take any $a = (a_r) \in w$. From equation (3.7) that $ax = (a_r x_r) \in cs$ whenever $x = (x_r) \in A_\lambda(f)(\Delta, \mathcal{M})$ iff $By \in c$ whenever $y = (y_r) \in f$, that is $(a_r) \in \{A_\lambda(f)(\Delta, \mathcal{M})\}^\beta$ iff $B \in (f : c)$. Therefore, by Lemma (3.5), we have $\{A_\lambda(f)(\Delta, \mathcal{M})\}^\beta = \cap_{i=1}^5 d_i$. $\square$

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