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Coincidence and common fixed points in metric space over Banach algebra

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Abstract

The purpose of this paper is to obtain coincidence and common fixed point theorems for two pairs of weakly commuting self mappings in cone metric space over unital Banach algebra. Moreover, an example is given in the support of the main result and show that the result is more general than the results in present literature.

Keywords: Common fixed points, cone metric space over Banach algebra
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1 Introduction

The Banach contraction mapping principle is one of the most influential sources in pure and applied mathematics. A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a contraction mapping if, for all $x, y \in X$, there is a constant $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y). \tag{1.1}$$

According to this principle, any mapping T satisfying the above inequality in a complete metric space will have a unique fixed point. Over the years, mathematicians generalized this principle in different directions in all kinds of spaces. Also, in the contemporary research, it remains a heavily investigated branch as a consequence of the strong applicability.The concept of cone metric space, as a meaningful generalization of metric spaces, was introduced in the work of Huang and Zhang [8] where they also established the Banach contraction mapping principle in such spaces. Afterwards, many authors have focused on fixed point problems in cone metric spaces ([1, 3, 4, 5, 6, 7, 10, 12]). Very recently, the notion of cone metric space over Banach algebra was introduced by Liu and Xu [11] and proved fixed point theorems in such spaces in a different way by restricting the contractive constants to be vectors and the relevant multiplications to be vector ones instead of usual real constants and scalar multiplications. And that they provided an example to explain the non-equivalence of fixed point results between the vectorial versions and scalar versions.

2 Preliminaries

Throughout this paper, we prove coincidence and common fixed point theorems of generalized mappings in the setting of cone metric space over Banach algebra. Furthermore, we give an examples to support our conclusions.

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Definition 2.1. [11] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:
(i) $\theta \precsim d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$for all $x, y \in X$;
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra $\mathcal{A}$.

Example 2.2. [11] Let $\mathcal{A} = R^2$, $P = \{(x, y) \in \mathcal{A} | x, y \geq 0\} \subset R^2$, $X = R$ and $d : X \times X \to \mathcal{A}$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.3. [11] Let (X, d) be a cone metric space over Banach algebra A, $x \in X$ and $\{x_n\}$ a sequence in X. Then
(i) $\{x_n\}$ converges to x whenever for every $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $lim_{n\to\infty} x_n = x or x_n \to x (n \to \infty)$.
(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geqslant N$.
(iii) (X, d) is complete cone metric space if every Cauchy sequence is convergent.

Definition 2.4. [14] Let P be a solid cone in a Metric space X. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $\theta \ll c$ there exists a natural number N such that $u_n \ll c$ for all $n > N$.

Definition 2.5. [14] Let P be a solid cone in a Banach algebra $\mathcal{A}$. Suppose that $k \in P$ and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.

Definition 2.6. [13] Let $\mathcal{A}$ be a Banach algebra with a unit e, $k \in \mathcal{A}$, then $lim_{n\to\infty} ||k^n||^{\frac{1}{n}}$ exists and the spectral radius $\rho(k)$ satisfies $\rho(k) = lim_{n\to\infty} ||k^n||^{\frac{1}{n}} = inf ||k^n||^{\frac{1}{n}}$.

If $\rho(k) < |\lambda|$, then $\lambda e - k$ is invertible in $\mathcal{A}$, moreover, $(\lambda e - k)^{-1} = \Sigma_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}$, where λ is a complex constant.

Lemma 2.7. [14] Let $\mathcal{A}$ be a Banach algebra with a unit e, $a, b \in \mathcal{A}$. If a commutes with b, then
(i)$\rho(a + b) \leq \rho(a) + \rho(b)$,
(ii) $\rho(ab) \leq \rho(a)\rho(b)$.

Lemma 2.8. [14] Let $\mathcal{A}$ be a Banach algebra with a unit e, $k \in \mathcal{A}$. If $0 \leq \rho(k) < 1$, then we have

$$\rho\big((e - k)^{-1}\big) \leq (1 - \rho(k))^{-1} \tag{2.1}$$

Definition 2.9. [1] Let $f, g : X \to X$ be mappings on a set$X$.
(i) If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g;
(ii) The pair (f, g) is called weakly compatible if f and g commute at all of their coincidence points, that is, $fgx = gfx$ for all $x \in C(f, g) = \{x \in X : fx = gx\}$.

3 The Main Results

Theorem 3.1. Let f, g, S and T be self mappings on a cone metric space (X, d) with a solid cone P over the unital Banach algebra $\mathcal{A}$ such that for all $x, y \in X$
(i) $f(x) \subseteq T(x)$ and $g(x) \subseteq S(x)$
(ii) there exists $p, q, r, s \in P$ with $\rho(q + s) + \rho(p + r + s) < 1$ or $\rho(r + t) + \rho(p + q + t) < 1$. such that

$$d(fx, gy) \leq pd(Sx, Ty) + qd(fx, Sx) + rd(gy, Ty) + sd(fx, Ty) + td(gy, Sx) \tag{3.1}$$

(iii)$q + s$ commute with $p + r + s$ and $r + t$ commute with $p + q + t$ respectively
If one of $f(X)$, $g(x)$, $S(X)$ and $T(X)$ is complete subspace of X. Then $\{f, S\}$ and $\{g, T\}$ have unique coincidence point in X. Moreover, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have unique common fixed point.

Proof . Let $x_0 \in X$ be arbitrary. Construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $fx_{2n} = Tx_{2n+1} = y_{2n+1}$ and $gx_{2n+1} = Sx_{2n+2} = y_{2n+2}$.

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(fx_{2n}, gx_{2n-1}) \\ &\leq pd(Sx_{2n}, Tx_{2n+1}) + qd(fx_{2n}, Sx_{2n}) + rd(gx_{2n+1}, Tx_{2n+1}) \\ &\quad +sd(fx_{2n}, Tx_{2n+1}) + td(gx_{2n+1}, Sx_{2n}) \\ &= pd(y_{2n}, y_{2n+1}) + qd(y_{2n+1}, y_{2n}) + rd(y_{2n+2}, y_{2n+1}) \\ &\quad +td(y_{2n+2}, y_{2n}) \end{aligned}$$

$$(e - r - t)d(y_{2n+1}, y_{2n+2}) \leq (p + q + t)d(y_{2n}, y_{2n+1})$$

Let $p + q + r + 2t = k$ and $r + t = k_1$,then $p + q + t = k - k_1$.We have

$$(e - k_1)d(y_{2n+1}, y_{2n+2}) \leq (k - k_1)d(y_{2n}, y_{2n+1}). \tag{3.2}$$

Since $\rho(k_1) < \rho(k_1) + \rho(k) < 1$.So $(e - k_1)$ is invertible and $(e - k_1)^{-1} = \sum_{i=0}^{\infty} k_1^i$. Eq. (3.2) reduces to

$$d(y_{2n+1}, y_{2n+2}) \leq (1 - k_1)^{-1}(k - k_1)d(y_{2n}, y_{2n+1}). \tag{3.3}$$

Let $h_1 = (1 - k_1)^{-1}(k - k_1)$ then (3.3) becomes

$$d(y_{2n+1}, y_{2n+2}) \leq h_1 d(y_{2n}, y_{2n+1}). \tag{3.4}$$

Again

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(fx_{2n}, gx_{2n-1}) \\ &\leq pd(Sx_{2n}, Tx_{2n-1}) + qd(fx_{2n}, Sx_{2n}) + rd(gx_{2n-1}, Tx_{2n-1}) \\ &\quad +sd(fx_{2n}, Tx_{2n-1}) + td(gx_{2n-1}, Sx_{2n}) \\ &= pd(y_{2n}, y_{2n-1}) + qd(y_{2n+1}, y_{2n}) + rd(y_{2n}, y_{2n-1}) \\ &\quad +sd(y_{2n+1}, y_{2n-1}) \end{aligned}$$

$$(e - q - s)d(y_{2n+1}, y_{2n}) \leq (p + r + s)d(y_{2n}, y_{2n-1}).$$

Let $q + s = k_2$ and $p + q + r + 2s = k'$,then $p + r + s = k' - k_2$.We have

$$(e - k_2)d(y_{2n+1}, y_{2n}) \leq (k' - k_2)d(y_{2n}, y_{2n-1}). \tag{3.5}$$

Since $\rho(k_2) < \rho(k_2) + \rho(k') < 1$. So $(e - k_2)$is invertible and $(e - k_2)^{-1} = \sum_{i=0}^{\infty} k_2^i$. Eq.(3.5) reduces to

$$d(y_{2n+1}, y_{2n}) \leq (1 - k_2)^{-1}(k' - k_2)d(y_{2n}, y_{2n-1}). \tag{3.6}$$

Let $h_2 = (1 - k_2)^{-1}(k' - k_2)$, then (3.6) becomes

$$d(y_{2n+1}, y_{2n}) \leq h_2 d(y_{2n}, y_{2n-1}). \tag{3.7}$$

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &\leq h_1 d(y_{2n}, y_{2n+1}) \\ &\leq h_1.h_2 d(y_{2n}, y_{2n-1}) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ &\leq h_1(h_1h_2)^n d(y_0, y_1). \end{aligned}$$

Similarly

$$d(y_{2n+1}, y_{2n}) \leq (h_1h_2)^n d(y_0, y_1).$$

$$\begin{aligned} d(y_{2n}, y_{2m+1}) &\leq (1 + h_1)\tfrac{(h_1h_2)^n}{1-h_1h_2} d(y_0, y_1). \\ d(y_{2n}, y_{2m}) &\leq (1 + h_1)\tfrac{(h_1h_2)^n}{1-h_1h_2} d(y_0, y_1). \\ d(y_{2n+1}, y_{2m}) &\leq (1 + h_2)h_1\tfrac{(h_1h_2)^n}{1-h_1h_2} d(y_0, y_1). \end{aligned}$$

For $n < m$, we have

$$\begin{aligned}
d(y_{2n+1}, y_{2m+1}) &\leq d(y_{2n+1}, y_{2n+2}) + ... + d(y_{2m}, y_{2n}) + ... + d(y_{2m}, y_{2m+1}). \\
&\leq \Big[h_1 \textstyle\sum_{i=n}^{m-1}(h_1h_2)^i + \sum_{i=n+1}^{m}(h_1h_2)^i\Big] d(y_0, y_1) \\
&\leq \Big[h_1 \tfrac{(h_1h_2)^n}{1-h_1h_2} + \tfrac{(h_1h_2)^{n+1}}{1-h_1h_2}\Big] d(y_0, y_1) \\
&= (1+h_2)h_1 \tfrac{(h_1h_2)^n}{1-h_1h_2} d(y_0, y_1).
\end{aligned}$$

Now $h_1 = (e-k_1)^{-1}(k-k_1)$ and since $(e-k_1)^{-1}$ commute with $(k-k_1)$, so $\rho(h_1) = \rho((e-k_1)^{-1}(k-k_1)) \leq \rho((e-k_1)^{-1})\rho((k-k_1)) \leq (1-\rho(k_1))^{-1}\rho(k-k_1) < 1$. We have $\| h_1^n \| \to 0 (n \to \infty)$. Similarly $(e-k_2)^{-1}$ commute with $(k'-k_2)$ and we have $\| h_2^n \| \to 0(n \to \infty)$.

Hence there exists $p > 0$ such that for $p < n < m$, we have

$$d(y_n, y_m) \leq max.\Big[(1+h_2)h_1\frac{(h_1h_2)^p}{1-h_1h_2}, (1+h_1)\frac{(h_1h_2)^p}{1-h_1h_2}\Big] d(y_0, y_1).$$

For any $c \in A$, there exists $K \in N$ such that for all $n \geq K$, we have $d(y_n, y_m) \ll c$ for all $\theta \ll c$. Hence $\{y_n\}$ c-Cauchy sequence in X. Suppose that $S(X)$ is complete. Then there exists a point u in $S(X)$ such that $Sx_{2n} = y_{2n} \to u$ as $n \to \infty$.

Consequently, we can find v in X such that $Sv = u$.
Now we shall show that $fv = u$. Now

$$\begin{aligned}
d(fv, u) &\leq d(fv, gx_{2n-1}) + d(gx_{2n-1}, u). \\
&\leq pd(Sv, Tx_{2n-1}) + qd(fv, Sv) + rd(gx_{2n-1}, Tx_{2n-1}). \\
&\quad sd(fv, Tx_{2n-1}) + td(gx_{2n-1}, Sv) + d(gx_{2n-1}, u) \\
&\leq pd(u, Tx_{2n-1}) + qd(fv.u) + r[d(gx_{2n-1}, u) + d(u, Tx_{2n-1})] \\
&\quad s[d(fv, u) + d(u, Tx_{2n-1})] + td(gx_{2n-1}, u) + d(gx_{2n-1}, u).
\end{aligned}$$

$$(e-q-s)d(fv, u) \leq (p+r+s)d(u, Tx_{2n-1}) + (1+r+t)d(gx_{2n-1}, u).$$

Since $(e-k_2)$ is invertible. This can be written as

$$d(fv, u) \leq (e-k_2)^{-1}(k'-k_2)d(u, Tx_{2n-1}) + (e-k_2)^{-1}(1+k_1)d(gx_{2n-1}, u). \tag{3.8}$$

Let $\theta \ll c$ be given. Since $y_n \to u$, there exists $N = N(c)$ such that for each $n > N$, we have $d(u, Tx_{2n-1}) \ll \frac{1-k_2}{k'-k_2}\frac{c}{2}$ and $d(gx_{2n-1}, u) \ll \frac{1-k_2}{1+k_1}\frac{c}{2}$. This implies $d(fv, u) \ll c$. Hence $fv = u$. Again, since $u \in T(X)$, we can find $w \in X$ such that $Tw = u$.
We shall show that $gw = u$.

$$\begin{aligned}
d(gw, u) &\leq d(fx_{2n}, gw) + d(fx_{2n}, u) \\
&\leq pd(Sx_{2n}, Tw) + qd(fx_{2n}, Sx_{2n}) + rd(gw, Tw) \\
&\quad +sd(fx_{2n}, Tw) + td(gw, Sx_{2n}) + d(fx_{2n}, u) \\
&\leq pd(Sx_{2n}, u) + q[d(fx_{2n}, u) + d(u, Sx_{2n})] \\
&\quad rd(gw, u) + sd(fx_{2n}, u) + t[d(gw, u) + d(u, Sx_{2n})]
\end{aligned}$$

$$(e-r-t)d(gw, u) \leq (1+q+s)d(fx_{2n}, u) + (p+q+t)d(Sx_{2n}, u).$$

This implies that

$$(e-k_1)d(gw, u) \leq (1+k_2)d(fx_{2n}, u) + (k-k_1)d(Sx_{2n}, u).$$

Since $(e-k_1)$ is invertible, above equation reduces to

$$d(gw, u) \leq (e-k_1)^{-1}(1+k_2)d(fx_{2n}, u) + (e-k_1)^{-1}(k-k_1)d(Sx_{2n}, u). \tag{3.9}$$

Since $y_n \to u$, so there exists $N = N(c)$ such that for all $n > N$, we have

$$d(fx_{2n}, u) \ll \frac{1-k_1}{1+k_2}\frac{c}{2}.$$

and

$$d(Sx_{2n},u) \ll \frac{1-k_2}{k'+k_2}\frac{c}{2}.$$

Equation (3.7) gives $d(gw,u) \ll c$ for all $\theta \ll c$. Hence $gw = u$. Thus $gw = Tw = fv = Sv = u$. Using (ii) we can easily show the uniqueness of coincidence point of pairs $\{f,S\}$ and $\{g,T\}$. Also it is easy to prove that if $\{f,S\}$ and $\{g,T\}$ are weakly compatible pairs, then f,g,S and T have unique common fixed point. □

Corollary 3.2. Let f,g and S be self mappings on a cone metric space (X,d) with a solid cone P over the unital Banach algebra $\mathcal{A}$ such that for all $x,y \in X$
(i) $f(x) \subseteq S(x)$ and $g(x) \subseteq S(x)$
(ii) there exists $p,q,r,s \in P$ with $\rho(q+s)+\rho(p+r+s) < 1$ or $\rho(r+t)+\rho(p+q+t) < 1$.

$$d(fx,gy) \leq pd(Sx,Sy)+qd(fx,Sx)+rd(gy,Sy)+sd(fx,Sy)+td(gy,Sx) \tag{3.10}$$

(iii)$q+s$ commute with $p+r+s$ and $r+t$ commute with $p+q+t$ respectively
If one of $f(X)$, $g(x)$ and $S(X)$ is complete subspace of X. Then f g and S have unique coincidence point in X. Moreover, if $\{f,S\}$ and $\{g,S\}$ are weakly compatible, then f,g and S have unique common fixed point.

Proof . Taking $S = T$ and construct sequences as $y_{2n} = fx_{2n} = Sx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$ □

Corollary 3.3. Let f and g be self mappings on a cone metric space (X,d) with a solid cone P over the unital Banach algebra $\mathcal{A}$ such that for all $x,y \in X$
(i) there exists $p,q,r,s \in P$ with $\rho(q+s)+\rho(p+r+s) < 1$ or $\rho(r+t)+\rho(p+q+t) < 1$.

$$d(fx,gy) \leq pd(x,y)+qd(fx,x)+rd(gy,y)+sd(fx,y)+td(gy,x) \tag{3.11}$$

(ii)$q+s$ commute with $p+r+s$ and $r+t$ commute with $p+q+t$ respectively
If one of $f(X)$ and $g(X)$ is complete subspace of X and if $\{f,g\}$ is weakly compatible, then f and g have unique common fixed point.

Taking $q = r$ and $s = t$, we have the following result:

Corollary 3.4. Let f and g be self mappings on a cone metric space (X,d) with a solid cone P over the unital Banach algebra $\mathcal{A}$. Let $p,q,r,s \in P$ with $\rho(p)+2\rho(q)+2\rho(s) < 1$. such that

$$d(fx,gy) \leq pd(x,y)+q[d(fx,x)+d(gy,y)]+s[d(fx,y)+d(gy,x)]. \tag{3.12}$$

for all $x,y \in X$. If one of $f(X)$ and $g(X)$ is complete subspace of X and a pair $\{f,g\}$ is weakly compatible, then f and g have unique common fixed point.

The condition of weakly compatibility can be replaced by continuity.

Taking $f = g$ in the above corollary, we have

Corollary 3.5. Let f be self mappings on a cone metric space (X,d) with a solid cone P over the unital Banach algebra $\mathcal{A}$. Let $p,q,r,s \in P$ with $\rho(p)+2\rho(q)+2\rho(s) < 1$.

$$d(fx,fy) \leq pd(x,y)+q[d(fx,x)+d(fy,y)]+s[d(fx,y)+d(fy,x)] \tag{3.13}$$

for all $x,y \in X$. If one of $f(X)$ is complete subspace of X. Then f have unique common fixed point in X.

Example 3.6. Let $\mathcal{A} = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} |a,b \in R\right\}$, $\|\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\| = |a|+|b|$. The multiplication is usual matrix multiplication. Then $\mathcal{A}$ is Banach algebra with a usual unit. Choose $X = [0,1]$, $P = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} |a,b \geq 0\right\}$. Let $d(x,y) = \begin{pmatrix} |x-y| & 2|x-y| \\ 0 & |x-y| \end{pmatrix}, \forall\ a,b \in X$. Then (X,d) is a complete cone metric space over $\mathcal{A}$ and P is a solid cone.Define the mappings $f,g,S,T : X \to X$ by $fx = \frac{x}{8}$, $Sx = \frac{x}{4}$ and $gx = \frac{x}{6}$, $Tx = \frac{x}{2}$. Set $p = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} \\ 0 & \frac{1}{4} \end{pmatrix}, q = \begin{pmatrix} \frac{1}{12} & \frac{1}{8} \\ 0 & \frac{1}{12} \end{pmatrix}, r = \begin{pmatrix} \frac{1}{12} & \frac{1}{10} \\ 0 & \frac{1}{12} \end{pmatrix}, s = \begin{pmatrix} \frac{1}{24} & \frac{1}{12} \\ 0 & \frac{1}{24} \end{pmatrix}, t = \begin{pmatrix} \frac{1}{25} & \frac{1}{5} \\ 0 & \frac{1}{25} \end{pmatrix}$.Then all the conditions of Theorem 3.1 are satisfied and f,g,S,T have a unique common fixed point $x = 0$ in X.

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