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C^*-algebra valued partial metric space and some fixed point and coincidence point results

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Abstract

In this paper, we introduce the concept of C^*-algebra valued partial metric as a generalization of partial metric and discuss the existence and uniqueness of fixed points for a self mapping defined on a C^*-algebra valued partial metric space. We use these results to obtain some coincidence point and common fixed point results in this setting. Some examples are provided to justify our results.

Keywords: Partial metric, C^*-algebra valued partial metric, C^*-algebra valued contraction, fixed point
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1 Introduction

It is well known that convergence of sequences and continuity of functions are two important concepts in real or complex analysis. Our main task in metric spaces is to introduce an abstract formulation of the notion of distance between two points of an arbitrary nonempty set. It is interesting to note that most of the central concepts of real or complex analysis can be generalized in metric spaces. Several authors successfully extended the notion of metric spaces in different directions (see [2, 4, 15, 16, 17]). In 1994, Matthews [14] gave the concept of a partial metric space while studying denotational semantics of data flow networks and proved the well known Banach Contraction Principle in this setting. Complete partial metric space is a useful framework to model several complex problems in theory of computation. The works of [3, 6, 9, 21] are viable and have opened new avenues for application in different fields of mathematics and applied sciences. In 2014, Z. Ma et al. [13] introduced the concept of C^*-algebra valued metric spaces by using the set of all positive elements of an unital C^*-algebra instead of the set of real numbers. Recently, a series of articles (see [5, 11, 12, 19] and references therein) have been dedicated to the improvement of fixed point theory in partial metric spaces. Motivated by some recent works on the extension of Banach contraction principle, we reformulated some important fixed point results in metric spaces to C^*-algebra valued partial metric spaces. As some consequences of our results, we obtain metric version of Banach Contraction Principle, Brian Fisher fixed point theorem and Kannan fixed point theorem. Moreover, we use our results to obtain some coincidence point and common fixed point results for pair of self mappings in this new framework. Finally, some examples are provided to illustrate the results.

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2 Some basic concepts

In this section, we begin with some basic facts and properties of C^*-algebras. Let $\mathbb{A}$ be an unital algebra with the unit I. An involution on $\mathbb{A}$ is a conjugate linear map $a \mapsto a^*$ on $\mathbb{A}$ such that $a^{**} = a$ and $(ab)^* = b^* a^*$ for all $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\| a^* \| = \| a \|$ for all $a \in \mathbb{A}$. A C^*-algebra is a Banach $*$-algebra such that $\| a^* a \| = \| a \|^2$ for all $a \in \mathbb{A}$. Let H be a Hilbert space and $B(H)$, the set of all bounded linear operators on H. Then, under the norm topology, $B(H)$ is a C^*-algebra.

Throughout this discussion, by $\mathbb{A}$ we always denote an unital C^*-algebra with the unit I. Set $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset [0, \infty)$, where $\sigma(x)$ is the spectrum of x. Using positive elements, one can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows:

$$x \preceq y \ \textit{if and only if} \ y - x \succeq \theta.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. From now on, by $\mathbb{A}_+$, we denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ and by $\mathbb{A}'$, we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$.

Lemma 2.1. [7, 18] Suppose that $\mathbb{A}$ is an unital C^*-algebra with a unit I.

(i) For any $x \in \mathbb{A}_+$, we have $x \preceq I \Leftrightarrow \| x \| \leq 1$.

(ii) If $a \in \mathbb{A}_+$ with $\| a \| < \frac{1}{2}$, then $I - a$ is invertible and $\| a(I - a)^{-1} \| < 1$.

(iii) Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$.

(iv) Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}'_+$ is an invertible operator, then $(I - a)^{-1} b \succeq (I - a)^{-1} c$.

Remark 2.2. It is worth mentioning that $x \preceq y \Rightarrow \| x \| \leq \| y \|$ for $x, y \in \mathbb{A}_+$. In fact, it follows from Lemma 2.1 (i).

Definition 2.3. [13] Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Definition 2.4. [14] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all x, y, $z \in X$:

(p1) $p(x, x) = p(y, y) = p(x, y) \Longleftrightarrow x = y$,

(p2) $p(x, x) \leq p(x, y)$,

(p3) $p(x, y) = p(y, x)$,

(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Definition 2.5. Let X be a nonempty set. Suppose the mapping $p : X \times X \to \mathbb{A}$ satisfies:

(i) $\theta \preceq p(x, y)$ for all x, $y \in X$;

(ii) $p(x, x) = p(y, y) = p(x, y) \Longleftrightarrow x = y$;

(iii) $p(x, x) \preceq p(x, y)$ for all x, $y \in X$;

(iv) $p(x, y) = p(y, x)$ for all x, $y \in X$;

(v) $p(x, y) \preceq p(x, z) + p(z, y) - p(z, z)$ for all $x, y, z \in X$.

Then p is called a C^*-algebra valued partial metric on X and $(X, \mathbb{A}, p)$ is called a C^*-algebra valued partial metric space.

It is worth mentioning that if $\mathbb{A} = \mathbb{C}$, then a C^*-algebra valued partial metric space becomes a partial metric space. Moreover, every C^*-algebra valued metric space is a C^*-algebra valued partial metric space but not conversely. The following examples support the above remark.

Example 2.6. Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H. Define $p : \mathbb{R}^+ \times \mathbb{R}^+ \to B(H)$ by $p(x, y) = max\,\{x, y\}I$, where I is the identity operator on H. Then $(\mathbb{R}^+, B(H), p)$ is a C^*-algebra valued partial metric space but it is not a C^*-algebra valued metric space.

Example 2.7. Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H and $X = \{[a, b] : a,\, b \in \mathbb{R},\, a \leq b\}$. For a fixed positive operator $T \in B(H)$, define $p : X \times X \to B(H)$ by $p([a, b], [c, d]) = (max\,\{b, d\} - min\,\{a, c\})T$. Then $(X, B(H), p)$ is a C^*-algebra valued partial metric space without being a C^*-algebra valued metric space.

Definition 2.8. Let $(X, \mathbb{A}, p)$ be a C^*-algebra valued partial metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x with respect to $\mathbb{A}$ if for any $\epsilon > 0$ there is n_0 such that for all $n > n_0$, $\| p(x_n, x) - p(x, x) \| \leq \epsilon$. We denote it by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.

(ii) (x_n) is Cauchy with respect to $\mathbb{A}$ if $\lim\limits_{n,m\to\infty} \| p(x_n, x_m) \|$ exists and is finite.

(iii) $(X, \mathbb{A}, p)$ is said to be a complete C^*-algebra valued partial metric space if every Cauchy sequence (x_n) in X with respect to $\mathbb{A}$ converges to a point x such that $\lim\limits_{n\to\infty} p(x_n, x) = p(x, x) = \lim\limits_{n,m\to\infty} p(x_n, x_m)$.

We now introduce the notion of a 0-complete C^*-algebra valued partial metric space. It is adapted from the ideas used in [20].

Definition 2.9. Let $(X, \mathbb{A}, p)$ be a C^*-algebra valued partial metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) is called 0-Cauchy with respect to $\mathbb{A}$ if $\lim\limits_{n,m\to\infty} \| p(x_n, x_m) \| = 0$.

(ii) $(X, \mathbb{A}, p)$ is said to be a 0-complete C^*-algebra valued partial metric space if every 0-Cauchy sequence (x_n) in X with respect to $\mathbb{A}$ converges to a point x such that $p(x, x) = \theta$, that is, $\lim\limits_{n\to\infty} p(x_n, x) = p(x, x) = \theta$.

Note that every complete C^*-algebra valued partial metric space is 0-complete. But the converse may not be true, in general. The following example supports the above contention.

Example 2.10. Let $X = [0, \infty) \cap \mathbb{Q}$ and $B(H)$ be the set of all bounded linear operators on a Hilbert space H. Define $p : X \times X \to B(H)$ by $p(x, y) = max\,\{x, y\}I$, where I is the identity operator on H. Then $(X, B(H), p)$ is 0-complete but it is not complete.

Definition 2.11. Let $(X, \mathbb{A}, p)$ be a C^*-algebra valued partial metric space. We call a mapping $f : X \to X$ a C^*-algebra valued contraction mapping on X if there exists $A \in \mathbb{A}$ with $\| A \| < 1$ such that

$$p(fx, fy) \preceq A^*\, p(x, y)A$$

for all $x, y \in X$.

Definition 2.12. Let $(X, \mathbb{A}, p)$ be a C^*-algebra valued partial metric space. A mapping $f : X \to X$ is called a C^*-algebra valued Fisher contraction if there exists $A \in \mathbb{A}'_+$ with $\| A \| < \frac{1}{2}$ such that

$$p(fx, fy) \preceq A\,[p(fx, y) + p(fy, x)]$$

for all $x, y \in X$.

Definition 2.13. Let $(X, \mathbb{A}, p)$ be a C^*-algebra valued partial metric space. A mapping $f : X \to X$ is called a C^*-algebra valued Kannan operator if there exists $A \in \mathbb{A}_+^{'}$ with $\| A \| < \frac{1}{2}$ such that

$$p(fx, fy) \preceq A\,[p(fx, x) + p(fy, y)]$$

for all $x, y \in X$.

3 Main Results

In this section, we prove some fixed point theorems for self mappings defined on a 0-complete C^*-algebra valued partial metric space and satisfying certain contraction condition.

Theorem 3.1. Let $(X, \mathbb{A}, p)$ be a 0-complete C^*-algebra valued partial metric space and $f : X \to X$ be a mapping. If there exists an $A \in \mathbb{A}$ with $\| A \| < 1$ such that

$$p(fx, fy) \preceq A^*\, p(x, y) A \tag{3.1}$$

for all $x, y \in X$, then f has a unique fixed point u(say) in X and $p(u, u) = \theta$.

Proof . If $A = \theta$, then f maps X into a single point and so f possesses a fixed point in X. Thus, without loss of generality, we can assume that $A \neq \theta$. Let $x_0 \in X$ be arbitrary and keep it fixed. We can construct a sequence (x_n) in X such that $x_n = fx_{n-1} = f^n x_0,\ n = 1, 2, 3, \cdots$. It is a well known fact that in a C^*-algebra $\mathbb{A}$, if $a, b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$[18].

For any $n \in \mathbb{N}$, we have by using condition (3.1) that

$$p(x_n, x_{n+1}) = p(fx_{n-1}, fx_n) \preceq A^* p(x_{n-1}, x_n) A. \tag{3.2}$$

By repeated use of condition (3.2), we get

$$p(x_n, x_{n+1}) \preceq (A^*)^n p(x_0, x_1) A^n = (A^n)^* A_0 A^n, \tag{3.3}$$

for all $n \in \mathbb{N}$, where $A_0 = p(x_0, x_1) \in \mathbb{A}_+$.

We assume that $A_0 \neq \theta$. Because, $A_0 = \theta \Rightarrow x_1 = x_0 \Rightarrow fx_0 = x_0$. Thus, x_0 becomes a fixed point of f.

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (3.3) and Lemma 2.1(i) that

$$\begin{aligned}
p(x_n, x_m) &\preceq p(x_n, x_{n+1}) + p(x_{n+1}, x_m) - p(x_{n+1}, x_{n+1}) \\
&\preceq p(x_n, x_{n+1}) + p(x_{n+1}, x_m) \\
&\preceq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\
&\preceq (A^*)^n A_0 A^n + (A^*)^{n+1} A_0 A^{n+1} + (A^*)^{n+2} A_0 A^{n+2} \\
&\preceq + \cdots + (A^*)^{m-1} A_0 A^{m-1} \\
&= \sum_{k=1}^{m-n} (A^*)^{n+k-1} A_0 A^{n+k-1} \\
&\preceq \sum_{k=1}^{m-n} \| (A^*)^{n+k-1} A_0 A^{n+k-1} \| \, I \\
&\preceq \| A_0 \| \sum_{k=1}^{m-n} \| A \|^{2(n+k-1)} \, I \\
&= \| A_0 \| \, \| A \|^{2n} \sum_{k=1}^{m-n} \left(\| A \|^2 \right)^{k-1} I \\
&\preceq \frac{\| A_0 \| \, \| A \|^{2n}}{1 - \| A \|^2} I, \ since \ \| A \| < 1 \\
&\to \theta \ as \ n \to \infty.
\end{aligned}$$

Therefore, (x_n) is a 0-Cauchy sequence with respect to $\mathbb{A}$. By 0-completeness of $(X, \mathbb{A}, p)$, there exists an $u \in X$ such that

$$\lim_{n\to\infty} p(x_n, u) = p(u, u) = \theta.$$

Using condition (3.1), we have

$$\begin{aligned} p(fu, u) &\preceq p(fu, x_n) + p(x_n, u) - p(x_n, x_n) \\ &\preceq p(fu, fx_{n-1}) + p(x_n, u) \\ &\preceq A^* p(u, x_{n-1})A + p(x_n, u) \\ &\to \theta \ as \ n \to \infty. \end{aligned}$$

This implies that $p(fu, u) = \theta$ and hence $fu = u$. Therefore, u is a fixed point of f and $p(u, u) = \theta$.

The next is to show that the fixed point is unique. Assume that there is another fixed point v of f in X. Then,

$$p(u, v) = p(fu, fv) \preceq A^* p(u, v)A$$

and so,

$$\begin{aligned} \| p(u, v) \| &\leq \| A^* p(u, v)A \| \\ &\leq \| A^* \| \| p(u, v) \| \| A \| \\ &= \| A \|^2 \| p(u, v) \| . \end{aligned}$$

Since $\| A \| < 1$, it follows that $p(u, v) = \theta$ i.e., $u = v$. Therefore, f has a unique fixed point u in X with $p(u, u) = \theta$. □

Remark 3.2. It is valuable to note that Banach contraction theorem in a complete metric space (X, d) can be obtained from Theorem 3.1 by taking $\mathbb{A} = \mathbb{C},\ p = d$. Thus, Theorem 3.1 is a generalization of Banach contraction theorem in metric spaces to C^*-algebra valued partial metric spaces.

Theorem 3.3. Let $(X, \mathbb{A}, p)$ be a 0-complete C^*-algebra valued partial metric space and the mapping $f : X \to X$ be such that

$$p(fx, fy) \preceq A\,[p(fx, y) + p(fy, x)] \tag{3.4}$$

for all $x, y \in X$, where $A \in \mathbb{A}'_+$ with $\| A \| < \frac{1}{2}$. Then, f has a unique fixed point u(say) in X and $p(u, u) = \theta$.

Proof . It follows from condition (3.4) that $A(p(fx, y) + p(fy, x))$ is a positive element.

Let $x_0 \in X$ be arbitrary and keep it fixed. We can construct a sequence (x_n) in X such that $x_n = fx_{n-1} = f^n x_0,\ n = 1, 2, 3, \cdots$.

For any $n \in \mathbb{N}$, we obtain by using condition (3.4) that

$$\begin{aligned} p(x_n, x_{n+1}) &= p(fx_{n-1}, fx_n) \\ &\preceq A\,[p(fx_{n-1}, x_n) + p(fx_n, x_{n-1})] \\ &= A\,[p(x_n, x_n) + p(x_{n+1}, x_{n-1})] \\ &\preceq A\,[p(x_n, x_n) + p(x_{n+1}, x_n) + p(x_n, x_{n-1}) - p(x_n, x_n)] \\ &= A\,[p(x_n, x_{n+1}) + p(x_{n-1}, x_n)] \end{aligned}$$

which implies that,

$$(I - A)p(x_n, x_{n+1}) \preceq Ap(x_{n-1}, x_n). \tag{3.5}$$

Since $A \in \mathbb{A}'_+$ with $\| A \| < \frac{1}{2}$, by applying Lemma 2.1, it follows that $(I-A)$ is invertible and $\| A(I-A)^{-1} \| = \| (I-A)^{-1}A \| < 1$. Moreover, by Lemma 2.1, $A \preceq I$ i.e., $I - A \succeq \theta$. As $A \in \mathbb{A}'_+$, we have $(I-A) \in \mathbb{A}'_+$. Furthermore, $(I-A)^{-1} \in \mathbb{A}'_+$. By using Lemma 2.1, we obtain from condition (3.5) that

$$p(x_n, x_{n+1}) \preceq (I-A)^{-1}A\, p(x_{n-1}, x_n) = \alpha\, p(x_{n-1}, x_n), \tag{3.6}$$

where $\alpha = (I-A)^{-1}A \in \mathbb{A}'_+$ with $\| \alpha \| < 1$. Therefore, $\alpha\, p(x_{n-1}, x_n) \succeq \theta$.

So, it must be the case that

$$\| p(x_n, x_{n+1}) \| \leq \| \alpha\, p(x_{n-1}, x_n) \| \leq \| \alpha \| \| p(x_{n-1}, x_n) \| . \tag{3.7}$$

By repeated use of condition (3.7), we get

$$\| p(x_n, x_{n+1}) \| \leq \| \alpha \|^n \| p(x_0, x_1) \| = \| \alpha \|^n \| B_0 \|, \tag{3.8}$$

for all $n \in \mathbb{N}$, where $B_0 = p(x_0, x_1) \in \mathbb{A}_+$.

For any $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} p(x_n, x_m) &\preceq p(x_n, x_{n+1}) + p(x_{n+1}, x_m) - p(x_{n+1}, x_{n+1}) \\ &\preceq p(x_n, x_{n+1}) + p(x_{n+1}, x_m) \\ &\preceq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m). \end{aligned}$$

By using Remark 2.2 and condition (3.8), we get

$$\begin{aligned} \| p(x_n, x_m) \| &\leq \| p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \| \\ &\leq \| p(x_n, x_{n+1}) \| + \| p(x_{n+1}, x_{n+2}) \| + \cdots + \| p(x_{m-1}, x_m) \| \\ &\leq \left(\| \alpha \|^n + \| \alpha \|^{n+1} + \cdots + \| \alpha \|^{m-1} \right) \| B_0 \| \\ &\leq \frac{\| \alpha \|^n \| B_0 \|}{1 - \| \alpha \|} \\ &\to 0 \ as\ n \to \infty. \end{aligned}$$

Therefore, (x_n) is a 0-Cauchy sequence with respect to $\mathbb{A}$. By 0-completeness of $(X, \mathbb{A}, p)$, there exists an $u \in X$ such that

$$\lim_{n \to \infty} p(x_n, u) = p(u, u) = \theta.$$

By using condition (3.4), we have

$$\begin{aligned} p(fu, u) &\preceq p(fu, x_n) + p(x_n, u) - p(x_n, x_n) \\ &\preceq p(fu, fx_{n-1}) + p(x_n, u) \\ &\preceq A\,[p(fu, x_{n-1}) + p(fx_{n-1}, u)] + p(x_n, u) \\ &\preceq A\,[p(fu, u) + p(u, x_{n-1}) - p(u, u) + p(x_n, u)] + p(x_n, u] \end{aligned}$$

which implies that,

$$\theta \preceq (I-A)\, p(fu, u) \preceq A\, p(x_{n-1}, u) + A\, p(x_n, u) + p(x_n, u).$$

Since $A \in \mathbb{A}'_+$ and $(I-A) \in \mathbb{A}'_+$ is an invertible element, by applying Lemma 2.1, we obtain

$$\begin{aligned} p(fu, u) &\preceq (I-A)^{-1}A\, p(x_{n-1}, u) + (I-A)^{-1}A\, p(x_n, u) + (I-A)^{-1}\, p(x_n, u) \\ &\to \theta \ as\ n \to \infty. \end{aligned}$$

This implies that $p(fu, u) = \theta$ i.e., $fu = u$ and hence u is a fixed point of f and $p(u, u) = \theta$.

The next is to show that the fixed point is unique. Assume that there is another fixed point v of f in X. Then,

$$\begin{aligned} p(u,v) &= p(fu,fv) \\ &\preceq A\,[\,p(fu,v)+p(fv,u)\,] \\ &= A\,[\,p(u,v)+p(v,u)\,] \end{aligned}$$

which implies that,

$$\begin{aligned} \| \, p(u,v) \, \| &\leq \| \, A \, \| \, [\, \| \, p(u,v) \, \| + \| \, p(v,u) \, \| \,] \\ &= 2 \, \| \, A \, \| \, \| \, p(u,v) \, \| \, . \end{aligned}$$

Since $\| \, A \, \| < \frac{1}{2}$, we have $\| \, p(u,v) \, \| = 0$ i.e., $u=v$. Therefore, f has a unique fixed point u in X with $p(u,u)=\theta$. □

Remark 3.4. We observe that Brian Fisher's theorem in a complete metric space (X,d) can be obtained from Theorem 3.3 by taking $\mathbb{A}=\mathbb{C}$, $p=d$. Thus, Theorem 3.3 is a generalization of Brian Fisher's theorem in metric spaces to C^*-algebra valued partial metric spaces.

Theorem 3.5. Let $(X,\mathbb{A},p)$ be a 0-complete C^*-algebra valued partial metric space and the mapping $f:X\to X$ be such that

$$p(fx,fy) \preceq A\,[p(fx,x)+p(fy,y)] \tag{3.9}$$

for all $x,y\in X$, where $A\in\mathbb{A}_+^{'}$ with $\| \, A \, \| < \frac{1}{2}$. Then, f has a unique fixed point u(say) in X and $p(u,u)=\theta$.

Proof . We observe that $A(p(fx,x)+p(fy,y))$ is a positive element.

Let $x_0\in X$ be arbitrary and keep it fixed. We can construct a sequence (x_n) in X such that $x_n=fx_{n-1}=f^n x_0,\ n=1,2,3,\cdots$.

For any $n\in\mathbb{N}$, we have by using condition (3.9) that

$$\begin{aligned} p(x_n,x_{n+1}) &= p(fx_{n-1},fx_n) \\ &\preceq A\,[p(fx_{n-1},x_{n-1})+p(fx_n,x_n)] \\ &= A\,p(x_n,x_{n-1})+A\,p(x_n,x_{n+1}) \end{aligned}$$

which implies that,

$$(I-A)\,p(x_n,x_{n+1}) \preceq A\,p(x_{n-1},x_n). \tag{3.10}$$

Since $A\in\mathbb{A}_+^{'}$ and $\| \, A \, \| < \frac{1}{2}$, by Lemma 2.1, it follows that $A\preceq I$ and $(I-A)$ is invertible with $\| \, A(I-A)^{-1} \, \| = \| \, (I-A)^{-1}A \, \| < 1$. Furthermore, $(I-A)$, $(I-A)^{-1}\in\mathbb{A}_+^{'}$ and so, $(I-A)^{-1}A\in\mathbb{A}_+^{'}$. Again, by using Lemma 2.1(iv), it follows from condition (3.10) that

$$p(x_n,x_{n+1}) \preceq (I-A)^{-1}A\,p(x_{n-1},x_n) = \alpha\,p(x_{n-1},x_n), \tag{3.11}$$

where $\alpha=(I-A)^{-1}A\in\mathbb{A}_+^{'}$ with $\| \, \alpha \, \| < 1$. Therefore, $\alpha\,p(x_{n-1},x_n)\succeq\theta$.

So, it must be the case that

$$\| \, p(x_n,x_{n+1}) \, \| \leq \| \, \alpha\,p(x_{n-1},x_n) \, \| \leq \| \, \alpha \, \| \, \| \, p(x_{n-1},x_n) \, \| \, . \tag{3.12}$$

By repeated use of condition (3.12), we get

$$\| \, p(x_n,x_{n+1}) \, \| \leq \| \, \alpha \, \|^n \, \| \, p(x_0,x_1) \, \| = \| \, \alpha \, \|^n \, \| \, B_0 \, \|,$$

for all $n\in\mathbb{N}$, where $B_0=p(x_0,x_1)\in\mathbb{A}_+$.

For any $m, n \in \mathbb{N}$ with $m > n$ and proceeding similarly to that of Theorem 3.3, we obtain

$$\| p(x_n, x_m) \| \leq \frac{\| \alpha \|^n \| B_0 \|}{1- \| \alpha \|} \to 0, \ as \ n \to \infty.$$

Therefore, (x_n) is a 0-Cauchy sequence with respect to $\mathbb{A}$. By 0-completeness of $(X, \mathbb{A}, p)$, there exists an $u \in X$ such that

$$\lim_{n\to\infty} p(x_n, u) = p(u, u) = \theta.$$

By using condition (3.9), we have

$$\begin{aligned} p(fu, u) &\preceq p(fu, x_n) + p(x_n, u) - p(x_n, x_n) \\ &\preceq p(fu, fx_{n-1}) + p(x_n, u) \\ &\preceq A\,[p(fu, u) + p(fx_{n-1}, x_{n-1})] + p(x_n, u) \\ &\preceq A\,[p(fu, u) + p(x_n, x_{n-1})] + p(x_n, u) \end{aligned}$$

which implies that,

$$\theta \preceq (I - A)\, p(fu, u) \preceq A\, p(x_n, x_{n-1}) + p(x_n, u).$$

Since $A \in \mathbb{A}_+^{'}$ and $(I - A) \in \mathbb{A}_+^{'}$ is an invertible element, by applying Lemma 2.1, we obtain

$$p(fu, u) \preceq (I - A)^{-1} A\, p(x_n, x_{n-1}) + (I - A)^{-1}\, p(x_n, u).$$

This implies that

$$\begin{aligned} \| p(fu, u) \| &\leq \| (I - A)^{-1} A \| \| p(x_n, x_{n-1}) \| + \| (I - A)^{-1} \| \| p(x_n, u) \| \\ &\leq \| (I - A)^{-1} A \| \| \alpha \|^{n-1} \| B_0 \| + \| (I - A)^{-1} \| \| p(x_n, u) \| \\ &\to 0 \ as \ n \to \infty. \end{aligned}$$

This implies that $p(fu, u) = \theta$ i.e., $fu = u$ and hence u is a fixed point of f and $p(u, u) = \theta$.

The next is to show that the fixed point is unique. Assume that there is another fixed point v of f in X with $p(v, v) = \theta$. Then,

$$\begin{aligned} p(u, v) &= p(fu, fv) \\ &\preceq A\,[p(fu, u) + p(fv, v)] \\ &= A\,[p(u, u) + p(v, v)] \\ &= \theta \end{aligned}$$

which implies that, $p(u, v) = \theta$ i.e., $u = v$. Therefore, f has a unique fixed point u in X with $p(u, u) = \theta$. □

Remark 3.6. We observe that Kannan's fixed point theorem in a complete metric space (X, d) can be obtained from Theorem 3.5 by taking $\mathbb{A} = \mathbb{C}$, $p = d$. Thus, Theorem 3.5 is a generalization of Kannan's fixed point theorem in metric spaces to C^*-algebra valued partial metric spaces.

Remark 3.7. The results of this study are obtained under the weaker assumption that the underlying C^*-algebra valued partial metric space is 0-complete. However, they also valid if the space is complete.

We give some examples to justify the validity of our results.

Example 3.8. Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H and $X = \{[2 - 3^{-n}, 2] : n \in \mathbb{N}\} \cup \{[2, 2 + 3^{-n}] : n \in \mathbb{N}\} \cup \{\{2\}\}$, where $\{2\} = [2, 2]$. We define $p : X \times X \to B(H)$ by $p([a, b], [c, d]) =$

$(max\,\{b,\,d\} - min\,\{a,\,c\})I$, where I is the identity operator on H. Then $(X, B(H), p)$ is a 0-complete C^*-algebra valued partial metric space. Define a mapping $f : X \to X$ by

$$fx = \begin{cases} [2, 2+3^{-(n+1)}], & if \;\; x = [2-3^{-n}, 2], \\ [2-3^{-(n+1)}, 2], & if \;\; x = [2, 2+3^{-n}], \\ \{2\}, & if \;\; x = \{2\}. \end{cases}$$

We now show that f satisfies the contraction condition (3.1) of Theorem 3.1.

Case-I: $x = [2-3^{-n}, 2],\; y = [2, 2+3^{-k}],\; n,\, k \in \mathbb{N}$.

In this case,

$$p(fx, fy) = p([2, 2+3^{-(n+1)}], [2-3^{-(k+1)}, 2]) = \frac{1}{3}\,(3^{-n} + 3^{-k})\,I = \frac{1}{3}\,p(x, y) = A^*\,p(x, y)\,A,$$

where $A = \frac{1}{\sqrt{3}}\,I \in B(H)$.

Case-II: $x = [2-3^{-n}, 2]$, $y = \{2\}$.

In this case,

$$p(fx, fy) = p([2, 2+3^{-(n+1)}], \{2\}) = \frac{1}{3}\;3^{-n}\,I = \frac{1}{3}\,p(x, y) = A^*\,p(x, y)\,A,$$

for $A = \frac{1}{\sqrt{3}}\,I$.

Case-III: $x = [2, 2+3^{-n}]$, $y = \{2\}$.

Then,

$$p(fx, fy) = p([2-3^{-(n+1)}, 2], \{2\}) = \frac{1}{3}\;3^{-n}\,I = \frac{1}{3}\,p(x, y) = A^*\,p(x, y)\,A,$$

for $A = \frac{1}{\sqrt{3}}\,I$.

Case-IV: $x = y$ is trivial.

Therefore,

$$p(fx, fy) = A^*\,p(x, y)\,A$$

for all $x, y \in X$, where $A \in B(H)$ with $\parallel A \parallel < 1$.

Thus, we have all the conditions of Theorem 3.1 and $\{2\}$ is the unique fixed point of f in X with $p(\{2\}, \{2\}) = O$, zero operator.

The following example shows that Theorem 3.1 is not applicable without contraction condition.

Example 3.9. Let $X = [0, 1]$ and $B(H)$ be the set of all bounded linear operators on a Hilbert space H. Define $p : X \times X \to B(H)$ by

$$p(x, y) = \begin{cases} max\,\{x,\, y\}\,I, & if \;\; x \neq y, \\ O, & if \;\; x = y, \end{cases}$$

where I and O are respectively the identity and zero operators on H. Then $(X, B(H), p)$ is a 0-complete C^*-algebra valued partial metric space. Let $f : X \to X$ be defined by

$$fx = \begin{cases} x^2, & if\ \ x \in [0, \frac{1}{3}), \\ 0, & if\ \ x \in [\frac{1}{3}, 1), \\ 1, & if\ \ x = 1. \end{cases}$$

Then,

$$p(f0, f1) = p(0, 1),$$

which shows that f does not satisfy the contraction condition. So, Theorem 3.1 is not applicable to this example. As a result, f possesses fixed point but it is not unique. In fact, 0 and 1 are fixed points of f in X.

The following examples show how Theorem 3.3 can be used.

Example 3.10. Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H and $X = \{[1, 1+5^{-n}] : n \in \mathbb{N}\} \cup \{[1-5^{-n}, 1] : n \in \mathbb{N}\} \cup \{\{1\}\}$, where $\{1\} = [1, 1]$. We define $p : X \times X \to B(H)$ by $p([a, b], [c, d]) = (max\,\{b, d\} - min\,\{a, c\})I$, where I is the identity operator on H. Then $(X, B(H), p)$ is a 0-complete C^*-algebra valued partial metric space. Define a mapping $f : X \to X$ by

$$fx = \begin{cases} [1-5^{-(n+1)}, 1], & if\ \ x = [1, 1+5^{-n}], \\ [1, 1+5^{-(n+1)}], & if\ \ x = [1-5^{-n}, 1], \\ \{1\}, & if\ \ x = \{1\}. \end{cases}$$

We now show that f satisfies the condition (3.4) of Theorem 3.3.

Case-I: $x = [1, 1+5^{-n}],\ y = [1-5^{-k}, 1],\ n, k \in \mathbb{N}$ with $n < k$.

Then,

$$p(fx, fy) = p([1-5^{-(n+1)}, 1], [1, 1+5^{-(k+1)}]) = \frac{1}{5}\,(5^{-n} + 5^{-k})\,I \prec \frac{2}{5}\,5^{-n}\,I,$$
$$p(fx, y) = p([1-5^{-(n+1)}, 1], [1-5^{-k}, 1]) = 5^{-(n+1)}\,I = \frac{1}{5}\,5^{-n}\,I,$$
$$p(fy, x) = p([1, 1+5^{-(k+1)}], [1, 1+5^{-n}]) = 5^{-n}\,I.$$

Therefore,

$$p(fx, fy) \prec \frac{2}{5}\,5^{-n}\,I = \frac{1}{3}\,(p(fx, y) + p(fy, x)) = A\,(p(fx, y) + p(fy, x)),$$

where $A = \frac{1}{3}I \in B(H)^{'}_{+}$ with $\| A \| = \frac{1}{3} < \frac{1}{2}$.

Case-II: $x = [1, 1+5^{-n}],\ y = [1-5^{-k}, 1],\ n, k \in \mathbb{N}$ with $n = k$.

Then,

$$p(fx, fy) = p([1-5^{-(n+1)}, 1], [1, 1+5^{-(k+1)}]) = \frac{1}{5}\,(5^{-n} + 5^{-k})\,I = \frac{2}{5}\,5^{-n}\,I,$$
$$p(fx, y) = p([1-5^{-(n+1)}, 1], [1-5^{-k}, 1]) = 5^{-n}\,I,$$
$$p(fy, x) = p([1, 1+5^{-(k+1)}], [1, 1+5^{-n}]) = 5^{-n}\,I.$$

Therefore,

$$p(fx, fy) = \frac{2}{5}\,5^{-n}\,I = \frac{1}{5}\,(p(fx, y) + p(fy, x)) \prec A\,(p(fx, y) + p(fy, x)),$$

where $A = \frac{1}{3}I \in B(H)'_+$ with $\| A \| = \frac{1}{3} < \frac{1}{2}$.

Case-III: $x = [1, 1 + 5^{-n}],\ y = [1 - 5^{-k}, 1],\ n, k \in \mathbb{N}$ with $n > k$.

Then,

$$p(fx, fy) = p([1 - 5^{-(n+1)}, 1], [1, 1 + 5^{-(k+1)}]) = \frac{1}{5}\,(5^{-n} + 5^{-k})\, I \prec \frac{2}{5}\, 5^{-k}\, I,$$
$$p(fx, y) = p([1 - 5^{-(n+1)}, 1], [1 - 5^{-k}, 1]) = 5^{-k}\, I,$$
$$p(fy, x) = p([1, 1 + 5^{-(k+1)}], [1, 1 + 5^{-n}]) = 5^{-(k+1)}\, I.$$

Therefore,

$$p(fx, fy) \prec \frac{2}{5}\, 5^{-k}\, I = \frac{1}{3}\,(p(fx, y) + p(fy, x)) = A\,(p(fx, y) + p(fy, x)),$$

where $A = \frac{1}{3}I \in B(H)'_+$ with $\| A \| = \frac{1}{3} < \frac{1}{2}$.

Case-IV: $x = [1, 1 + 5^{-n}]$, $y = \{1\}$.

In this case,

$$p(fx, fy) = p([1 - 5^{-(n+1)}, 1], \{1\}) = \frac{1}{5}\; 5^{-n}\, I,$$
$$p(fx, y) = p([1 - 5^{-(n+1)}, 1], \{1\}) = 5^{-(n+1)}\, I,$$
$$p(fy, x) = p(\{1\}, [1, 1 + 5^{-n}]) = 5^{-n}\, I.$$

Therefore,

$$p(fx, fy) = \frac{1}{5}\, 5^{-n}\, I = \frac{1}{6}\,(p(fx, y) + p(fy, x)) \prec A\,(p(fx, y) + p(fy, x)),$$

where $A = \frac{1}{3}I \in B(H)'_+$ with $\| A \| = \frac{1}{3} < \frac{1}{2}$.

Case-V: $x = [1 - 5^{-n}, 1]$, $y = \{1\}$.

Then,

$$p(fx, fy) = p([1, 1 + 5^{-(n+1)}], \{1\}) = \frac{1}{5}\; 5^{-n}\, I,$$
$$p(fx, y) = p([1, 1 + 5^{-(n+1)}], \{1\}) = 5^{-(n+1)}\, I,$$
$$p(fy, x) = p(\{1\}, [1 - 5^{-n}, 1]) = 5^{-n}\, I.$$

Therefore,

$$p(fx, fy) = \frac{1}{5}\, 5^{-n}\, I = \frac{1}{6}\,(p(fx, y) + p(fy, x)) \prec A\,(p(fx, y) + p(fy, x)),$$

where $A = \frac{1}{3}I \in B(H)'_+$ with $\| A \| = \frac{1}{3} < \frac{1}{2}$.

Case-VI: $x = y$ is trivial.

Therefore,

$$p(fx, fy) \preceq A\,(p(fx, y) + p(fy, x)),$$

for all $x, y \in X$, where $A = \frac{1}{3}I \in B(H)'_+$ with $\| A \| < \frac{1}{2}$.

Thus, we have all the conditions of Theorem 3.3 and $\{1\}$ is the unique fixed point of f in X with $p(\{1\}, \{1\}) = O$, zero operator.

Example 3.11. Let $X = [0, \infty) \cap \mathbb{Q}$ and $B(H)$ be the set of all bounded linear operators on a Hilbert space H. Define $p : X \times X \to B(H)$ by $p(x, y) = max\,\{x, y\}I$ for all $x, y \in X$, where I is the identity operator on H. Then $(X, B(H), p)$ is a 0-complete C^*-algebra valued partial metric space. Let $f : X \to X$ be defined by

$$fx = \frac{x^3}{5(1+x^2)}, \ \forall x \in X.$$

For $x, y \in X$, we have

$$\begin{aligned}
p(fx, fy) &= max\left\{\frac{x^3}{5(1+x^2)}, \frac{y^3}{5(1+y^2)}\right\} I \\
&\preceq \left[\frac{x^3}{5(1+x^2)} + \frac{y^3}{5(1+y^2)}\right] I \\
&= \frac{1}{5}\left[\frac{x^2}{(1+x^2)}x + \frac{y^2}{(1+y^2)}y\right] I \\
&\preceq \frac{1}{5}(x+y)I \\
&\preceq \frac{1}{5}\left[max\,\{\frac{x^3}{5(1+x^2)}, y\}\,I + max\,\{\frac{y^3}{5(1+y^2)}, x\}\,I\,\right] \\
&= \frac{1}{5}I\,(p(fx, y) + p(fy, x)) \\
&= A\,(p(fx, y) + p(fy, x)),
\end{aligned}$$

where $A = \frac{1}{5}I \in B(H)_+^{'}$ with $\| A \| = \frac{1}{5} < \frac{1}{2}$.

Thus, all the conditions of Theorem 3.3 hold true and 0 is the unique fixed point of f in X with $p(0, 0) = O$, zero operator.

Here we present some examples showing the use of Theorem 3.5.

Example 3.12. Let $X = [0, \infty) \cap \mathbb{Q}$ and $B(H)$ be the set of all bounded linear operators on a Hilbert space H. Define $p : X \times X \to B(H)$ by $p(x, y) = max\,\{x, y\}I$ for all $x, y \in X$, where I is the identity operator on H. Then $(X, B(H), p)$ is a 0-complete C^*-algebra valued partial metric space. Let $f : X \to X$ be defined by

$$fx = \frac{x^2}{3(1+x)}, \ \forall x \in X.$$

For $x, y \in X$, we have

$$\begin{aligned}
p(fx, fy) &= max\left\{\frac{x^2}{3(1+x)}, \frac{y^2}{3(1+y)}\right\} I \\
&\preceq \left[\frac{x^2}{3(1+x)} + \frac{y^2}{3(1+y)}\right] I \\
&= \frac{1}{3}\left[\frac{x}{(1+x)}x + \frac{y}{(1+y)}y\right] I \\
&\preceq \frac{1}{3}(x+y)I \\
&= \frac{1}{3}I\,(p(fx, x) + p(fy, y)) \\
&= A\,(p(fx, x) + p(fy, y)),
\end{aligned}$$

where $A = \frac{1}{3}I \in B(H)_+^{'}$ with $\| A \| = \frac{1}{3} < \frac{1}{2}$.

Thus, all the conditions of Theorem 3.5 hold true and 0 is the unique fixed point of f in X with $p(0, 0) = O$, zero operator.

Example 3.13. Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H and $X = \{[3, 3+5^{-n}] : n \in \mathbb{N}\} \cup \{[3-5^{-n}, 3] : n \in \mathbb{N}\} \cup \{\{3\}\}$, where $\{3\} = [3, 3]$. We define $p : X \times X \to B(H)$ by $p([a,b],[c,d]) = (max\,\{b,\, d\} - min\,\{a,\, c\})I$, where I is the identity operator on H. Then $(X, B(H), p)$ is a 0-complete C^*-algebra valued partial metric space. Define a mapping $f : X \to X$ by

$$fx = \begin{cases} [3-5^{-(n+1)}, 3], & if \;\; x = [3, 3+5^{-n}], \\ [3, 3+5^{-(n+1)}], & if \;\; x = [3-5^{-n}, 3], \\ \{3\}, & if \;\; x = \{3\}. \end{cases}$$

We now show that f satisfies the condition (3.9) of Theorem 3.5.

Case-I: $x = [3, 3+5^{-n}],\; y = [3-5^{-k}, 3],\; n,\, k \in \mathbb{N}.$

Then,

$$p(fx, fy) = p([3-5^{-(n+1)}, 3], [3, 3+5^{-(k+1)}]) = \frac{1}{5}\,(5^{-n} + 5^{-k})\,I,$$

$$p(fx, x) = p([3-5^{-(n+1)}, 3], [3, 3+5^{-n}]) = (5^{-n} + 5^{-(n+1)})\,I = \frac{6}{5}\,5^{-n}\,I,$$

$$p(fy, y) = p([3, 3+5^{-(k+1)}], [3-5^{-k}, 3]) = (5^{-(k+1)} + 5^{-k})\,I == \frac{6}{5}\,5^{-k}\,I.$$

Therefore,

$$p(fx, fy) = \frac{1}{5}\,(5^{-n} + 5^{-k})\,I = \frac{1}{6}\,(p(fx,x) + p(fy,y)) = A\,(p(fx,x) + p(fy,y)),$$

where $A = \frac{1}{6}I \in B(H)^{'}_{+}$ with $\parallel A \parallel = \frac{1}{6} < \frac{1}{2}$.

Case-II: $x = [3, 3+5^{-n}]$, $y = \{3\}$.

In this case,

$$p(fx, fy) = p([3-5^{-(n+1)}, 3], \{3\}) = \frac{1}{5}\;5^{-n}\,I,$$

$$p(fx, x) = p([3-5^{-(n+1)}, 3], [3, 3+5^{-n}]) = \frac{6}{5}\,5^{-n}\,I,$$

$$p(fy, y) = p(\{3\}, \{3\}) = O.$$

Therefore,

$$p(fx, fy) = \frac{1}{6}\,(p(fx,x) + p(fy,y)) = A\,(p(fx,x) + p(fy,y)),$$

where $A = \frac{1}{6}I \in B(H)^{'}_{+}$ with $\parallel A \parallel = \frac{1}{6} < \frac{1}{2}$.

Case-III: $x = [3-5^{-n}, 3]$, $y = \{3\}$.

Then,

$$p(fx, fy) = p([3, 3+5^{-(n+1)}], \{3\}) = \frac{1}{5}\;5^{-n}\,I,$$

$$p(fx, x) = p([3, 3+5^{-(n+1)}], [3-5^{-n}, 3]) = \frac{6}{5}\,5^{-n}\,I,$$

$$p(fy, y) = p(\{3\}, \{3\}) = O.$$

Therefore,

$$p(fx, fy) = A\,(p(fx,x) + p(fy,y)),$$

where $A = \frac{1}{6}I \in B(H)'_{+}$ with $\| A \| = \frac{1}{6} < \frac{1}{2}$.

Case-IV: $x = y$ is trivial and in this case

$$p(fx, fy) \preceq A\,(p(fx, x) + p(fy, y)),$$

where $A = \frac{1}{6}I \in B(H)'_{+}$.

Therefore,

$$p(fx, fy) \preceq A\,(p(fx, x) + p(fy, y)),$$

for all x, $y \in X$, where $A = \frac{1}{6}I \in B(H)'_{+}$ with $\| A \| < \frac{1}{2}$.

Thus, we have all the conditions of Theorem 3.5 and $\{3\}$ is the unique fixed point of f in X with $p(\{3\}, \{3\}) = O$, zero operator.

4 Some Coincidence Point Results

Definition 4.1. [1] Let f and g be self mappings of a set X. If $y = fx = gx$ for some x in X, then x is called a coincidence point of f and g and y is called a point of coincidence of f and g.

Definition 4.2. [10] The mappings f, $g : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$g(fx) = f(gx) \textit{ whenever } fx = gx.$$

Proposition 4.3. [1] Let f and g be weakly compatible self maps of a nonempty set X. If f and g have a unique point of coincidence $y = fx = gx$, then y is the unique common fixed point of f and g.

We state the following lemma which is a key result in this section.

Lemma 4.4. [8] Let X be a nonempty set and $f : X \to X$ a function. Then there exists a subset $G \subseteq X$ such that $f(G) = f(X)$ and $f : G \to X$ is one-to-one.

Theorem 4.5. Let $(X, \mathbb{A}, p)$ be a C^*-algebra valued partial metric space and the mappings f, $g : X \to X$ satisfy the following condition

$$p(fx, fy) \preceq B^* p(gx, gy) B \tag{4.1}$$

for all x, $y \in X$, where $B \in \mathbb{A}$ with $\|B\| < 1$. If $f(X) \subseteq g(X)$ and $g(X)$ is a 0-complete subspace of X, then f and g have a unique point of coincidence u(say) in $g(X)$ with $p(u, u) = \theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof . By Lemma 4.4, there exists $G \subseteq X$ such that $g(G) = g(X)$ and $g : G \to X$ is one-to-one. Define $h : g(G) \to g(G)$ by $h(gx) = fx$. This is possible as $f(X) \subseteq g(X)$. Then h is well defined, as g is one-to-one on G.

For all gx, $gy \in g(G)$, we obtain from condition (4.1) that

$$p(h(gx), h(gy)) = p(fx, fy) \preceq B^* p(gx, gy) B.$$

Since $g(G) = g(X)$ is 0-complete, by Theorem 3.1, there exists a unique $gx_0 \in g(X)$ such that $h(gx_0) = gx_0 = u$, say with $p(u, u) = \theta$. That is, $fx_0 = gx_0 = u$. Hence, f and g have a unique point of coincidence u in $g(X)$.
If f and g are weakly compatible, then by Proposition 4.3 it follows that f and g have a unique common fixed point in $g(X)$. □

Corollary 4.6. Let $(X, \mathbb{A}, p)$ be a 0-complete C^*-algebra valued partial metric space and let $g : X \to X$ be an onto mapping satisfying

$$p(x, y) \preceq B^* p(gx, gy) B$$

for all x, $y \in X$, where $B \in \mathbb{A}$ with $\|B\| < 1$. Then g has a unique fixed point u(say) in X with $p(u, u) = \theta$.

Proof . The proof follows from Theorem 4.5 by taking $f = I$. $\square$

The following theorem is a consequence of Theorem 3.3 and Lemma 4.4.

Theorem 4.7. Let $(X, \mathbb{A}, p)$ be a C^*-algebra valued partial metric space and the mappings f, $g : X \to X$ satisfy the following condition

$$p(fx, fy) \preceq B[p(fx, gx) + p(fy, gy)]$$

for all x, $y \in X$, where $B \in \mathbb{A}'_+$ and $\|B\| < \frac{1}{2}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a 0-complete subspace of X, then f and g have a unique point of coincidence u(say) in $g(X)$ with $p(u, u) = \theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

The following theorem is a consequence of Theorem 3.5 and Lemma 4.4.

Theorem 4.8. Let $(X, \mathbb{A}, p)$ be a C^*-algebra valued partial metric space and the mappings f, $g : X \to X$ satisfy the following condition

$$p(fx, fy) \preceq B[p(fx, gy) + p(fy, gx)]$$

for all x, $y \in X$, where $B \in \mathbb{A}'_+$ and $\|B\| < \frac{1}{2}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a 0-complete subspace of X, then f and g have a unique point of coincidence u(say) in $g(X)$ with $p(u, u) = \theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

As an application of Theorem 4.5, we can discuss the existence of unique fixed point of an expansive mapping in partial metric spaces.

Theorem 4.9. Let (X, p) be a 0-complete partial metric space and let $g : X \to X$ be an onto mapping. If there exists $k > 1$ such that

$$p(gx, gy) \geq k\, p(x, y)$$

for all x, $y \in X$, then g has a unique fixed point u(say) in X with $p(u, u) = 0$.

Proof . The proof follows from Theorem 4.5 by taking $f = I$ and $\mathbb{A} = \mathbb{C}$. $\square$

5 An Application

In this section, we give an application of Theorem 4.5. In fact, we consider an operator equation and discuss its unique solution.

Example 5.1. Let H be a Hilbert space, $B(H)$ be the set of bounded linear operators on H and let $g : B(H) \to B(H)$ be a bijective map. If $T_1, T_2, \cdots, T_n, \cdots \in B(H)$ with $\sum_{n=1}^{\infty} \| T_n \|^2 < 1$ and $Q \in B(H)_+$, then the operator equation

$$g(X) - \sum_{n=1}^{\infty} T_n^* g(X) T_n = Q$$

has a unique solution in $B(H)$.

Proof . Choose a positive operator $T \in B(H)$. For $X, Y \in B(H)$, we define

$$p(X, Y) = \| X - Y \| T.$$

Then $(B(H), B(H), p)$ is a 0-complete C^*-algebra valued partial metric space.
Define $F : B(H) \longrightarrow B(H)$ by

$$F(X) = \sum_{n=1}^{\infty} T_n^* g(X) T_n + Q.$$

Then,

$$\begin{aligned} p(F(X),F(Y)) &\preceq \left(\sum_{n=1}^{\infty} \| T_n \|^2\right) \| g(X)-g(Y) \| T \\ &= \left(\sum_{n=1}^{\infty} \| T_n \|^2\right) p(g(X),g(Y)) \\ &= B^* \, p(g(X),g(Y)) \, B \end{aligned}$$

for all $X,Y \in B(H)$ and $B = \left(\sum_{n=1}^{\infty} \| T_n \|^2\right)^{\frac{1}{2}} I \in B(H)$ with $\| B \| < 1$. As g is onto, all the hypotheses of Theorem 4.5 are fulfilled. By applying Theorem 4.5, there exists a unique $g(X_0)$ in $B(H)$ such that $g(X_0) = F(X_0)$. Thus, there exists a unique $X_0 \in B(H)$, g being one to one such that

$$g(X_0) = \sum_{n=1}^{\infty} T_n^* g(X_0) T_n + Q.$$

□

Remark 5.2. In particular, if $g = I$ then the operator equation

$$X - \sum_{n=1}^{\infty} T_n^* X T_n = Q$$

has a unique solution in $B(H)$. Thus, there exists a unique $X_0 \in B(H)$ such that

$$X_0 = \sum_{n=1}^{\infty} T_n^* X_0 T_n + Q.$$

Since Q is a positive operator, we have

$$X_0^* = \sum_{n=1}^{\infty} T_n^* X_0^* T_n + Q.$$

Uniqueness of X_0 implies that $X_0 = X_0^*$ and hence the solution is a self-adjoint or Hermitian operator.

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