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Rings domination in graphs

Saja Saeed Abed*, M. N. Al-Harere

Department of Applied Sciences, University of Technology, Iraq

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Abstract

The aim of this paper is to introduce a new parameter of domination in graphs called "Rings Domination Number". Some properties and important boundaries of the rings dominating set and rings domination number have been discussed. Also, this number for certain graphs have been determined. Furthermore, some operations on two graphs as join, composition, cross product and corona, have been introduced and determined the rings domination numbers for each one of them.

Keywords: Rings dominating set, Rings domination number, Certain graphs, Operation on two graphs
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1 Introduction

In this paper, a finite, simple, and undirected graph is dealt with. The reader can find all terms used throughout this work in [3]. A dominating set of the graph G is a set of vertices of the graph G say D such that each vertex in $V-D$ is adjacent to at least one vertex in D. Claude Berge in [13] is the first one who defined the notion of the domination number of a graph G. This number is denoted by $\gamma(G)$, and it is defined as the minimum cardinality of all dominating sets. The term dominating set and domination number was first used by Ore in [26]. In recent years, the domination number plays an important role in solving many life problems. It deals with various fields in graph theory as a topological graph [5, 18], fuzzy graph [20, 25, 30, 31, 32], labeled graph [21, 27, 28], game theory [22, 23] and others. In [8] the definition of domination is introduced under the condition that every vertex dominates exactly two vertices. In [2, 4, 6, 7, 9, 11, 19, 24] the number of dominated vertices has been specified. Another type of domination number put a condition on the $V-D$ set in the graph [1, 3, 17] or the condition on the sets D and $V-D$ [12, 29]. There is a paper that is studying different types of domination on the same graph, and comparing domination numbers for these types [10].

Here, a new definition is introduced called "rings domination" such that the conditions are put on $V-D$ set. Some fundamental results on rings domination are presented. Also, some operations on two graphs like a join, composition, cross-product and corona, have been introduced and determined rings domination numbers for each of them. For more details on other types of dominations, you may refer to [15, 16].

Definition 1.1. [14] Let $G_1 \times G_2$ denotes to the Cartesian product of two graphs G_1 and G_2 which is having vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2), and (v_1, v_2) of G are adjacent if and only if, either

*Corresponding author
Email address: as.20.23@grad.uotechnology.edu.iq (Saja Saeed Abed)

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$$u_1 = v_1 \text{ and } u_2v_2 \in E(G_2) \text{ or } u_2 = v_2 \text{ and } u_1v_1 \in E(G_1).$$

Definition 1.2. [14] The join $G_1 + G_2$ of G_1 and G_2 is the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Definition 1.3. [14] Let $G_1 \bigodot G_2$ denotes to the corona of two disjoint graphs G_1 and G_2 of orders r and s, respectively. This graph is obtained by taking one copy of the graph G_1 and r copies of the graph G_2, and then joining the j^{th} vertex of the graph G_1 to each vertex in the j^{th} copy of the graph G_2.

Lemma 1.4. [14] Let G be a graph such that each vertex has a degree at least 2, this graph contains a cycle.

2 Rings Domination

Definition 2.1. A dominating set $D \subset V(G)$ in G is a rings dominating set if each vertex $v \in V - D$ is adjacent to at least two vertices in $V - D$.

Definition 2.2. Let $G = (V, E)$ be a graph, if D is a rings dominating set, then D is called a **minimal rings dominating set**, if it has no proper rings dominating subset. A **minimum rings dominating set** is a rings dominating set of smallest size in a given graph.

Definition 2.3. The minimum cardinality of all minimal rings dominating set is denoted by $\gamma_{ri}(G)$ and it is called the **rings domination number** of a graph G.

Example 2.4.

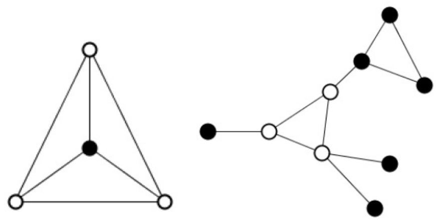

Figure 1: The rings domination in a graph

Observation 2.5. For a rings dominating set D to any graph G of order n, we have:

1. The order of G is $n \geq 4$.
2. For each vertex $v \in V - D$, $deg(v) \geq 3$.
3. $1 \leq D \leq n - 3$
4. $3 \leq |V - D| \leq n - 1$
5. $1 \leq \gamma_{ri} \leq |D| \leq n - 3$.

Proposition 2.6. Trees have no rings dominating set.

Proof . According to Observation 2.5(2) and Lemma 1.4, $G[V - D]$ has a cycle. □

Observation 2.7. If G is a graph of order n and has rings dominating set, then

$\gamma_{ri}(G) > p+l+t$ where $p, l,$ and t is the number of pendants, isolated and vertices with degree 2 in G, respectively.

Observation 2.8.

1. $\gamma_{ri}(K_n) = \gamma_{ri}(W_n) = 1,\ n \geq 4,\ n \geq 3$, respectively.
2. $S_n,\ C_n$, and N_n has no rings dominating set.

Theorem 2.9. The size of graph $G(n, m)$ has rings domination number $\gamma_{ri}(G)$ is $2(n - \gamma_{ri}) \leq m \leq \frac{n(n-1)}{2}$.

Proof . Let D be a γ_{ri}–set of G, then there are two cases.

Case 1. The lower bound occurs where $G[D]$ is a null graph according to Observation 2.5 and Lemma 1.4, while $G[V - D]$ has at least $|V - D|$ edges. Furthermore, there are $|V - D|$ edges between $G[D]$ and $G[V - D]$, since $D\gamma_{ri}$–set. Hence, the graph G contains as few edges as possible. Then the number of edges is $|V - D| + |V - D| = 2(n - \gamma_{ri})$. Thus, in general $m \geq n$.

Case 2. To find the upper bound, this case occurs where G is a complete graph. □

Thus, from the two cases above, the result is obtained.

Proposition 2.10. $\gamma_{ri}(k_{n,m}) = 2, \ n, m \geq 3$.

Proof . Let V_1 and V_2 be the sets of n and m vertices respectively of the graph $k_{n,m}$. Let $v \in V_1$ and $u \in V_2$ then v is adjacent to every vertex in V_2, and u is adjacent to every vertex in V_1. Let $D = \{v, u\}$ a dominating set. Therefore, the remaining vertices in $V - D$ have a degree of two at least. Therefore, D is the minimum rings dominating set. Hence, $\gamma_{ri}(k_{n,m}) = 2$. □

Proposition 2.11. For complete 3-partite graph $k_{n,m,l}$

$$\gamma_{ri}(k_{n,m,l}) = \begin{cases} 1 & if \ \ n, m \geq 2, \ l = 1 \\ 2 & if \ \ n, m, l \geq 2 \end{cases}$$

Proof . Let the sets X, Y, and Z be the partite of a graph $k_{n,m,l}$ of orders n, m, l respectively. So, there are two cases as follows.

Case 1. If $n, m \geq 2, \ l = 1$, then let $D = \{z\}$, where $z \in Z$. It is obvious that the vertex z is adjacent to all other vertices in the graph $k_{n,m,l}$. Furthermore, there is an edge between every vertex in the set X with at least two vertices of the set Y and vice versa. Thus, $\gamma_{ri}(k_{n,m,l}) = 1$.

Case 2. If $n, m, l \geq 2$, then let the set D contains two vertices such that each vertex come from a different set. Without a loss of generality, assume that $D = \{x, y\}, \ x \in X$ and $y \in Y$. It is clear that D is a dominating set of the graph $k_{n,m,l}$, and each vertex in the sets $V - D$ is adjacent to at least two vertices of the set $V - D$. Thus, $\gamma_{ri}(k_{n,m,l}) = 2$.

Thus, from each of the cases above, the result is obtained. □

3 Rings Domination Number of Some Operation on Two Graphs

Theorem 3.1. For a graph $G = P_n[P_m]$ $n, m \geq 2$ then

$$\gamma_{ri}(G) = \begin{cases} 1, & if \ \ n = 2 \ or \ 3 \ and \ m = 2 \ or \ 3 \\ \left\lceil \frac{m}{3} \right\rceil, & if \ \ n = 2 \ or \ 3 \ and \ m > 4 \\ 2\left\lceil \frac{m}{4} \right\rceil, & if \ \ m \equiv 0, 2, 3 (mod 4) \ and \ n, m > 3 \\ 2\left\lceil \frac{m}{4} \right\rceil - 1, & if \ \ m \equiv 1 (mod 4) \ and \ n, m > 3 \end{cases}$$

Proof . Let the vertex set of $P_n[P_m]$ be labeled as follows.

$V(P_n[P_m]) = \{v_i^j, \ i = 1, 2, 3, ..., m \ \forall \ j = 1, 2, ..., n\}$. (as shown in Fig. 2)

Let $P_i^j = \{v_i^j; \ j = 1, 2, ..., n\}, \ \forall \ i = 1, 2, 3, ..., m$ be a path as an induced subgraph. It is clear that every induced subgraph which is induced by two paths (P_i^j and P_{i+1}^j) contains an induced subgraph isomorphic to a complete bipartite graph. So, each vertex in each path dominates the path that is before and after it, but does not dominate the non-adjacent vertices which lie in the same path. Therefore, four cases will be discussed as follows.

Case1. If $n = 2$ or 3 and $m = 2$ or 3, then

- a. If $n = m = 2$ then $P_2[P_2] \equiv K_4$, so $\gamma_{ri}(G) = 1$, according to Observation 2.8.

- b. For $P_2[P_3]\&P_3[P_2]\&P_3[P_3]$: in three cases the degree for every center vertex is equal to $n-1$, then $\gamma_{ri}(G)=1$.

Case 2. If $n=2$ or 3 and $m>4$, then let $D=\left\{v^2_{i+3k},\ i=1,2,...,\ \left\lceil\frac{m}{3}\right\rceil\right\}$. One can easily conclude that D is a dominating set and has minimum cardinality. Therefore, $\gamma_{ri}(G)=\left\lceil\frac{m}{3}\right\rceil$.

Case 3. If $m\equiv 0,2,3(mod4)$ and $n,m\geq 4$, then

- a. If $n\equiv 0,3(mod4)$, let $D=\left\{v^1_{4i-2},v^1_{4i-1}\ i=1,2,...,\left\lceil\frac{m}{4}\right\rceil\right\}$. It is clear that as mentioned in the above cases, the vertex v^1_2 dominates all vertices in the induced subgraphs P^j_1 and P^j_3, but does not dominate the non-adjacent vertices in the induced subgraphs P^j_2. Now, the vertex v^1_3 dominates all vertices in P^j_2 and all vertices in the induced subgraphs P^j_4. Thus, the set $\{v^1_2,v^1_3\}$ dominates all vertices in subgraph generated by the vertices of the paths $\{P^j_i,\ i=1,2,3,4\}$. Also, it is clear that the set $\{v^1_2,v^1_3\}$ is the minimum cardinality which dominates the vertices of the paths $\{P^j_i,\ i=1,2,3,4\}$. Thus, by following the same procedure for the other paths, the result is $\gamma_{ri}(G)=2\left\lceil\frac{m}{4}\right\rceil$.
- b. If $n\equiv 0,3(mod4)$, then let $D=\left\{v^1_{4i-2},v^1_{4i-1}\ i=1,2,...,\left\lceil\frac{m}{4}\right\rceil\right\}\cup\left\{v^1_{m-1},v^1_m\right\}$. In the similar technique in case 3(a), the set $D=\left\{v^1_{4i-2},v^1_{4i-1}\ i=1,2,...,\left\lceil\frac{m}{4}\right\rceil-1\right\}$ is a minimum dominating set of the graph $P_n[P_m]$ excluding the last two paths P^j_{m-1} and P^j_m. Therefore, the set D is a γ_{ri}-set to the graph $P_n[P_m]$. Thus, $\gamma_{ri}(G)=2\left\lceil\frac{m}{4}\right\rceil$.

Case 4. If $n\equiv 1(mod4)$, then let $D=\left\{v^1_{4i-2},v^1_{4i-1}\ i=1,2,...,\left\lceil\frac{m}{4}\right\rceil-1\right\}\cup\left\{v^1_{m-1}\right\}$. Again, in the similar technique in case 3(a), the set $D_1=\left\{v^1_{4i-2},v^1_{4i-1}\ i=1,2,...,\left\lceil\frac{m}{4}\right\rceil-1\right\}$ is a minimum dominating set of the graph $P_n[P_m]$ except the last path P^j_{m-1}. So, add the vertex v^1_{m-1} to keep two conditions, the first of them is it dominates all vertices in the path P^j_{m-1} and all others vertices in its path, are dominated by the vertex v^1_{m-2} that belongs to the set D_1. Thus, $\gamma_{ri}(G)=2\left\lceil\frac{m}{4}\right\rceil-1$.

Therefore, from each of the cases above, the proof is done. □

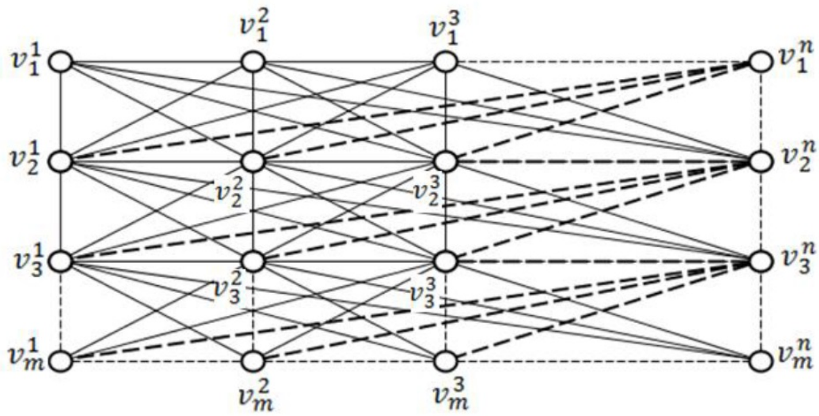

Figure 2: The graph $G=P_n[P_m]$.

Theorem 3.2. For a ladder graph $G\equiv P_n\times P_2,\ \gamma_{ri}(G)=4+2\left\lceil\frac{n-4}{3}\right\rceil$.

Proof . The ladder graph contains two induced subgraphs, each one as a path, The first path say $P_{n_1}=\{v_i,\ i=1,2,...,n\}$ and the second is $P_{n_2}=\{u_i,\ i=1,2,...,n\}$ and each vertex (v_i) in P_{n_1} is adjacent to a corresponding vertex (u_i) in P_{n_2}. It is clear that the vertices $\{v_1,v_n,u_1,u_n\}$ belong to every rings dominating set, according to Observation 2.5(2). These vertices dominate the adjacent vertices $\{v_2,v_{n-1},u_2,u_{n-1}\}$. Thus, the remaining vertices are still not dominated, and they are $S=\{v_3,v_4,...,v_{n-2},u_1,u_1,...,u_{n-2}\}$. So, there are three cases depending on the number of vertices in each path as follows.

Case 1. If $n-4\equiv 0(mod3)$, then let $D_1=\left\{v_{3i+4},u_{3i+4},\ i=0,...,\left\lceil\frac{n-4}{3}\right\rceil-1\right\}$. It is clear that D_1 is the minimum dominating set of induced subgraph generated by S. Thus, $D=D_1\cup\{v_1,v_n,u_1,u_n\}$ is the minimum dominating set of graph G. Therefore, $\gamma_{ri}(G)=4+|D|=4+2\left\lceil\frac{n-4}{3}\right\rceil$.

Case 2. If $n-4\equiv 1(mod3)$, then let $D_1=\left\{v_{3i+4},u_{3i+4},\ i=0,...,\left\lceil\frac{n-4}{3}\right\rceil-2\right\}$. It is clear that D_1 is the minimum dominating set of all vertices in induced subgraph generated by S except the two vertices $\{v_{n-2},u_{n-2}\}$. Now, if these vertices are added to the set D_1, then these vertices are dominating themselves, but in this case the two vertices $\{v_{n-1}, u_{n-1}\}$ lie out the dominating set and this contradicts with definition of rings domination. Therefore, the set $D = D_1 \cup \{v_{n-3}, u_{n-3}\}$ is minimum rings dominating set with γ_{ri}-set (as an example, see Fig. 3). Thus, $\gamma_{ri}(G) = 4 + |D| = 4 + 2\left\lceil \frac{n-4}{3} \right\rceil$.

Case 3. In the same manner in the previous two cases, the set $D = D_1 \cup \{v_{n-3}, u_{n-3}\}$ is a minimum rings dominating set with γ_{ri}-set.

Therefore, from all cases above, the proof is obtained. □

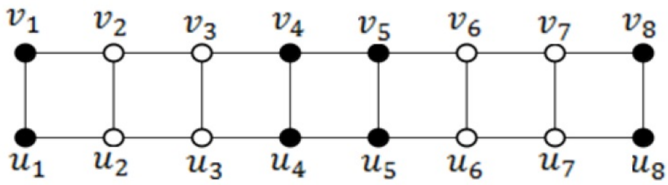

Figure 3: The minimum rings dominating set of ladder graph of order 16.

Theorem 3.3. For any two graphs G_1 and G_2 of order n and m respectively, then

$$\gamma_{ri}(G_1 + G_2) = \begin{cases} 1, & if \begin{cases} n = 1 \ and \ \delta(G_2) \geq 2, \ or \\ n = m = 2 \ and \ \delta(G_i) = 1, \ i = 1, 2, \ or \\ G_1 \equiv K_2 \ and \ \delta(G_2) \geq 1 \end{cases} \\ 2, & if \begin{cases} G_1 \equiv N_2 \ and \ \delta(G_2) \geq 2, \ or \\ n, m \geq 3 \end{cases} \end{cases}$$

Proof . The cases are as follows.

Case 1. If $n = 1$ and $\delta(G_2) \geq 2$, then the vertex v of the graph G_1 dominates all vertices in graph $G_1 + G_2$, in addition to each vertex in the join graph, except the vertex v, which is adjacent to other two vertices at least in $V(G_1 + G_2) - \{v\}$. Thus, $\gamma_{ri}(G_1 + G_2) = 1$.

Case 2. If $n = m = 2$ and $\delta(G_i) = 1$, $i = 1, 2$, then $G_1 + G_2 \equiv K_4$. Thus, $\gamma_{ri}(G_1 + G_2) = 1$, according to observation 2.8.

Case 3. If $G_1 \equiv K_2$ and $\delta(G_2) \geq 1$, then let $v \in G_1$, then this vertex is adjacent to all vertices in graph $G_1 + G_2$. Thus, in the same manner in case 1, the result is obtained.

Case 4. If $G_1 \equiv N_2$ and $\delta(G_2) \geq 2$, then let $V(G_1) = \{v_1, v_2\}$ and suppose that $D = \{v_1, v_2\}$. It is obvious that D is a minimum rings dominating set of graph $G_1 + G_2$ and each vertex in $V - D$ is adjacent to other two vertices at least in$V - D$, since $\delta(G_2) \geq 2$. Thus, $\gamma_{ri}(G_1 + G_2) = 2$.

Case 5. If $n, m \geq 3$, then the graph $G_1 + G_2$ contains an induced subgraph, isomorphic to complete bipartite graph. Thus, $\gamma_{ri}(G_1 + G_2) = 2$ according to Proposition 2.10.

Therefore, from all cases above the proof is done. □

Theorem 3.4. Let G_1 and G_2 be any graphs of order n and m respectively such that $\delta(G_2) \geq 2$, then $\gamma(G_1 \odot G_2) = |G_1|$.

Proof . Let $V(G_1) = \{v_1, v_2, ..., v_n\}$ and let $D = \{v_1, v_2, ..., v_n\}$. It is obvious that the set D is a minimal dominating set, since $D - v_i$ is not dominating because it does not dominate all vertices of a copy of G_2 that join with the vertex v_i. Furthermore, the set D is minimum and each vertex in $V - D$ is adjacent to other two vertices at least, since $\delta(G_2) \geq 2$. Thus, the result is obtained. □

4 Conclusion

Throughout this paper, a new parameter of domination in graphs called "rings domination number" has been introduced. Many results about the boundary of the number of edges, rings dominating set, and rings domination number have been presented. Additionally, the value of this number to some operations on two graphs has been determined.

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