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Remarks on uniformly convexity with applications in $A - G - H$ inequality and entropy

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Abstract

In this work, we shall give an upper bound for Jensen's inequality (for uniformly convex functions). Also, we introduce a refinement for the generalized $A-G-H$ inequality. Applying those results in information theory and obtain bounds for entropy.

Keywords: Jensen's inequality, entropy, arithmetic and geometric means, uniformly convex function, $A - G - H$ inequality
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1 Introduction and Basic notions

Arithmetic and geometric means are used in making estimates or approximations. In this section we study the properties of the arithmetic and geometric means.

Definition 1.1. Let $x_1, ..., x_n \in I$ be points, and let $p_1, ..., p_n \in [0, 1]$ be coefficients such that $\sum_{i=1}^n p_i = 1$. The sum $\sum_{i=1}^n p_i x_i$ is called the convex combination of points x_i.

Definition 1.2. [2] Let f be a real function on $I := [a, b]$. Then f is uniformly convex with modulus $\phi : R_{\geq 0} \longrightarrow [0, +\infty)$ if is increasing, vanishes only at 0, and

$$f(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)\phi(|x-y|) \leq \alpha f(x) + (1-\alpha)f(y)$$

for every $\alpha \in [0, 1]$ and $x, y \in [a, b]$.

For $\mathbf{x} := \{x_i\}_{i=1}^n \subseteq I$ and $\mathbf{p} := \{p_1, ..., p_n\} \subseteq (0, 1]$ define

$$A(\mathbf{p}, \mathbf{x}) := \sum_{i=1}^n p_i x_i, \;\; G(\mathbf{p}, \mathbf{x}) = \prod_{i=1}^n x_i^{p_i}, \;\; H(\mathbf{p}, \mathbf{x}) := (\sum_{i=1}^n \frac{p_i}{x_i})^{-1},$$

$$G(a, b) = \sqrt{ab}, \;\; L(a, b) := \frac{b-a}{\log b - \log a}, \;\; I(a, b) := \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}.$$

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Theorem 1.3. [7] (The Theorem of Arithmetic and Geometric Means) Let $\{x_1, ..., x_n\}$ be a positive sequence of real numbers. The geometric mean of n positive real numbers is always less than or equal to their arithmetic mean, i.e. $G_n := (\prod_{i=1}^n x_i)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^n x_i}{n} := A_n$.

Theorem 1.4. [14] If f is convex on $[a, b]$, then for any $\{p_i\},\ p_1, ..., p_n > 0$ and $\{x_i\} \subseteq I$, we have

$$\sum_{i=1}^n p_i f(x_i) - f(\sum_{i=1}^n p_i x_i) \leq f(a) + f(b) - 2f(\frac{a+b}{2}) = S_f(a, b). \tag{1.1}$$

Theorem 1.5. [15] If f is convex on I, and $p, q > 0$, $p + q = 1$, then

$$\sum_{i=1}^n p_i f(x_i) - f(\sum_{i=1}^n p_i x_i) \leq \max_p\{pf(a) + qf(b) - f(pa + qb)\} := T_f(a, b).$$

Theorem 1.6. [15] Let $0 < a < b$, $\mathbf{x} = \{x_i\}_{i=1}^n \subseteq I$. Then

$$1 \leq \frac{A(\mathbf{p}, \mathbf{x})}{G(\mathbf{p}, \mathbf{x})} \leq \frac{L(a, b)I(a, b)}{G^2(a, b)} := \Lambda_1(a, b). \tag{1.2}$$

Theorem 1.7. [15] Let $0 < a < b$, $\mathbf{x} = \{x_i\}_{i=1}^n \subseteq I$. Then

$$1 \leq \frac{G(\mathbf{p}, \mathbf{x})}{H(\mathbf{p}, \mathbf{x})} \leq \Lambda_1(a, b). \tag{1.3}$$

2 main results

In this section we continue with a refinement of theorems from [13, 15].

Lemma 2.1. [11] If $a > 0$ and $f : [a, b] \longrightarrow \mathbb{R}$ defined by $f(x) = \log(\frac{1}{x})$, then f is uniformly convex with modulus $\phi(r) = \frac{r^2}{2b^2}$.

Lemma 2.2. [11] If $a \geq 0$ and $f : [a, b] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x\log(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

then f is uniformly convex with modulus $\phi(r) = \frac{r^2}{2b}$.

Lemma 2.3. If f is differentiable and uniformly convex with modulus ϕ on $[a, b]$, then

$$f(x) \geq f(t) + (x - t)f'(t) + \phi(|x - t|),$$

for every $x, t \in I$.

Proof . First, suppose that $t < x$. For every $r(t < r < x)$, we have

$$\begin{aligned} f(r) &= f(\frac{x-r}{x-t}t + \frac{r-t}{x-t}x) \\ &\leq \frac{x-r}{x-t}f(t) + \frac{r-t}{x-t}f(x) - \frac{(x-r)(r-t)}{(x-t)^2}\phi(x-t). \end{aligned}$$

Hence,

$$\frac{f(r) - f(t)}{r - t} \leq \frac{f(x) - f(t)}{x - t} - \frac{x - r}{(x - t)^2}\phi(x - t),$$

for every $r(t<r<x)$. As $r \longrightarrow t^+$,

$$f'(t_+) \leq \frac{f(x)-f(t)}{x-t} - \frac{1}{x-t}\phi(x-t).$$

Thus,

$$f(x) \geq f(t) + (x-t)f'(t_+) + \phi(x-t). \tag{2.1}$$

Similarly,

$$f(x) \geq f(t) + (x-t)f'(t_-) + \phi(t-x), \tag{2.2}$$

for every $x<t$. The claim follows from (2.1) and (2.2). □

Lemma 2.4. If f is uniformly convex with modulus ϕ on $[a,b]$ and $0 \leq p,q \leq 1$; $p+q=1$, then

$$\begin{aligned} pf(a)+qf(b)-f(qa+pb) &\leq f(a)+f(b)-2f(\frac{a+b}{2}) \\ &\quad - pq\phi(b-a) - \frac{1}{2}\phi(|(b-a)(p-q)|). \end{aligned}$$

Proof . Let $p+q=1$, so

$$\begin{aligned} f(\frac{a+b}{2}) &= f(\frac{1}{2}(pa+qb)+\frac{1}{2}(qa+pb)) \\ &\leq \frac{1}{2}f(pa+qb)+\frac{1}{2}f(qa+pb)-\frac{1}{4}\phi(|(b-a)(p-q)|). \end{aligned}$$

Define $I_{pq} := pf(a)+qf(b)-f(qa+pb)$,

$$\begin{aligned} I_{pq} &= f(a)+f(b)-(qf(a)+pf(b))-f(qa+pb) \\ &\leq f(a)+f(b)-f(qa+pb)-pq\phi(b-a)-f(qa+pb) \\ &\leq f(a)+f(b)-pq\phi(b-a)-2f(\frac{a+b}{2})-\frac{1}{2}\phi(|(b-a)(p-q)|). \end{aligned}$$

□

Theorem 2.5. If f is uniformly convex with modulus ϕ on I, then

$$\begin{aligned} J_f(\mathbf{p},\mathbf{x}) :&= \sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \\ &\leq \max_p\{pf(a)+qf(b)-f(pa+qb)\} \\ &\quad - \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a) := \tilde{T}_f(a,b), \end{aligned}$$

where $p,q>0$ and $p+q=1$.

Proof . Since $\{x_i\}_i \subseteq [a,b]$, there is a sequence $\{\lambda_i\}_i (0 \leq \lambda_i \leq 1)$, such that $x_i = \lambda_i a + (1-\lambda_i)b$. Hence, following

Simic's method [15], we get

$$\begin{aligned}
J_f(\mathbf{p},\mathbf{x}) &= \sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \\
&= \sum_{i=1}^{n} p_i f(\lambda_i a + (1-\lambda_i)b) - f(\sum_{i=1}^{n} p_i(\lambda_i a + (1-\lambda_i)b)) \\
&\leq \sum_{i=1}^{n} p_i(\lambda_i f(a) + (1-\lambda_i)f(b) - \lambda_i(1-\lambda_i)\phi(b-a)) \\
&- f(a\sum_{i=1}^{n} p_i\lambda_i + b\sum_{i=1}^{n} p_i(1-\lambda_i)) \\
&= f(a)\sum_{i=1}^{n} p_i\lambda_i + f(b)\sum_{i=1}^{n} p_i(1-\lambda_i) - \sum_{i=1}^{n} p_i\lambda_i(1-\lambda_i)\phi(b-a) \\
&- f(a\sum_{i=1}^{n} p_i\lambda_i + b\sum_{i=1}^{n} p_i(1-\lambda_i)).
\end{aligned}$$

Denoting $p := \sum_{i=1}^{n} p_i\lambda_i$ and $q := 1 - \sum_{i=1}^{n} p_i\lambda_i$. Consequently,

$$J_f(\mathbf{P},\mathbf{x}) \leq pf(a) + qf(b) - f(pa+qb) - \sum_{i=1}^{n} p_i\lambda_i(1-\lambda_i)\phi(b-a), \tag{2.3}$$

On the other hand, since $\lambda_i = \frac{b-x_i}{b-a}$,

$$\begin{aligned}
\sum_{i=1}^{n} p_i\lambda_i(1-\lambda_i)\phi(b-a) &\geq \sum_{i=1}^{n} p_i \min_{1\leq i\leq n}\{\lambda_i\}(1-\max_{1\leq i\leq n}\{\lambda_i\})\phi(b-a) \\
&= \min_{1\leq i\leq n}\{\lambda_i\}(1-\max_{1\leq i\leq n}\{\lambda_i\})\phi(b-a) \\
&= \frac{(b-\max_i\{x_i\})(\min_i\{x_i\}-a)}{(b-a)^2}\phi(b-a).
\end{aligned} \tag{2.4}$$

Together, (2.3) and (2.4) imply

$$\begin{aligned}
\sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) &\leq pf(a) + qf(b) - f(pa+qb) \\
&- \frac{(b-\max_i\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a) \\
&\leq \max_P\{pf(a) + qf(b) - f(pa+qb)\} \\
&- \frac{(b-\max_i\{x_i\})(\min_i\{x_i\}-a)}{(b-a)^2}\phi(b-a).
\end{aligned}$$

□

Theorem 2.6. If f is differentiable and uniformly convex with modulus ϕ on $I \subset D_f$, then

$$\begin{aligned}
&\max_p\{pf(a) + qf(b) - f(pa+qb)\} \\
&\leq \frac{1}{4}(b-a)(f'(b)-f'(a)) - \min_p\{p\phi(q(b-a)) + q\phi(p(b-a))\}.
\end{aligned}$$

Proof . By the use of Lemma 2.3 we have

$$f(x) \geq f(t) + (x-t)f'(t) + \phi(|x-t|),$$

for every $x, t \in I$. So,

$$f(pa+qb) \geq f(a) + q(b-a)f'(a) + \phi(q(b-a)),$$

and

$$f(pa+qb) \geq f(b) + p(b-a)f'(b) + \phi(p(b-a)).$$

Hence,

$$pf(pa+qb) \geq pf(a) + pq(b-a)f'(a) + p\phi(q(b-a)),$$

and

$$qf(pa+qb) \geq qf(b) + pq(b-a)f'(b) + q\phi(p(b-a)).$$

Therefore,

$$\begin{aligned}&pf(a)+qf(b)-f(pa+qb)\\&\leq pq(b-a)(f'(b)-f'(a)) - \{p\phi(q(b-a)) + q\phi(p(b-a))\}.\end{aligned}$$

Thus,

$$\begin{aligned}&\max_p\{pf(a)+qf(b)-f(pa+qb)\}\\&\leq \max_p\{pq(b-a)(f'(b)-f'(a)) - p\phi(q(b-a)) - q\phi(p(b-a))\}\\&\leq \frac{1}{4}(b-a)(f'(b)-f'(a)) - \min_p\{p\phi(q(b-a)) + q\phi(p(b-a))\}.\end{aligned}$$

□

Corollary 2.7. If f is differentiable and uniformly convex with modulus ϕ on $I \subset D_f$, then

$$\begin{aligned}J_f(\mathbf{p},\mathbf{x}) \leq &\frac{1}{4}(b-a)(f'(b)-f'(a)) - \min_p\{p\phi(q(b-a)) + q\phi(p(b-a))\}\\&- \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a) := \tilde{R}_f(a,b).\end{aligned} \tag{2.5}$$

Proof . The Corollary follows from Theorems 2.5 and 2.6. □

Remark 2.8. Since $\tilde{R}_f(a,b) \leq \frac{1}{4}(b-a)(f'(b)-f'(a))$, the estimation (2.5) is better than ([13], Theorem 2.2 *(I) or [15], Theorem D (i))*.

Theorem 2.9. For an arbitrary uniformly convex function f and $I \subset D_f$ there exist a sequence $\{\mathbf{x_0}\} \subset I$ and an associated weight sequence $\mathbf{p_0}$, such that $J_f(\mathbf{p_0},\mathbf{x_0}) = \tilde{T}_f(a,b)$.

Proof . The proof is similar to the proof of ([15], Theorem E). □

In view of ([15], Theorem D), we prove the following theorem.

Theorem 2.10. If f is differentiable and uniformly convex with modulus ϕ on $I \subset D_f$, then

$$\tilde{T}_f(a,b) \leq \tilde{C}(f)\tilde{S}_f(a,b), \tag{2.6}$$

where

$$\tilde{C}(f) := \sup_{a,b,p}\{\frac{pf(a)+qf(b)-f(pa+qb) - \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a)}{f(a)+f(b)-2f(\frac{a+b}{2}) - J^{\phi,\overline{x}}(a,b)}\},$$

$$\begin{aligned} J^{\phi,\overline{x}}(a,b) :&= \frac{1}{2}\phi(|a+b-2\sum_{i=1}^{n}p_i x_i|) \\ &+ \frac{(b-\sum_{i=1}^{n}p_i x_i)(\sum_{i=1}^{n}p_i x_i - a)}{(b-a)^2}\phi(b-a) \\ &+ \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a), \end{aligned}$$

and $\tilde{S}_f(a,b) := f(a)+f(b)-2f(\frac{a+b}{2}) - J^{\phi,\overline{x}}(a,b)$.

Proof . Let p and q be two non-negative arbitrary points with $p+q=1$,

$$K_{pq} := pf(a)+qf(b)-f(pa+qb) - \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a)$$

and $\tilde{S}_f(a,b) := f(a)+f(b)-2f(\frac{a+b}{2}) - J^{\phi,\overline{x}}(a,b)$. So,

$$\begin{aligned} K_{pq} &= \frac{Kpq}{\tilde{S}_f(a,b)} \times \tilde{S}_f(a,b) \\ &\le \sup_{a,b,p}\{\frac{Kpq}{\tilde{S}_f(a,b)}\} \times \tilde{S}_f(a,b) \\ &= \tilde{C}(f)\tilde{S}_f(a,b). \end{aligned}$$

Since p is an arbitrary point in $[0,1]$, we obtain $\tilde{T}_f(a,b) \le \tilde{C}(f)\tilde{S}_f(a,b)$. $\square$

Remark 2.11. With the use of Lemma 2.4, we obtain $\tilde{C}(f) \le 1$ and $\tilde{S}_f(a,b) \le S_f(a,b)$. So the estimation (2.6) is better than (1.1) and ([13], Theorem 2.2 *(II))*.

Theorem 2.12. Let f be differentiable and uniformly convex with modulus ϕ on $I \subset D_f$, then

$$\begin{aligned} \tilde{T}_f(a,b) = \frac{f(b)-f(a)}{b-a}\Theta_f(a,b) + \frac{bf(a)-af(b)}{b-a} \\ - f(\Theta_f(a,b)) - \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a), \end{aligned}$$

where $\Theta_f(a,b)$ is the Lagrange mean value of numbers a,b defined by

$$\Theta_f(a,b) := (f')^{-1}(\frac{f(b)-f(a)}{b-a}).$$

Proof . Let

$$\begin{aligned} g(p) :&= pf(a)+(1-p)f(b) - f(pa+(1-p)b) \\ &- \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a). \end{aligned}$$

Therefore $p_0 = \frac{b-\Theta(a,b)}{b-a}$ is the only critical point of g, and $g(p_0)$ is the maximum value of the function g. $\square$

3 Applications

The convex theory becomes essential in entropy bounds [8]-[15]. In this section we apply the results of previous section in the $A-G-H$ inequality, applying those results in information theory and obtain bounds for Shannon's entropy.

Theorem 3.1. Let $0<a<b$, $\mathbf{x}=\{x_i\}_{i=1}^{n} \subseteq I$. Then

$$1 \le \frac{A(\mathbf{p},\mathbf{x})}{G(\mathbf{p},\mathbf{x})} \le \frac{L(a,b)I(a,b)}{G^2(a,b)} e^{-\frac{(b-\max_i\{x_i\})(\min_i\{x_i\}-a)}{2b^2}} := \tilde{\Lambda}_1(a,b). \tag{3.1}$$

Proof . Applying Theorems 2.5 and 2.12 with $f(x)=-\log x$ and $\phi(r)=\frac{r^2}{2b^2}$, obtain

$$\begin{aligned}0\le \log\{\frac{A(\mathbf{p},\mathbf{x})}{G(\mathbf{p},\mathbf{x})}\} &\le \log(p_0a+(1-p_0)b)-p_0\log a-(1-p_0)\log b\\ &-\frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{2b^2}\\ &=\log\{\frac{L(a,b)I(a,b)}{G^2(a,b)}e^{-\frac{(b-\max_i\{x_i\})(\min_i\{x_i\}-a)}{2b^2}}\},\end{aligned}$$

where $p_0=\frac{b-\Theta_{-\log x}(a,b)}{b-a}=\frac{b-L(a,b)}{b-a}$. Which completes the proof. □

Remark 3.2. Since $\tilde{\Lambda}_1(a,b)\le\Lambda_1(a,b)$, (3.1) is a better bound than (1.2), for $A-G$ inequality.

Theorem 3.3. Let $0<a<b$, $\mathbf{x}=\{x_i\}_{i=1}^n\subseteq I$. Then

$$1\le\frac{G(\mathbf{p},\mathbf{x})}{H(\mathbf{p},\mathbf{x})}\le\frac{L(a,b)I(a,b)}{G^2(a,b)}e^{-\frac{(b\min_i\{x_i\}-1\})(1-a\max_i\{x_i\})}{2b^2\min_i\{x_i\}\max_i\{x_i\}}}:=\tilde{\Lambda}_2(a,b). \tag{3.2}$$

Proof . By the change of variable $x_i\longrightarrow\frac{1}{x_i}$, we have

$$A(\mathbf{p},\frac{\mathbf{1}}{\mathbf{x}})=\frac{1}{H(\mathbf{p},\mathbf{x})},\ G(\mathbf{p},\frac{\mathbf{1}}{\mathbf{x}})=\frac{1}{G(\mathbf{p},\mathbf{x})}\text{ and }\tilde{\Lambda}_1(\frac{1}{a},\frac{1}{b})=\tilde{\Lambda}_2(a,b).$$

Hence the proof easily follows from Theorem 3.1. □

The combination of Theorems 2.5 and 2.12 yields the following theorem.

Theorem 3.4. Let $0<a<b$, $\mathbf{x}=\{x_i\}_{i=1}^n\subseteq I$. Then

$$\begin{aligned}0&\le\sum_{i=1}^n p_ix_i\log x_i-(\sum_{i=1}^n p_ix_i)\log(\sum_{i=1}^n p_ix_i)\\ &\le I(a,b)-\frac{G^2(a,b)}{L(a,b)}-\frac{(b-\max_i\{x_i\})(\min_i\{x_i\}-a)}{2b}.\end{aligned}$$

Proof . Suppose that $f(t)=t\log t$. Then

$$J_f(\mathbf{p},\mathbf{x})=\sum_{i=1}^n p_ix_i\log x_i-(\sum_{i=1}^n p_ix_i)\log(\sum_{i=1}^n p_ix_i), \tag{3.3}$$

${f'}^{-1}(x)=e^{x-1}$ and

$$\begin{aligned}\Theta_f(a,b)&=(f')^{-1}(\frac{f(b)-f(a)}{b-a})\\ &=e^{\frac{b\log b-a\log a}{b-a}-1}\\ &=\frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}=I(a,b).\end{aligned}$$

Also,

$$\frac{bf(a)-af(b)}{b-a}=\frac{ba\log a-ab\log b}{b-a}=-\frac{G^2(a,b)}{L(a,b)},$$

$\frac{f(b)-f(a)}{b-a}=\frac{b\log b-a\log a}{b-a}:=\alpha(a,b)$ and

$$\begin{aligned}f(\Theta_f(a,b))&=\Theta_f(a,b)\log(\Theta_f(a,b))\\ &=\frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}\log(\frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}})\\ &=I(a,b)\times\alpha(a,b)-I(a,b).\end{aligned}$$

On the other hand, by Theorem 2.12,

$$\begin{aligned}\tilde{T}_f(a,b) &= \max_{0<p<1}\{g(p)\} = g(p_0)\\ &= \frac{f(b)-f(a)}{b-a}\Theta_f(a,b) + \frac{bf(a)-af(b)}{b-a}\\ &- f(\Theta_f(a,b)) - \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a),\end{aligned}$$

where $p_0 = \frac{b-\Theta(a,b)}{b-a}$ and

$$\begin{aligned}g(p) &= pf(a) + (1-p)f(b) - f(pa+(1-p)b)\\ &- \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a).\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{T}_f(a,b) &= \frac{f(b)-f(a)}{b-a}\Theta_f(a,b) + \frac{bf(a)-af(b)}{b-a}\\ &- f(\Theta_f(a,b)) - \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\phi(b-a)\\ &= \alpha(a,b)\times I(a,b) - \frac{G^2(a,b)}{L(a,b)} - (I(a,b)\times\alpha(a,b) - I(a,b))\\ &- \frac{(b-\max\{x_i\})(\min\{x_i\}-a)}{(b-a)^2}\times\frac{(b-a)^2}{2b}.\end{aligned}$$

Hence,

$$\tilde{T}_f(a,b) = I(a,b) - \frac{G^2(a,b)}{L(a,b)} - \frac{(b-\max_i\{x_i\})(\min_i\{x_i\}-a)}{2b}. \tag{3.4}$$

Thus, the result follows from (3.3) and (3.4), Lemma 2.2 and Theorems 2.5 and 2.12. □

Definition 3.5. Let $P = (p_1, ..., p_n)$ and $Q = (q_1, ..., q_n)$ are two positive probability distributions. The Shannon entropy of P is defined by $H(P) := \sum_{i=1}^n p_i \log \frac{1}{p_i}$. Also the relative entropy or Kullback-Leibler distance between two positive probability distributions P and Q, is defined as

$$D(P \parallel Q) := \sum_{i=1}^n p_i \log \frac{p_i}{q_i}.$$

Remark 3.6. 1. $\tilde{\Lambda}_1(a,b) = \Lambda_1(a,b)$ if and only if $a = \min\{x_i\}_i$ or $b = \max\{x_i\}_i$.
2. $\tilde{\Lambda}_2(a,b) = \Lambda_1(a,b)$ if and only if $\min\{x_i\}_i = \frac{1}{b}$ or $\max\{x_i\}_i = \frac{1}{a}$.

From the above remark and ([15], Theorem I) we conclude the following estimates.

Theorem 3.7. Define $m := \min_i\{\frac{p_i}{q_i}\}$, $M := \max_i\{\frac{p_i}{q_i}\}$, $\mu := \min_i\{p_i\}$ and $\nu := \max_i\{p_i\}$. Then

1. $0 \le D(P \parallel Q) \le \log(\tilde{\Lambda}_1(m,M)) = \log(\Lambda_1(m,M))$.
2. $0 \le \log n - H(P) \le \log(\tilde{\Lambda}_2(\mu,\nu)) = \log(\Lambda_1(\mu,\nu))$.

Proof . This is an easy consequence of Remark 3.6 and ([15], Theorem I). □

4 Conclusions

The topic of this paper is precisely the extension of results from [15] considering the class of uniformly convex functions, and improve the upper bound for the $A-G-H$ inequality. Also, we establish some results related to the bounds of the Shannon entropy.

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