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A numerical approach for solving a class of nonlinear fractional integro-differential equation with weakly singular kernel by alternative legendary polynomials

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Abstract

In this paper, we propose a new spectral approach based on alternative Legendre polynomials for solving nonlinear fractional integro-differential equations with weakly singular kernel. To do this, by the help of operational matrices of fractional integration and product based on these polynomials, we reduce the considered problem to a system of algebraic equations. Also, we investigate the error analysis of the proposed scheme. Finally, we present some numerical examples to show the high accuracy and validity of the new work.

Keywords: Alternative Legendre polynomials, nonlinear fractional integro-differential equations, operational matrix, error analysis
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1 Introduction

Fractional calculus is a useful tool for understanding the complex world. The significance of fractional calculus has been demonstrated to be very effective in various phenomena, such as diffusion processes, viscoelastic materials, long-range interactions, etc. It turns out that fractional calculus provide many helpful features that offer interesting solutions to system modeling and control, optimization algorithm design, and machine learning. The fractional order calculus (FOC) was unexplored in engineering, because of its inherent complexity, the apparent self-sufficiency of the integer order calculus (IOC), and the fact that it does not have a fully acceptable geometrical or physical interpretation. The integral and integro-differential equations play an important role in the modeling and analyzing many problems in the fields of mechanics, engineering, physics, chemistry [1, 5, 4, 22, 8, 23, 9, 12, 15]. Also, some of these types of equations can be appear in the field of dynamics of interfaces between nanoparticles and substrates [20], radioactive equilibrium [13], heat condition equation [27] and etc. As we know, in general, determining the analytical solutions of these equations are either difficult or complex. Therefore, the development of efficient and accurate numerical methods to solve these equations is inevitable. Over the last decades, several numerical methods for solving these equations have been proposed by researchers. Among these numerical techniques, spectral approaches based on operational

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matrices are a class of spatial discretization for obtaining the approximate solution for various integral and differential equations. To study some numerical methods for solving these equations, one can refer to [11, 28, 7, 2, 16, 10, 26]. In this work, we consider a class of nonlinear fractional integro-differential equations with weakly singular kernel of the following form

$$ {}_0^C D_t^{\alpha} y(t) = g(t) + p(t)y(t) + \lambda_1 \int_0^t (t-s)^{-\beta} y^m(s)ds + \lambda_2 \int_0^t s^{\alpha}(t-s)^{-\beta} y(s)ds, \tag{1.1} $$

$$ y^{(i)}(0) = y_0^{(i)}, \qquad i = 0, 1, \cdots, [\alpha]-1, \tag{1.2} $$

where α, β are positive real constants, $p(t)$ and $g(t)$ are known functions defined on the interval $I(T) := [0,T], y_0^{(i)}$ $(i = 0, 1, \cdots, [\alpha]-1)$ are given real numbers, $[\alpha]$ is the ceiling function of α, D_t^{α} is the Caputo fractional differential operator of order α and λ_1, λ_2, m are positive integer numbers and also $y(t)$ is the unknown function to be determined. Because of the importance of fractional integro- differential equations, many studies such as the existence and uniqueness of solution for these equations have been done in [6, 25]. Local and global existence and uniqueness results for the solution of fractional differential equations have been studied in [18] and [19], respectively. Nemati et al. [21] applied modification of hat functions to solve nonlinear fractional integro-differential equations (1.1),(1.2) in cases $\lambda_1 = 1$ and $\lambda_2 = 0$. The authors of [31] investigated collocation schemes to solve problems (1.1),(1.2) in cases $\lambda_1 = 1, \lambda_2 = 0$ and $m = 1$ (linear case). In [14], the authors applied a hybrid collocation method for solving fractional integro-differential equations with a weakly singular kernel. Also, Wang et al. [29] introduced the second kind Chebyshev wavelets schemes for solving the fractional integro-differential equations with weakly singular kernel.
The present paper is organized in six sections. In Section 2, we give preliminaries of the fractional calculus. Section 3 is devoted to introduce some basic definitions and the operational matrices of integration and product based on alternative Legendre polynomials. In Section 4, we propose a numerical approach to solve problems (1.1),(1.2) by the help of operational matrices derived in previous section. Also, in this section, we present the error estimation for the proposed method. In Section 5, to show the applicability and the accuracy of the proposed method, we give some numerical examples. Finally, in Section 7 the conclusion of this article is presented.

2 Overview of the fractional calculus

There are several definitions of fractional integration and differentiation. Among the types of fractional derivatives that are used in fractional calculus, we use the Caputo fractional derivative. The essentials of the theory of fractional calculus and the main definitions and some facts applied in this article are presented here.

Definition 2.1. [25] A real function $f(t), t > 0$ is assumed to be in the space $C_{\mu}, \mu \in \mathbb{R}$ if there exists a real number $(q > \mu)$ such that $f(t) = t^q g(t)$, where $g(t) \in C[0, \infty]$, and it is said to be in the space C_{μ}^m if $f^{(m)} \in C_{\mu}, m \in \mathbb{N}$.

Definition 2.2. [25] Caputo fractional derivative of order α is defined as follows

$$ (D^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \qquad n-1 < \alpha \le n, \quad n \in \mathbb{N}, \tag{2.1} $$

where α is the order of the derivative and n is the smallest integer greater than α.

Definition 2.3. [25] The Riemann-Liouville fractional integral of order α $(\alpha > 0)$ of function $y(t) \in C_{\mu}, \mu \ge -1$ is denoted by I^{α}, which defined by

$$ D^{-\alpha} y(t) = I^{\alpha} y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(x)}{(t-x)^{1-\alpha}} dx, & \alpha > 0, \\ y(t), & \alpha = 0. \end{cases} \tag{2.2} $$

Theorem 2.4. The supposition is that the continuous function $\zeta(t)$ has a fractional derivative of order α thus, we have:

$$ D_t^{\alpha} I_t^{\beta} \zeta(t) = \begin{cases} I_t^{\beta-\alpha} \zeta(t), & \alpha < \beta, \\ \zeta(t), & \alpha = \beta, \\ D_t^{-\beta+\alpha} \zeta(t), & \alpha > \beta, \end{cases} \tag{2.3} $$

$$I_t^{\alpha} D_t^{\alpha} \zeta(t) = \zeta(t) - \sum_{k=0}^{n-1} \zeta^{(k)}(0^+) \frac{t^k}{k!}, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}, \tag{2.4}$$

$$D_t^{\alpha} I_t^{\alpha} \zeta(t) = \begin{cases} \zeta(t), & n-1 < \alpha \leq n, \quad n \in \mathbb{N}, \\ D_t^{\alpha} I_t^{\alpha} \zeta(t) + \zeta(0), & 0 < \alpha < 1. \end{cases} \tag{2.5}$$

The Caputo derivative and Riemann-Liouville integral present the following properties [24] for $f \in C_{\alpha}$, $\alpha \geq -1$, $\mu \geq 1$, $\eta \geq 0$, $\beta > -1$:

$$\begin{aligned} & I^{\mu} \in C_0, \\ & I^{\eta} I^{\delta} f(t) = I^{\delta} I^{\eta} f(t) \\ & I^{\delta} I^{\eta} f(t) = I^{\delta+\eta} f(t), \\ & D^{\delta} D^{\eta} f(t) = D^{\delta+\eta} f(t), \\ & D^{\delta} I^{\delta} f(t) = f(t), \\ & I^{\delta} D^{\delta} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(o^+), \quad m-1 < \delta < m, \quad m \in \mathbb{N}. \end{aligned} \tag{2.6}$$

Furthermore,

$$I^{\delta} X^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\delta+\beta+1)} X^{\delta+\beta}. \tag{2.7}$$

3 Some basic concepts of alternative Legendre polynomials (ALPs)

3.1 Properties of ALPs

The set $P_n = \{P_{nk} : k = 0, 1, \cdots, n\}$ of ALPs of degree n are defined by explicit formula on the interval $[0, 1]$ (see [3]) as follows:

$$P_{nk}(t) = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} \binom{n+k+j+1}{n-k} t^{k+j}, \quad k = 0, 1, \cdots, n. \tag{3.1}$$

In the other hand, by considering the weighting function $w(t) = 1$, they are the orthogonal function on the interval $[0, 1]$. We note that the ALPs satisfy the orthogonality relationships as

$$\int_0^1 P_{nk}(t) P_{nl}(t) dt = \begin{cases} \frac{1}{k+l+1}, & k = l, \\ 0, & otherwhise, \end{cases} \tag{3.2}$$

for $k, l = 0, 1, \cdots, n$.
We can reproduce equation (3.1) with Rodrigues's type as

$$P_{nk}(t) = \frac{1}{(n-k)!} \frac{1}{t^{k+1}} \frac{d^{n-k}}{dt^{n-k}} \left(t^{n+k+1} (1-t)^{n-k} \right), \quad k = 0, 1, \cdots, n. \tag{3.3}$$

So, we have

$$\int_0^1 P_{nk}(t) dt = \int_0^1 t^n dt = \frac{1}{n+1}, \quad k = 0, 1, \cdots, n. \tag{3.4}$$

Here, we note that each element of the set $P_n = \{P_{nk}\}_{k=0}^{n}$ are the polynomials of degree $k, 0 \leq k \leq n$. For example, in the following and for $n = 3$, we give the first few alternative Legendre polynomials

$$\begin{aligned} & P_{30}(t) = 4 - 30t + 60t^2 - 35t^3, \qquad && P_{31}(t) = 10t - 30t^2 + 21t^3, \\ & P_{32}(t) = 6t^2 - 7t^3, && P_{33}(t) = t^3. \end{aligned}$$

3.2 Function approximation

Consider $P_n = \{P_{nk}\}_{k=0}^{n} \subset H = L^2[0,1]$ is a set of ALPs and suppose that $Y = Span\{P_{nk}(t) : k = 0, 1, \cdots, n\}$. Also, f be arbitrary function from H. So, Y is a finite dimensional subspace of H. Therefore, based on Weierstrass theorem states that every continuous function f on interval $[a, b]$ can be uniformly approximated by a polynomial function. So, f has a unique best approximation in Y named $f^*(t)$ such that

$$\forall y(t) \in Y : ||f - f^*||_2 \leq ||f - y||_2. \tag{3.5}$$

This implies that

$$\forall y(t) \in Y : \langle y, f - f^* \rangle = 0, \tag{3.6}$$

where $\langle ., . \rangle$ denotes inner product. Therefore, any arbitrary function $f \in H = L^2[0,1]$ can be approximated in terms of ALPs. So, there exists a set of unique coefficient $\{c_k : k = 0, 1, \cdots, n\}$ such that

$$f(t) \approx f^*(t) = \sum_{k=0}^{n} c_k P_{nk}(t), \tag{3.7}$$

Here coefficient c_k can be calculated by the following formula

$$c_k = \frac{\langle f, P_{nk} \rangle}{\langle P_{nk}, P_{nk} \rangle} = (2k+1)\langle f, P_{nk} \rangle, \quad k = 0, 1, \cdots, n, \tag{3.8}$$

and

$$\langle f, f \rangle = \int_0^1 f^2(t)dt. \tag{3.9}$$

Clearly, equation (3.7) can be written by the matrix form as follows:

$$f(t) \simeq \sum_{k=0}^{n} c_k P_{nk}(t) = C^t \Phi(t), \tag{3.10}$$

where

$$C = [c_0, c_1, \cdots, c_n], \tag{3.11}$$

and

$$\Phi(t) = [P_{n0}(t), P_{n1}(t), \cdots, P_{nn}(t)]^T. \tag{3.12}$$

In the order hand, we can be written equation (3.12) by the following formula

$$\Phi(t) = QX_t, \tag{3.13}$$

where

$$X_t = [1, t, t^2, \cdots, t^n]^T, \tag{3.14}$$

and Q is the upper triangular matrix defined by

$$Q = [a_{kj}], \quad k, j = 0, 1, \cdots, n,$$

$$a_{kj} = \begin{cases} 0, & 0 \leq j < k, \\ (-1)^{j-k}\binom{n-k}{j-k}\binom{n+j+1}{n-k}, & k \leq j \leq n. \end{cases} \tag{3.15}$$

3.3 Operational matrices of the ALPs

In this part of the study, we describe the analytic expression of operational matrices of fractional integration and product based on the ALPs.

To compute the operational matrices of fractional integration and product, we need to introduce some properties of ALPs as the following. Let

$P_{ni}(t)=\sum_{r=0}^{n} p_r^{(i)}t^r$, $P_{nj}(t)=\sum_{r=0}^{n} p_r^{(i)}t^r$ and $P_{nk}(t)=\sum_{r=0}^{n} p_r^{(k)}t^r$ are ith, jth and kth ALPs, respectively. Therefore, we have

$$1)\quad P_{nk}(t)P_{nj}(t)=\sum_{r=0}^{2n} q_r^{(k,j)}t^r,$$

$$2)\quad q_r^{(k,j)}=\begin{cases}\sum_{l=0}^{r} p_l^{(k)}p_{r-l}^{(j)}, & r\le n,\\ \sum_{l=r-n}^{n} p_l^{(k)}p_{r-l}^{(j)}, & r>n,\end{cases} \tag{3.16}$$

$$3)\quad \int_0^1 t^r P_{nk}(t)dt=\sum_{l=0}^{n-k}\frac{(-1)^l\binom{n-k}{l}\binom{n+k+l+1}{n-k}}{k+l+r+1},\qquad k=0,1,\cdots,n, \tag{3.17}$$

$$4)\quad \int_0^1 P_{ni}(t)P_{nj}(t)P_{nk}(t)dt=\sum_{r=0}^{2n} q_r^{(k,j)}\sum_{l=0}^{n-i}\frac{(-1)^l\binom{n-i}{l}\binom{n+i+l+1}{n-i}}{i+l+r+1}. \tag{3.18}$$

By applying the fractional operator I_t^{α} defined by equation (2.2) on the polynomial $P_{nk}(t)$ and then by using equation (2.7), we get

$$\begin{aligned} I^{\alpha}P_{nk}(t)&=I^{\alpha}\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}\binom{n+k+j+1}{n-k}t^{k+j}\\ &=\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}\binom{n+k+j+1}{n-k}I^{\alpha}t^{k+j}\\ &=\sum_{j=0}^{n-k}(-1)^j\frac{\Gamma(k+j+1)}{\Gamma(k+j+\alpha+1)}\binom{n-k}{j}\binom{n+k+j+1}{n-k}t^{\alpha+k+j},\quad k=0,1,\cdots,n \end{aligned} \tag{3.19}$$

Now, by using equation (3.10), one can approximate $t^{\alpha+k+j}$ in terms of ALPs as follows:

$$t^{k+j+\alpha}\simeq\sum_{r=0}^{n} c_{kjr}p_{nr}(t). \tag{3.20}$$

Clearly, we can obtain the coefficients c_{kjr} by using equation (3.8) as follows:

$$\begin{aligned} c_{kjr}&=(2r+1)\int_0^1 y(t)P_{nr}dt\\ &=(2r+1)\int_0^1 t^{\alpha+k+j}\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}\binom{n+r+l+1}{n-r}t^{r+1}dt\\ &=(2r+1)\sum_{l=0}^{n-r}\frac{(-1)^l\binom{n-r}{l}\binom{n+r+l+1}{n-r}}{k+j+r+l+\alpha+1},\quad r=0,1,\cdots,n. \end{aligned} \tag{3.21}$$

Substituting (3.20) into (3.19), we have

$$I^{\alpha}p_{nk}(t)=\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}\binom{n+k+j+1}{n-k}\frac{\Gamma(k+j+1)}{\Gamma(k+j+\alpha+1)}\sum_{r=0}^{n} c_{kjr}p_{nr}(t), \tag{3.22}$$

for $k=0,1,\cdots,n$. By using equations (3.21) and (3.22), we get

$$\begin{aligned} I^{\alpha}p_{nk}(t)&=\sum_{r=0}^{n}(2r+1)\left[\sum_{j=0}^{n-k}\frac{(-1)^j\binom{n-k}{j}\binom{n+k+j+1}{n-k}\Gamma(k+j+1)}{\Gamma(k+j+\alpha+1)}\sum_{l=0}^{n-r}\frac{(-1)^l\binom{n-r}{l}\binom{n+r+l+1}{n-r}}{k+j+r+l+\alpha+1}\right]p_{nr}(t)\\ &=\sum_{r=0}^{n}\theta_{kr}^{(\alpha)}p_{nr}(t), \end{aligned} \tag{3.23}$$

where

$$\theta_{kr}^{(\alpha)} = (2r+1)\left[\sum_{j=0}^{n-k} \frac{(-1)^j \binom{n-k}{j}\binom{n+k+j+1}{n-k}\Gamma(k+j+1)}{\Gamma(k+j+\alpha+1)} \sum_{l=0}^{n-r} \frac{(-1)^l \binom{n-r}{l}\binom{n+r+l+1}{n-r}}{k+j+r+l+\alpha+1}\right]. \tag{3.24}$$

So, we conclude that

$$I_t^{\alpha}\Phi(t) = P^{(\alpha)}\Phi(t), \tag{3.25}$$

where $P^{(\alpha)}$ is operational matrix of fractional integration based on the ALPs defined as follows

$$P^{(\alpha)} = [\theta_{kr}^{(\alpha)}], \quad k, r = 0, 1, \cdots, n. \tag{3.26}$$

In the process of solving all forms of differential and integral equations numerically, we require to evaluate $\Phi(t)\Phi^T(t)C$ in terms of $P_{n,k}, k = 0, 1, \cdots, n$ where C is an arbitrary $n \times 1$ vector. So, we have

$$\Phi(t)\Phi^T(t)C \simeq \tilde{C}\Phi(t). \tag{3.27}$$

Here, $\tilde{C} = [\tilde{c}_{ik}]_{i,k=0}^{n}$ is called the operational matrix of product of order $(n+1) \times (n+1)$. The elements of matrix $\tilde{C}$ can be obtain based on ALPs over the interval $[0, 1]$. Obviously, we have

$$\Phi(t)\Phi^T(t)C = \begin{bmatrix} \sum_{j=0}^{n} c_j p_{n0}(t) p_{nj}(t) \\ \sum_{j=0}^{n} c_j p_{n1}(t) p_{nj}(t) \\ \vdots \\ \sum_{j=0}^{n} c_j p_{nn}(t) p_{nj}(t) \end{bmatrix} \tag{3.28}$$

By approximating $p_{ni}(t)p_{nj}(t),\ i, j = 0, 1, \cdots, n$ in terms of ALPs, we have

$$p_{ni}(t)p_{nj}(t) \simeq \sum_{k=0}^{n} a_{ijk} p_{nk}(t). \tag{3.29}$$

Also, by using equation (3.8) and (3.18), we have

$$a_{ijk} = (2k+1)\int_0^1 p_{ni}(t)p_{nj}(t)p_{nk}(t)dt = (2k+1)\tau_{ijk}, \tag{3.30}$$

where

$$\tau_{ijk} = \int_0^1 p_{ni}(t)p_{nj}(t)p_{nk}(t)dt. \tag{3.31}$$

Therefore, we obtain

$$\begin{aligned} \sum_{j=0}^{n} c_j p_{ni}(t) p_{nj}(x) &\simeq \sum_{j=0}^{n} c_j \sum_{k=0}^{n} (2k+1)\tau_{ijk} p_{nk}(t) \\ &= \sum_{k=0}^{n} \big((2k+1)\sum_{j=0}^{n} c_j \tau_{ijk}\big) p_{nk}(t), \\ &= \sum_{k=0}^{n} \tilde{c}_{ik} p_{nk}(t), \end{aligned} \tag{3.32}$$

for $i = 0, 1, \cdots, n$. Substituting (3.32) into (3.28), we get the following equation

$$\Phi(t)\Phi^T(t)C \simeq \begin{bmatrix} \sum_{k=0}^{n} \tilde{c}_{0k} p_{nk}(t) \\ \sum_{k=0}^{n} \tilde{c}_{1k} p_{nk}(t) \\ \vdots \\ \sum_{k=0}^{n} \tilde{c}_{nk} p_{nk}(t) \end{bmatrix} = \tilde{C}\Phi(t). \tag{3.33}$$

Clearly, from (3.33), we have

$$\tilde{C} = [\tilde{c}_{ik}], \quad i, k = 0, 1, \cdots, n, \tag{3.34}$$

where

$$\tilde{c}_{ik} = (2k+1)\sum_{j=0}^{n} c_j \tau_{ijk},$$

$$\tau_{ijk} = \int_0^1 p_{ni}(t)p_{nj}(t)p_{nk}(t)dt. \tag{3.35}$$

Obviously, for $n=2$ and $a=1$, we have

$$C = [c_0, c_1, c_2]^T, \quad \Phi(t) = [P_{2\ 0}(t), b_{2\ 1}(t), b_{2\ 2}(t)]^T, \tag{3.36}$$

$$\tilde{C} = \begin{bmatrix} \frac{1}{105}(195c_0 + 25c_1 + 9c_2) & \frac{1}{35}(25c_0 + 5c_1 - 8c_2) & \frac{1}{21}(9c_0 - 8c_1 + 3c_2) \\ \frac{1}{105}(25c_0 + 5c_1 - 8c_2) & \frac{1}{35}(5c_0 + 15c_1 + 11c_2) & \frac{1}{21}(-8c_0 + 11c_1 - 5c_2) \\ \frac{1}{105}(9c_0 - 8c_1 + 3c_2) & \frac{1}{35}(-8c_0 + 11c_1 - 5c_2) & \frac{1}{21}(3c_0 - 5c_1 + 15c_2) \end{bmatrix}$$

4 Analysis of the numerical implementation

In this section, by the help of the operational matrices of integration and product of the ALPs as well as the collocation method, we convert (1.1) with the condition (1.2) into a system of algebraic equations. To do this, we approximate $D_t^\alpha y(t)$ as follows:

$$D_t^\alpha y(t) = C^T \Phi(t). \tag{4.1}$$

By implementation of operator I_t^α (4.1), we conclude that

$$y(t) = C^T P^{(\alpha)} \Phi(t) + y_0(t), \tag{4.2}$$

where $y_0(t) = \sum_{k=0}^{n-1} y_0^{(k)} \frac{t^k}{k!}$. Also, by approximating $y_0(t)$, we get

$$y_0(t) = X^T \Phi(t). \tag{4.3}$$

In the order hand, we can obtain the following equation

$$\begin{aligned} &y(t) = C^T P^{(\alpha)}\Phi(t) + X^T\Phi(t) = (C^T P^{(\alpha)} + X^T)\Phi(t), \\ &y^2(t) = (C^T P^{(\alpha)} + X^T)\tilde{C}_1\Phi(t), \\ &\cdots, \\ &y^m(t) = (C^T P^{(\alpha)} + X^T)\tilde{C}_{m-1}\Phi(t). \end{aligned} \tag{4.4}$$

where $\tilde{C}_1$ is the operational matrix of product of the $(n+1)\times 1$ vector $C^T P^{(\alpha)} + X^T$. Clearly, the order of $\tilde{C}_1$ is $(n+1)\times(n+1)$.

Now, we need to approximate the integral parts of (1.1) in the matrix form. So, by using (3.13) and (4.4), we can approximate the following integral part as

$$\int_0^t (t-s)^{-\beta} y^m(s)ds \simeq (C^T P^{(\alpha)} + X^T)\tilde{C}_{m-1} Q \int_0^t \frac{X_s}{(t-s)^\beta} ds. \tag{4.5}$$

Clearly, we have

$$\int_0^t \frac{s^r}{(t-s)^\beta} ds = \frac{\Gamma(1-\beta)\Gamma(r+1)}{\Gamma(r-\beta+2)} t^{(r-\beta+1)}, \qquad r = 0, 1, \cdots \tag{4.6}$$

therefore, by using (4.6) and (4.5), we obtain

$$\begin{aligned} \int_0^t \frac{X_s}{(t-s)^\beta} ds &= \Big[\int_0^t \frac{1}{(t-s)^\beta} ds, \int_0^t \frac{s}{(t-s)^\beta} ds, \cdots, \int_0^t \frac{s^r}{(t-s)^\alpha} ds, \cdots\Big]^T \\ &= \Big[\frac{\Gamma(1-\beta)\Gamma(1)}{\Gamma(-\beta+2)} t^{(-\beta+1)}, \frac{\Gamma(1-\beta)\Gamma(2)}{\Gamma(-\beta+3)} t^{(-\beta+2)}, \cdots\Big]^T \\ &= T_1 \Pi_1, \end{aligned} \tag{4.7}$$

where Π_1 is a vector as

$$\Pi_1 = [t^{-\beta+1}, t^{-\beta+2}, \cdots, t^{r-\beta+1}, \cdots], \quad r = 0, 1, \cdots \tag{4.8}$$

Also T_1 is an infinite diagonal matrix with elements $\frac{\Gamma(1-\beta)\Gamma(r+1)}{\Gamma(r-\beta+2)}$, $r=0,1,\cdots$.
By approximating $t^{r-\beta+1}, r=0,1,\cdots$ in terms of ALPs, we have

$$t^{r-\beta+1}=\sum_{l=1}^{\infty}c_{r,l}\Phi_l(t)=\partial_r Q X_t, \quad \partial_r=[c_{r,0},c_{r,1},\cdots], \quad r=0,1,\cdots \tag{4.9}$$

Also, by using (4.9), we obtain

$$\Pi_1=[\partial_0 QX_t,\partial_1 QX_t,\cdots,\partial_r QX_t,\cdots]=\Upsilon_1 QX_t, \quad \Upsilon=[\partial_0,\partial_1,\cdots,\partial_r,\cdots]^T, \quad r=0,1,\cdots \tag{4.10}$$

Substituting (4.10) into (4.8), we have

$$\int_0^t \frac{X_s}{(t-s)^\beta}ds=T_1\Upsilon_1 QX_t=T_1\Upsilon_1\Phi(t). \tag{4.11}$$

So, by using (4.11), we rewrite (4.5) as the following form

$$\int_0^t (t-s)^{-\beta}y^m(s)ds \simeq (C^TP^{(\alpha)}+X^T)\tilde{C}_{m-1}QT_1\Upsilon_1\Phi(t). \tag{4.12}$$

Similarly, we can compute the second integral in the side of (1.1) as follows:

$$\int_0^t s^\alpha(t-s)^{-\beta}y(s)ds \simeq (C^TP^{(\alpha)}+X^T)QT_2\Upsilon_2\Phi(t). \tag{4.13}$$

Finally, substituting equations (4.1), (4.12) and (4.13) into (1.1), the main problem convert into a system of algebraic equations as follows:

$$C^T\Phi(t)=g(t)+(C^TP^{(\alpha)}+X^T)[p(t)+\lambda_1\tilde{C}_{m-1}QT_1\Upsilon_1+\lambda_2QT_2\Upsilon_2]\Phi(t). \tag{4.14}$$

For finding unknown vector C, we collocate (4.14) in $n+1$ nodal points t_1 named the Guass- Chelyshkov points [3]

$$G_n=\{t_l|P_{n+1,0}(t_l)=0, \quad l=0,1,\cdots,n\}. \tag{4.15}$$

Now, by replacing the nodes t_l instead of t in (4.14), we can obtain vector C by solving the following system

$$C^T\Phi(t_l)=g(t_l)+(C^TP^{(\alpha)}+X^T)[p(t_l)+\lambda_1\tilde{C}_{m-1}QT_1\Upsilon_1+\lambda_2QT_2\Upsilon_2]\Phi(t_l). \tag{4.16}$$

for $l=0,1,\cdots,n$. Finally, the approximate solution of problem (1.1),(1.2) can be obtained as follows:

$$y_n(t)=C^TP^{(\alpha)}\Phi(t)+y_0(t)=(C^TP^{(\alpha)}+X^T)\Phi(t). \tag{4.17}$$

If we expand $g(t)$ in terms of ALPs as the follows:

$$g(t)\approx G^T\Phi(t),$$

we get the following equation

$$C^T\Phi(t)=G^T\Phi(t)+(C^TP^{(\alpha)}+X^T)[p(t)+\lambda_1\tilde{C}_{m-1}QT_1\Upsilon_1+\lambda_2QT_2\Upsilon_2]\Phi(t) \tag{4.18}$$

or

$$C^T=G^T+(C^TP^{(\alpha)}+X^T)[p(t)+\lambda_1\tilde{C}_{m-1}QT_1\Upsilon_1+\lambda_2QT_2\Upsilon_2]. \tag{4.19}$$

Obviously, we can solve the above system without the need for the collocation method and obtain the unknown vector C. So, we give the approximate solution of problem (1.1),(1.2) as follows:

$$y_n(t)=C^TP^{(\alpha)}\Phi(t)+y_0(t)=(C^TP^{(\alpha)}+X^T)\Phi(t). \tag{4.20}$$

5 Convergence analysis of the presented method

In this section, we consider convergence analysis of the proposed method in the previous section for solving the following problem:

$$ {}_0^C D_t^\delta y(t) = g(t) + p(t)y(t) + \lambda_1 \int_0^t (t-s)^{-\beta} y^m(s)ds + \lambda_2 \int_0^t s^\alpha (t-s)^{-\beta} y(s)ds, \tag{5.1} $$

$$ y^{(i)} = y_0^{(i)}, \quad i = 0, 1, \cdots, [\alpha] - 1 $$

Here, we use the following norm for any arbitrary function $f \in C[0,1]$

$$ \|f\|_\infty = \max_{0 \le t \le 1} |f(t)|. \tag{5.2} $$

Suppose that $y_n(t) = C^T \Phi(t)$ is the approximate solution of (5.1) in terms of ALPs.

Theorem 5.1. (Convergence) Suppose $y_n(t)$ and $y(t) \in L^2[0,1]$ are the approximate solution obtained by the proposed method in Section 4 and the exact solution of (5.1), respectively. Also, let $y(t), p(t)$ and $g(t)$ be continuous function. If

$$ \|p\|_\infty \frac{1}{\Gamma(\delta+1)} + |\lambda_1| \frac{\Gamma(1-\beta)\Gamma(1)}{\Gamma(-\beta+\delta+1)} + |\lambda_2| B(1-\beta, 1+\alpha) \frac{\Gamma(2-\beta+\alpha)}{\Gamma(2-\beta+\alpha+\delta)} < 1, \tag{5.3} $$

we have

$$ \|y - y_n\|_\infty = \max_{0 \le t \le 1} |y(t) - y_n(t)| \to 0, $$

as $n \to \infty$.

Proof . By applying (2.6) on (5.1), we get the following equation

$$ \begin{aligned} y(t) = &\sum_{k=0}^{n-1} \frac{t^k}{k!} y^{(k)}(0^+) + I^\delta g(t) + I^\delta p(t)y(t) \\ &+ \lambda_1 I^\delta \int_0^t (t-s)^{-\beta} y^m(s)ds + \lambda_2 I^\delta \int_0^t s^\alpha (t-s)^{-\beta} y(s)ds. \end{aligned} \tag{5.4} $$

By substituting $y_n(t) = C^T \Phi(t)$ instead of $y(t)$ in (5.4), we have

$$ \begin{aligned} y_n(t) = &\sum_{k=0}^{n-1} \frac{t^k}{k!} y^{(k)}(0^+) + I^\delta g(t) + I^\delta p(t)y_n(t) \\ &+ \lambda_1 I^\delta \int_0^t (t-s)^{-\beta} y_n^m(s)ds + \lambda_2 I^\delta \int_0^t s^\alpha (t-s)^{-\beta} y_n(s)ds. \end{aligned} \tag{5.5} $$

By subtracting (5.5) from (5.4), we have:

$$ \begin{aligned} y(t) - y_n(t) = &I^\delta p(t)\big(y(t) - y_n(t)\big) + \lambda_1 I^\delta \int_0^t (t-s)^{-\beta} \big(y^m(s) - y_n^m(s)\big)ds \\ &+ \lambda_2 I^\delta \int_0^t s^\alpha (t-s)^{-\beta} \big(y(t) - y_n(t))\big)ds. \end{aligned} \tag{5.6} $$

Therefore, we can write:

$$\begin{aligned}
|y(t)-y_n(t)| &\leq |I^\delta p(t)\big(y(t)-y_n(t)\big)| + |\lambda_1|\Big|I^\delta \int_0^t (t-s)^{-\beta}\big(y^m(s)-y_n^m(s)\big)ds\Big| \\
&\quad + |\lambda_2|\Big|I^\delta \int_0^t s^\alpha (t-s)^{-\beta}\big(y(t)-y_n(t)\big)ds\Big| \\
&\leq I^\delta |p(t)|\big|\big(y(t)-y_n(t)\big)\big| + |\lambda_1| I^\delta \int_0^t (t-s)^{-\beta}|\big(y^m(s)-y_n^m(s)\big)|ds \\
&\quad + |\lambda_2| I^\delta \int_0^t s^\alpha (t-s)^{-\beta}\big|\big(y(t)-y_n(t)\big)\big|ds \\
&\leq I^\delta |p(t)||y-y_n| + |\lambda_1| I^\delta \int_0^t (t-s)^{-\beta}|y-y_n|ds \\
&\quad + |\lambda_2| I^\delta \int_0^t s^\alpha (t-s)^{-\beta}|y-y_n|ds \\
&\leq \|p(t)\|_\infty \|y-y_n\|_\infty I^\delta 1 + |\lambda_1| I^\delta \int_0^t (t-s)^{-\beta}\|y-y_n\|_\infty ds \\
&\quad + |\lambda_2| I^\delta \int_0^t s^\alpha (t-s)^{-\beta}\|y-y_n\|_\infty ds. \qquad (5.7)
\end{aligned}$$

Hence

$$\begin{aligned}
|y(t)-y_n(t)| &\leq \|p(t)\|_\infty \|y-y_n\|_\infty I^\delta 1 + |\lambda_1| I^\delta \int_0^t (t-s)^{-\beta}\|y-y_n\|_\infty ds + |\lambda_2| I^\delta \int_0^t s^\alpha (t-s)^{-\beta}\|y-y_n\|_\infty ds \\
&\leq \|y-y_n\|_\infty \left(\|p(t)\|_\infty I^\delta 1 + |\lambda_1| I^\delta \int_0^t (t-s)^{-\beta} ds + |\lambda_2| I^\delta \int_0^t s^\alpha (t-s)^{-\beta} ds\right) \\
&= \|y-y_n\|_\infty \left(\|p(t)\|_\infty I^\delta 1 + |\lambda_1| \frac{\Gamma(1-\beta)\Gamma(1)}{\Gamma(-\beta+2)} I^\delta t^{-\beta+1} + |\lambda_2| B(1-\beta, 1+\alpha) I^\delta t^{1-\beta+\alpha}\right) \\
&\leq \|y-y_n\|_\infty \left(\|p(t)\|_\infty \frac{1}{\Gamma(\delta+1)} + |\lambda_1| \frac{\Gamma(1-\beta)\Gamma(1)}{\Gamma(-\beta+\alpha+\delta+1)} t^{-\beta+1+\delta} + |\lambda_2| B(1-\beta, 1+\alpha) \frac{\Gamma(2-\beta+\alpha)}{\Gamma(2-\beta+\alpha+\delta)} t^{1-\beta+\alpha+\delta}\right) \\
&\leq \|y-y_n\|_\infty \left(\|p(t)\|_\infty \frac{1}{\Gamma(\delta+1)} + |\lambda_1| \frac{\Gamma(1-\beta)\Gamma(1)}{\Gamma(-\beta+\alpha+\delta+1)} + |\lambda_2| B(1-\beta, 1+\alpha) \frac{\Gamma(2-\beta+\alpha)}{\Gamma(2-\beta+\alpha+\delta)}\right) \qquad (5.8)
\end{aligned}$$

So, we have the following inequality

$$\|y-y_n\|_\infty \leq \|y-y_n\|_\infty \Big(\|p\|_\infty \tfrac{1}{\Gamma(\delta+1)} + |\lambda_1| \tfrac{\Gamma(1-\beta)\Gamma(1)}{\Gamma(-\beta+\delta+1)} + |\lambda_2| B(1-\beta, 1+\alpha) \tfrac{\Gamma(2-\beta+\alpha)}{\Gamma(2-\beta+\alpha+\delta)}\Big), \qquad (5.9)$$

and hence

$$\|y-y_n\|_\infty \Big(\|p\|_\infty \tfrac{1}{\Gamma(\delta+1)} + |\lambda_1| \tfrac{\Gamma(1-\beta)\Gamma(1)}{\Gamma(-\beta+\delta+1)} + |\lambda_2| B(1-\beta, 1+\alpha) \tfrac{\Gamma(2-\beta+\alpha)}{\Gamma(2-\beta+\alpha+\delta)}\Big) \leq 0. \qquad (5.10)$$

By considering (5.2), we conclude that

$$\|y-y_n\|_\infty = \max_{0\leq t\leq 1} |y(t)-y_n(t)| \to 0, \qquad (5.11)$$

as $n \to \infty$. □

6 Illustrative examples

In this section, some numerical examples are presented to illustrate the accuracy and the applicability of the proposed method. Also, in this section, we consider the following errors:

$$\begin{aligned}
\|y-y_n\|_\infty &= \max_{0\leq t\leq 1} |y(t)-y_n(t)|, \\
e_n &= \max_{0\leq i\leq 1} |y(t_i)-y_n(t_i)|, \\
\zeta_n &= \log_2\left(\frac{e_n}{e_{2n}}\right), \qquad (6.1)
\end{aligned}$$

where $t_i = ih$, y and y_n are the exact solution and the approximate solution, respectively. ζ_n is an estimate of the convergence order [21].

Example 6.1. [21] As the first example, we consider a nonlinear fractional order integro-differential equation with weakly singular kernel as follows:

$$ {}_0^C D_t^{\frac{2}{3}} y(t) = g(t) + p(t)y(t) + \lambda_1 \int_0^t (t-s)^{-\frac{1}{2}} y^2(s)ds, \quad 0 \le t \le 1, $$

where

$$ p(t) = t, \quad g(t) = \frac{3\Gamma(\frac{1}{2})}{4\Gamma(\frac{11}{6})} t^{\frac{5}{6}} - t^{\frac{5}{2}} - \frac{32}{35} t^{\frac{7}{2}}, $$

with initial condition $y(0) = 0$ and the exact solution $y(t) = t^{\frac{3}{2}}$.

First, we solved this example by the proposed method for different values of n.
The operational matrices of fractional integration and product are obtained for $n = 2$ in the following forms:

$$ P^{(\alpha)} = P^{(\frac{2}{3})} = \begin{bmatrix} 0.186597 & 0.516378 & 0.384617 \\ -0.0415082 & 0.343871 & 0.64928 \\ 0.00571214 & -0.0377001 & 0.439835 \end{bmatrix}, $$

$$ \tilde{C}_{m-1} = \tilde{C}_1 = \begin{bmatrix} \frac{1}{105}(195c_0 + 25c_1 + 9c_2) & \frac{1}{35}(25c_0 + 5c_1 - 8c_2) & \frac{1}{21}(9c_0 - 8c_1 + 3c_2) \\ \frac{1}{105}(25c_0 + 5c_1 - 8c_2) & \frac{1}{35}(5c_0 + 15c_1 + 11c_2) & \frac{1}{21}(-8c_0 + 11c_1 - 5c_2) \\ \frac{1}{105}(9c_0 - 8c_1 + 3c_2) & \frac{1}{35}(-8c_0 + 11c_1 - 5c_2) & \frac{1}{21}(3c_0 - 5c_1 + 15c_2) \end{bmatrix} $$

and

$$ c_0 = 0.0321013, \quad c_1 = 0.442571, \quad c_2 = 1.88036 $$

Therefore, the approximate solution of this example in terms of ALPs is as follows:

$$ y_2(t) = -0.00491842 + 0.411169t + 0.620983t^2. $$

Figure 1, compares the exact solution with the approximate solution obtained by proposed method for $n = 10$. We display the absolute error for $n = 10$ in Figure 2 over the interval $[0, 1]$.

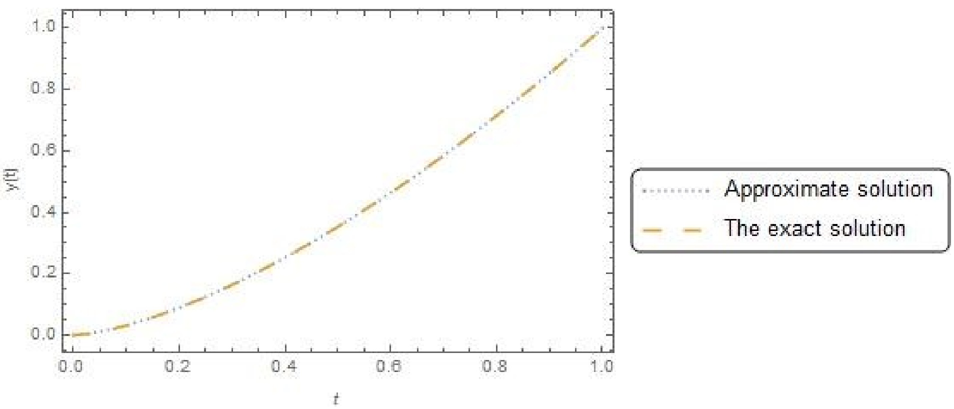

Figure 1: Comparison of the approximate and exact solutions with $n = 10$ and $m = 2$ for Example 6.1

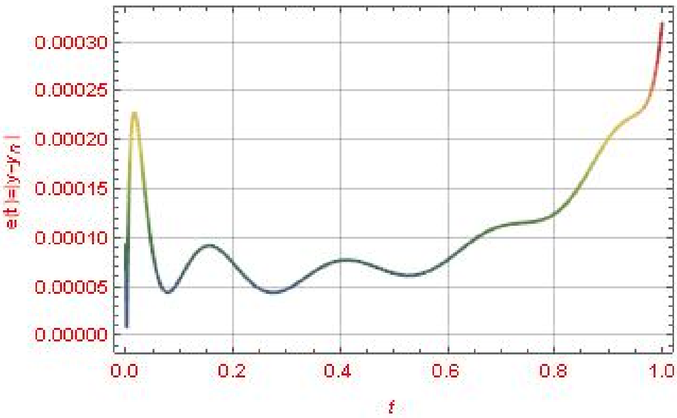

Figure 2: Plot of the absolute error with $n = 10$ and $m = 2$ for Example 6.1

In Table 1, the obtained results are compared with the method of [21] (MHFM) for different values of n at some selected grid points.

Table 1: Comparison of the absolute errors between the proposed method and the method of [21] for different values of n and $m = 2$ for Example 6.1.

The method of [21]					Proposed method	
$n = 64$	$n = 32$	$n = 16$	$n = 8$	t	$n = 8$	$n = 16$
0.00000	0.00000	0.00000	0.00000	0.000	0.00000	0.00000
$1.70440E-5$	$5.782571E-5$	$1.54710E-4$	$1.59756E-3$	0.125	$7.79604E-5$	$1.18748E-5$
$1.49408E-5$	$5.27913E-5$	$1.84256E-4$	$5.69812E-4$	0.250	$1.67278E-4$	$1.15360E-5$
$1.50946E-5$	$5.38168E-5$	$1.93902E-4$	$6.53198E-4$	0.375	$8.89137E-5$	$1.47671E-5$
$1.69314E-5$	$6.05455E-5$	$2.20346E-4$	$8.56779E-4$	0.500	$1.72782E-4$	$1.61787E-5$
$2.08999E-5$	$7.47583E-5$	$2.72394E-4$	$8.88166E-4$	0.625	$1.88116E-4$	$1.74240E-5$
$2.83408E-5$	$1.01259E-5$	$3.67651E-4$	$1.42594E-3$	0.750	$2.09224E-4$	$2.54256E-5$
$4.22319E-5$	$1.50631E-5$	$5.43833E-4$	$1.81991E-3$	0.875	$3.71892E-4$	$3.89983E-5$
$6.92324E-5$	$2.46509E-5$	$8.84780E-4$	$3.30803E-3$	1.000	$6.45901E-4$	$6.83567E-5$

Table 2: Comparison of e_n between the proposed method and the method of [21] for different values of n and $m = 2$ for Example 6.1.

n	e_n(The method of [21])	e_n(Proposed method)	Computing time (Proposed method)
2	$7.093035E-2$	$2.72338E-2$	0.145
4	$1.39991E-2$	$5.17108E-3$	0.158
8	$3.30803E-3$	$6.45901E-4$	0.254
10	$4.21471E-3$	$3.18577E-4$	0.314
12	$7.47821E-4$	$1.76609E-4$	0.348
16	$8.84780E-4$	$6.84373E-5$	0.478

Example 6.2. [31] For the second example, consider the linear fractional order integro-differential equation with weakly singular kernel as follows

$$ {}_0^C D_t^{\frac{1}{3}} y(t) = g(t) + p(t)y(t) + \lambda_1 \int_0^t (t-s)^{-\frac{1}{2}} y(s)ds, \quad 0 \le t \le 1, $$

$$ p(t) = \frac{-32}{35} t^{\frac{1}{2}}, \quad \lambda_1 = 1, \lambda_2 = 0, \quad g(t) = \frac{6}{\Gamma(\frac{11}{3})} t^{\frac{8}{3}} + \Big(\frac{32}{35} - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{17}{6})} \Big) t^{\frac{11}{6}} + \Gamma(\frac{7}{3})t, $$

with initial condition $y(0) = 0$ and the exact solution $y(t) = t^3 + t^{\frac{4}{3}}$.

Table 3: Comparison of numerical results for different values of n for Example 6.2.

t	y_8	y_{10}	y_{12}	y_{16}	Exact solution
0	-0.000903	-0.000524	-0.000333	-0.000161	0
0.2	0.125009	0.124972	0.124944	0.124965	0.124961
0.4	0.358674	0.358741	0.358713	0.358721	0.358723
0.6	0.722083	0.722053	0.722058	0.722057	0.72206
0.8	1.25466	1.25464	1.225466	1.25465	1.25465
1	2.00012	2.00006	2.00003	2.00001	2

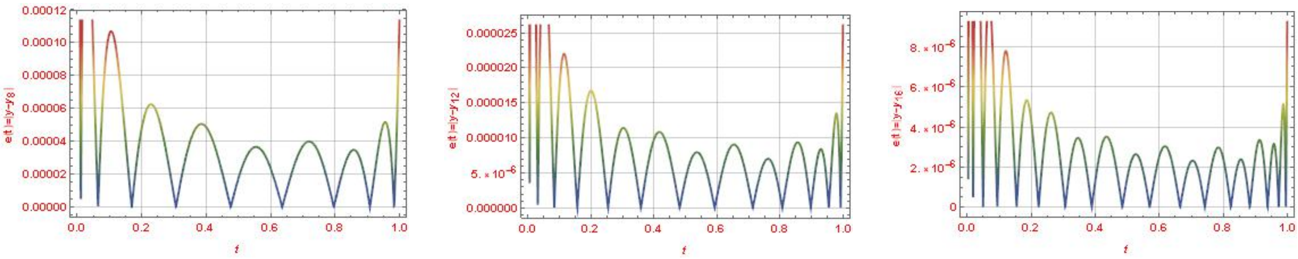

Figure 3: Plots of the absolute errors for $n = 8, 12, 16$, $\lambda_1 = 1$ and $m = 1$ for Example 6.2.

Table 4: Comparison of absolute errors between the proposed method and the method of [21] for different values of n and $m = 1$ for Example 6.2.

The method of [21]					Proposed method	
$n = 64$	$n = 32$	$n = 16$	$n = 8$	t	$n = 8$	$n = 16$
0.00000	0.00000	0.00000	0.00000	0.000	0.00000	0.00000
$3.39892E-6$	$1.25593E-5$	$6.81733E-5$	$9.44601E-5$	0.125	$8.99379E-5$	$7.54902E-6$
$2.43485E-6$	$1.23046E-5$	$6.21373E-5$	$3.14107E-4$	0.250	$5.58767E-5$	$4.25036E-6$
$2.57556E-6$	$1.27213E-5$	$6.11849E-5$	$1.40102E-4$	0.375	$4.96901E-5$	$1.63611E-6$
$2.76220E-6$	$1.34327E-5$	$6.23334E-5$	$2.53578E-4$	0.500	$1.73906E-5$	$1.49249E-6$
$2.97850E-6$	$1.43297E-5$	$6.47637E-5$	$2.32363E-4$	0.625	$7.78011E-6$	$2.93472E-6$
$3.21794E-6$	$1.53667E-5$	$6.81214E-5$	$2.38092E-4$	0.750	$3.26190E-5$	$8.29279E-7$
$3.47739E-6$	$1.65207E-5$	$7.22219E-5$	$3.15569E-4$	0.875	$3.07487E-5$	$1.04465E-6$
$3.75521E-6$	$1.77792E-5$	$7.69578E-5$	$2.43856E-4$	1.000	$1.17337E-4$	$1.14567E-5$

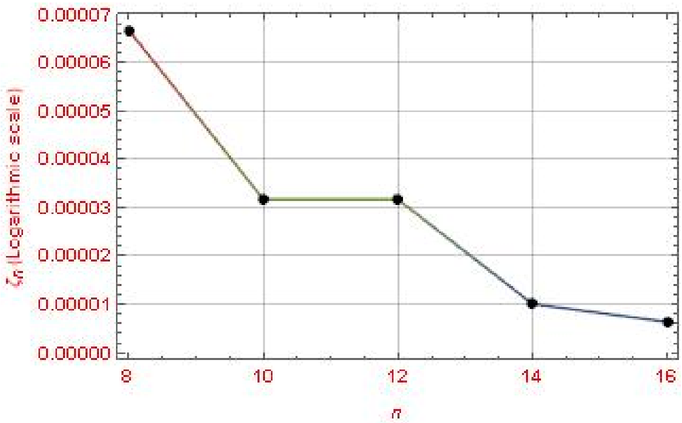

Figure 4: ζ_n on Logarithmic scale, $\lambda_1 = 1$ and $m = 1$ for Example 6.2

Table 5: Comparison of e_n between the proposed method and the method of [21] for different values of n and $m = 2$ for Example 6.2.

n	e_n(The method of [21])	e_n(Proposed method)	Computing time (Proposed method)
2	$2.35051E-2$	$1.90023E-2$	0.296
4	$1.14967E-3$	$1.21439E-3$	0.301
8	$3.15569E-4$	$1.17335E-4$	0.327
10	$6.15412E-4$	$5.61497E-5$	0.378
12	$7.57810E-5$	$3.05871E-5$	0.402
16	$8.90424E-5$	$1.14567E-5$	0.428

Example 6.3. As third example which was proposed in [21], consider the nonlinear fractional order integro-differential equation with weakly singular kernel of the following form

$$ {}_0^C D_t^{\alpha} y(t) = g(t) + \lambda_1 \int_0^t (t-s)^{-\frac{1}{2}} y^2(s) ds, \quad 0 \leq t \leq 1, $$

where

$$ g(t) = 3t^2 - \frac{\sqrt{\pi}\Gamma(7)}{\Gamma(\frac{15}{2})} t^{\frac{13}{2}}, \quad \lambda_1 = 1, \quad \lambda_2 = 0, $$

with initial condition $y(0) = 0$ and the exact solution for $\alpha = 1$ is $y(t) = t^3$. The absolute error for $n = 6$ is shown in Figure 5. Obviously, extremely high accuracy can be seen.

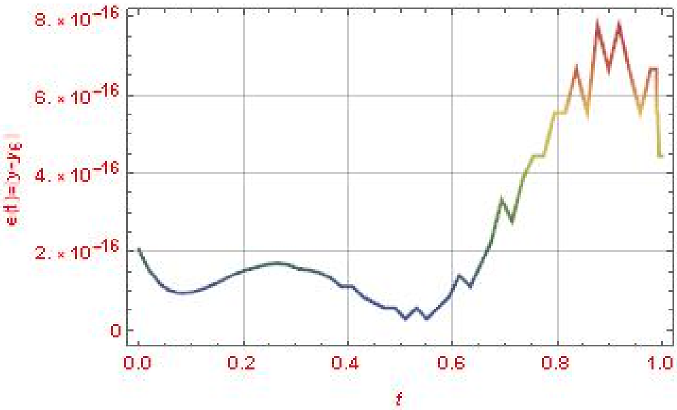

Figure 5: Plot of the absolute error for $n = 6$, with $\alpha = 1, \lambda_1 = 1$ and $m = 2$ for Example 6.3

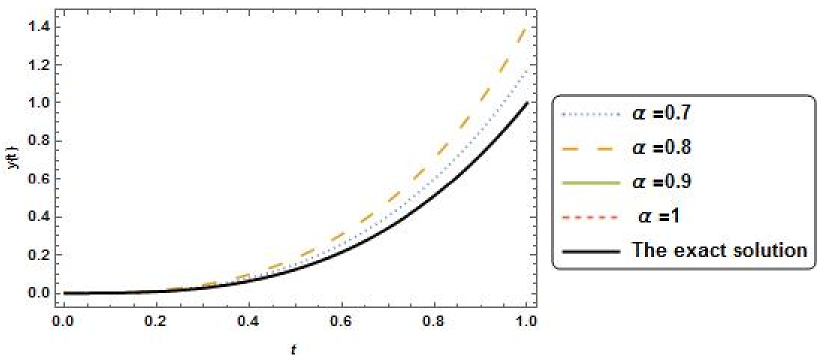

Figure 6: Plot of the approximate solution $y_6(t)$ with $\alpha = 0.7, 0.8, 0.9, 1, \lambda_1 = 1$ and $m = 2$ for Example 6.3

Table 6: Comparison of e_n between the proposed method and the method of [21] for different values of $n, \alpha = 1$ and $m = 2$ for Example 6.3.

n	e_n(The method of [21])	e_n(Proposed method)	Computing time (Proposed method)
2	$3.52633E-2$	$5.41208E-2$	0.178
4	$4.81704E-3$	$1.07462E-4$	0.245
5	$7.10457E-3$	$8.08607E-6$	0.312
6	$3.12470E-4$	$6.66134E-16$	0.345
8	$3.15596E-4$	$3.24185E-16$	0.379

Table 7: Comparison of absolute errors between the proposed method and the method of [21] for different values of $n, \alpha = 1$ and $m = 2$ for Example 6.3.

	The proposed method	The method of [21]	The proposed method	The method of [21]	The proposed method
t	n=6	n=8	n=8	n=16	n=16
0.000	0.00000	0.00000	0.00000	0.00000	0.00000
0.125	$1.06252E-16$	$3.14015E-7$	$1.28283E-15$	$4.75615E-9$	$3.75003E-11$
0.250	$1.66533E-16$	$8.61334E-7$	$6.97359E-16$	$1.46318E-7$	$3.09495E-11$
0.375	$1.24900E-16$	$3.91922E-6$	$9.78384E-16$	$7.47778E-7$	$2.40549E-11$
0.500	$5.55112E-17$	$2.65040E-5$	$8.60423E-16$	$2.25373E-6$	$3.83279E-11$
0.625	$1.38778E-16$	$5.09917E-5$	0.00000	$5.19464E-6$	$3.59824E-11$
0.750	$4.44089E-16$	$1.34129E-4$	$2.72007E-15$	$1.01464E-5$	$1.68068E-11$
0.875	$7.77156E-16$	$2.09299E-4$	$1.11022E-16$	$1.76618E-5$	$2.11803E-10$
1.000	$4.44089E-16$	$3.89313E-4$	$3.24185E-14$	$2.81379E-5$	$4.94162E-10$

Figure 7: ζ_n on Logarithmic scale with $\alpha = 1, \lambda_1 = 1$ and $m = 2$ for Example 6.3

7 Conclusion and future work

In this articles a spectral method based on the operational matrices of integration and product of ALPs for solving a class of nonlinear fractional integro- differential equation with weakly singular kernel is presented. Also, error analysis is investigated and accuracy is shown in 3 numerical experiments. The convergence of the proposed method is confirmed by the results presented in Tables 1–7 and Figures 1–7. It can be concluded that the method proposed in this paper is a suitable method to solve such problems.

For the next work, the proposed method can be used to solve for nonlinear fractional integro-differential equation with weakly singular kernel in two-dimensional.

References

[1] D.D. Bainov, M.B. Dimitrova and A.B. Dishliev, *Oscillation of the bounded solutions of impulsive differential-difference equations of second order*, Appl. Math. Comput. **114** (2000), no. 1, 61–68.

[2] Y. Chen, M. Yi and C. Yu, *Error analysis for numerical solution of fractional differential equation by Haar wavelets method*, J. Comput. Sci. **3** (2012), no. 5, 367–73.

[3] V.S. Chelyshkov, *Alternative orthogonal polynomials and quadratures*, Electron. Trans. Numer. Anal. **25** (2006), no. 7, 17–26.

[4] V. Daftardar-Gejji and S. Bhalekar, *Solving multi-term linear and non-linear diffusion–wave equations of fractional order by Adomian decomposition method*, Appl. Math. Comput. 202(1) (2008) 113–20.

[5] L.M. Delves and J. Mohamed, *Computational methods for integral equations*, CUP Archive, 1988.

[6] K. Diethelm and N.J. Ford, *Analysis of fractional differential equations*, J. Math. Anal. Appl. **265** (2002), no. 2, 229–48.

[7] M. Ebadi and E. Hashemizadeh, *A new approach based on the Zernike radial polynomials for numerical solution of the fractional diffusion-wave and fractional Klein–Gordon equations*, Physica Scripta **93** (2018), no. 12, 125–202.

[8] E. Hashemizadeh and B. Basirat, *An efficient computational method for the system of linear Volterra integral equations by means of hybrid functions*, Math. Sci. **4** (2011), 355–368.

[9] E. Hashemizadeh and M. Rostami, *Numerical solution of Hammerstein integral equations of mixed type using the Sinc-collocation method*, Appl. Math. Comput. **279** (2015), 31–39.

[10] SH. Guodong, Y. Gong and Y. Mingxu, *Alternative legendre polynomials method for nonlinear fractional integro-differential equations with weakly singular kernel*, J. Math. **2021** (2021).

[11] H. Jafari, H. Tajadodi and DA. Baleanu, *Numerical approach for fractional order Riccati differential equation using B-spline operational matrix*, Fract. Calc. Appl. Anal. **18** (2015), no. 2, 387–99.

[12] MK. Kadalbajoo and KK. Sharma, *Numerical analysis of singularly perturbed delay differential equations with layer behavior*, Appl. Math. Comput. **157** (2004), no. 1, 11–28.

[13] P. Kythe and P. Puri, *Computational methods for linear integral equations*, Springer Science & Business Media, 2011.

[14] X. Ma and C. Huang, *Numerical solution of fractional integro-differential equations by a hybrid collocation method*, Appl. Math. Comput. **219** (2013), no. 12, 6750–60.

[15] K. Maleknejad and E. Hashemizadeh, *A numerical approach for Hammerstein integral equations of mixed type using operational matrices of hybrid functions*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **73** (2011), 95–104.

[16] H.R. Marzban and M. Razzaghi, *Solution of multi-delay systems using hybrid of block-pulse functions and Taylor series*, J. Sound Vib. **292** (2006), no. 3-5, 954–63.

[17] Z. Meng, M. Yi, J. Huang and L. Song, *Numerical solutions of nonlinear fractional differential equations by alternative Legendre polynomials*, Appl. Math. Comput. **336** (2018), 454–64.

[18] SM. Momani, *Local and global existence theorems on fractional integro-differential equations*, J. Fract. Calc. **18** (2000), 81–6.

[19] S. Momani and A. Jameel, *Al-Azawi SO. local and global uniqueness theorems on fractional integro-differential equations via Bihari's and Gronwall's inequalities*, Soochow J. Math. **33** (2007), no. 4, 619.

[20] S. Nemati, S. Sedaghat and I. Mohammadi, *A fast numerical algorithm based on the second kind Chebyshev polynomials for fractional integro-differential equations with weakly singular kernels*, J. Comput. Appl. Math. **15** (2016), no. 308, 231–42.

[21] S. Nemati and P.M. Lima, *Numerical solution of nonlinear fractional integro-differential equations with weakly singular kernels via a modification of hat functions*, Appl. Math. Comput. **327** (2018), 79–92.

[22] S. Noeiaghdam, A. Dreglea, H. Isik and M.A. Suleman, *A Comparative Study between Discrete Stochastic Arithmetic and Floating-Point Arithmetic to Validate the Results of Fractional Order Model of Malaria Infection*, Math. **9** (2021), no. 12, 1435.

[23] S. Noeiaghdam, S. Micula and J.J. Nieto, *A Novel Technique to Control the Accuracy of a Nonlinear Fractional Order Model of COVID-19: Application of the CESTAC Method and the CADNA Library*, Math. **9** (2021), no. 12, 1321.

[24] I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier. 1998.

[25] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering. 1998.

[26] A. Saadatmandi, *A new operational matrix for solving fractional-order differential equations*, Comput. Math. Appl. **59** (2010), 1326-–1336.

[27] B.Q. Tang and X.F. Li, *Solution of a class of Volterra integral equations with singular and weakly singular kernels*, Appl. Math. Comput. **199** (2008), no. 2, 406–13.

[28] S.K. Vanani and F. Soleymani, *Tau approximate solution of weakly singular Volterra integral equations*, Math. Comput. Model. **57** (2013), no. 3-4, 494–502.

[29] Y. Wang and L. Zhu, *SCW method for solving the fractional integro-differential equations with a weakly singular kernel*, Appl. Math. Comput. **275** (2016), 72–80.

[30] M. Yi and J. Huang, *CAS wavelet method for solving the fractional integro-differential equation with a weakly singular kernel*, Int. J. Comput. Math. **92** (2015), no. 8, 1715–28.

[31] J. Zhao, J. Xiao and N.J. Ford, *Collocation methods for fractional integro-differential equations with weakly singular kernels*, Numer. Algorithms **65** (2014), no. 4, 723–43.