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The reverse diamond-α Minkowski inequality on time scales

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Abstract

In this paper, we give a new version of the reverse Minkowski inequality on time scales by applying diamond-α calculus for two parameters p, q and a weight function ν.

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1 Introduction

The famous Minkowski inequality states that, for $p \geq 1$, if

$$0 < \int_a^b h^p(\tau)d\tau < \infty \quad and \quad 0 < \int_a^b \phi^p(\tau)d\tau < \infty$$

then

$$\left(\int_a^b (h(\tau)+\phi(\tau))^p d\tau\right)^{\frac{1}{p}} \leq \left(\int_a^b h^p(\tau)d\tau\right)^{\frac{1}{p}} + \left(\int_a^b \phi^p(\tau)d\tau\right)^{\frac{1}{p}}.$$

B.Benaissa presented a reverse Minkowski's inequality [3, Theorem 2.1]. For any $h, \phi > 0,\ \lambda > 0$, if $p \geq 1$ and

$$0 < c < m \leq \frac{\lambda h(\tau)}{\phi(\tau)} \leq M \quad for\ all\ \tau \in [a,b],$$

then

$$\begin{aligned} \frac{M+\lambda}{\lambda(M-c)}\left(\int_a^b (\lambda h(\tau)-c\phi(\tau))^p d\tau\right)^{\frac{1}{p}} &\leq \left(\int_a^b h^p(\tau)d\tau\right)^{\frac{1}{p}} + \left(\int_a^b \phi^p(\tau)d\tau\right)^{\frac{1}{p}} \\ &\leq \frac{m+\lambda}{\lambda(m-c)}\left(\int_a^b (\lambda h(\tau)-c\phi(\tau))^p d\tau\right)^{\frac{1}{p}}. \end{aligned} \tag{1.1}$$

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When we replace λ by 1 in the inequality (1.1) we obtain [5, Theorem 2.2].
When we put $\lambda = 1$ and $c = 1$ in the inequality (1.1) we obtain [6, Ttheorem 1.5].

The aim of this paper is to present a new version of the reverse Minkowski inequality to an arbitrary time scale and to extend some continuous inequalities and their corresponding discrete analogues by introducing weight function and two positive parameters p, q.

2 Auxiliary results

We introduce the diamond-alpha dynamic derivative and diamond-alpha dynamic integration. The comprehensive development of the calculus of the diamond-alpha derivative and diamond-alpha integration is given in ([2], [4]). Let $\mathbb{T}$ be a time scale and $h(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}$, we define the diamond-alpha derivative $h^{\diamondsuit_\alpha}(t)$ by

$$h^{\diamondsuit_\alpha}(t) = \alpha h^{\Delta}(t) + (1-\alpha) h^{\nabla}(t), \quad 0 \le \alpha \le 1.$$

Thus h is diamond-alpha differentiable if and only if h is Δ and ∇ differentiable.

Definition 2.1. Let $a, b \in \mathbb{T}$, and $h : \mathbb{T} \to \mathbb{R}$. Then, the diamond-$\alpha$ integral from a to b of h is defined by

$$\int_a^b h(\tau)\diamondsuit_\alpha \tau = \alpha \int_a^b h(\tau)\Delta\tau + (1-\alpha)\int_a^b h(\tau)\nabla\tau, \quad 0 \le \alpha \le 1.$$

provided that there exist Δ and ∇ integrals of f on $\mathbb{T}$.

It is clear that the diamond-α integral of h exists when h is a continuous function. Let a, b, $c \in \mathbb{T}$, $\lambda, \beta \in \mathbb{R}$ and h, ϕ be continuous functions on $[a, b] \cap \mathbb{T} = [a, b]_{\mathbb{T}}$. Then the following properties hold:

1 $\int_a^b (\lambda h(\tau) + \beta\phi(\tau))\diamondsuit_\alpha \tau = \lambda \int_a^b h(\tau)\diamondsuit_\alpha\tau + \beta \int_a^b \phi(\tau)\diamondsuit_\alpha\tau.$

2 $\int_a^b h(\tau)\diamondsuit_\alpha\tau = -\int_b^a h(\tau)\diamondsuit_\alpha\tau, \quad \int_a^a h(\tau)\diamondsuit_\alpha\tau = 0.$

3 $\int_a^b h(\tau)\diamondsuit_\alpha\tau = \int_a^c h(\tau)\diamondsuit_\alpha\tau + \int_c^b h(\tau)\diamondsuit_\alpha\tau.$

4 If $h(\tau) \ge 0$ for all $\tau \in [a, b]_{\mathbb{T}}$, then $\int_a^b h(\tau)\diamondsuit_\alpha\tau \ge 0.$

5 If $h(\tau) \le \phi(\tau)$ for all $\tau \in [a, b]_{\mathbb{T}}$, then $\int_a^b h(\tau)\diamondsuit_\alpha\tau \le \int_a^b \phi(\tau)\diamondsuit_\alpha\tau.$

6 If $h(\tau) \ge 0$ for all $\tau \in [a, b]_{\mathbb{T}}$, then $h(\tau) = 0$ if only if $\int_a^b h(\tau)\diamondsuit_\alpha\tau = 0.$

Lemma 2.2. [1, Theorem 1.1.21]. Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and h, ϕ be two positive functions. If $\frac{1}{p} + \frac{1}{p'} = 1$ with $p > 1$, then

$$\int_a^b h(\tau)\phi(\tau)\diamondsuit_\alpha\tau \le \left(\int_a^b h^p(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}} \left(\int_a^b \phi^{p'}(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p'}}. \tag{2.1}$$

3 Preliminaries

We state the following proposition and lemma which are useful in the proof of main theorem

Proposition 3.1. [3] Let $0<\beta<m\leq M$ and $\lambda>0$. Then

$$\frac{M+\lambda}{\lambda(M-\beta)}\leq\frac{m+\lambda}{\lambda(m-\beta)}$$

Lemma 3.2. Let $0<p\leq q<\infty$ and $h,\,\nu$ are nonnegative and continuous functions on $[a,\,b]_{\mathbb{T}}$ and we suppose that $0<\int_a^b h^q(\tau)\nu(\tau)\diamondsuit_\alpha\tau<\infty$, then

$$\left(\int_a^b h^p(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}}\leq\left(\int_a^b \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{q-p}{qp}}\left(\int_a^b h^q(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}}. \tag{3.1}$$

Proof . By Hölder integral inequality (2.1) for $\frac{q}{p}\geq 1$, we have

$$\begin{aligned}\int_a^b h^p(\tau)\nu(\tau)\diamondsuit_\alpha\tau &= \int_a^b\left(\nu^{\frac{q-p}{q}}(\tau)\right)\left(h^p(\tau)\nu^{\frac{p}{q}}(\tau)\right)\diamondsuit_\alpha\tau\\ &\leq\left(\int_a^b \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{q-p}{q}}\left(\int_a^b h^q(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{p}{q}}.\end{aligned}$$

□

4 Main Results

We will assume that the functions $h,\,\phi$ *and* ν in the statements of the theorems are positive and continuous on $[a,\,b]_{\mathbb{T}}$.

4.1 Reverse Minkowski's Inequality

We suppose that

$$0<\int_a^b h^p(\tau)\nu(\tau)\diamondsuit_\alpha\tau<\infty\ and\ 0<\int_a^b \phi^q(\tau)\nu(\tau)\diamondsuit_\alpha\tau<\infty,\ for\ all\ 0<p\leq q.$$

Theorem 4.1. Let $\alpha\in[0,\,1]$, $\lambda>0$, $0<p\leq q<\infty$ and $\mathbb{T}$ be a time scale, $a,\,b\in\mathbb{T}$ with $a<b$. Let $h,\phi>0$ be continuous functions on $[a,\,b]_{\mathbb{T}}$ and $0<K_{(p,q)}=:\left(\int_a^b \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{q-p}{p}}<\infty$. If

$$0<\beta<m\leq\frac{\lambda h(\tau)}{\phi(\tau)}\leq M\quad for\ all\ \tau\in[a,b]_{\mathbb{T}}, \tag{4.1}$$

then

$$\begin{aligned}&\frac{M+\lambda}{\lambda(M-\beta)}\left(\int_a^b\{\lambda h(\tau)-\beta\phi(\tau)\}^p\,\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}}\\ &\leq\left(\int_a^b h^p(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}}+\left(K_{(p,q)}\int_a^b\phi^q(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}}\\ &\leq\frac{m+\lambda}{\lambda(m-\beta)}\left(K_{(p,q)}\int_a^b\{\lambda h(\tau)-\beta\phi(\tau)\}^q\,(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}}.\end{aligned} \tag{4.2}$$

Proof . From thehypothesis (4.1) we get

$$0 < \frac{1}{\beta} - \frac{1}{m} \le \frac{1}{\beta} - \frac{\phi(\tau)}{\lambda h(\tau)} \le \frac{1}{\beta} - \frac{1}{M},$$

as a result

$$\frac{M}{M-\beta} \le \frac{\lambda h(\tau)}{\lambda h(\tau) - \beta\phi(\tau)} \le \frac{m}{m-\beta},$$

thus

$$\frac{M}{\lambda(M-\beta)}(\lambda h(\tau) - \beta\phi(\tau)) \le h(\tau) \le \frac{m}{\lambda(m-\beta)}(\lambda h(\tau) - \beta\phi(\tau)).$$

Integrating on $[a,b]_{\mathbb{T}}$ and using the properties (4) and (5), we get

$$\begin{aligned}&\frac{M}{\lambda(M-\beta)}\left(\int_a^b \{\lambda h(\tau) - \beta\phi(\tau)\}^p \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}} \le \left(\int_a^b h^p(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}}\\ &\le \frac{m}{\lambda(m-\beta)}\left(\int_a^b \{\lambda h(\tau) - \beta\phi(\tau)\}^p \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}},\end{aligned}$$

now applying the inequality (3.1) on the last obtained inequality, we get

$$\begin{aligned}&\frac{M}{\lambda(M-\beta)}\left(\int_a^b \{\lambda h(\tau) - \beta\phi(\tau)\}^p \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}} \le \left(\int_a^b h^p(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}}\\ &\le \frac{m}{\lambda(m-\beta)}\left(\int_a^b \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{q-p}{qp}}\left(\int_a^b \{\lambda h(\tau) - \beta\phi(\tau)\}^q \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}}.\end{aligned} \tag{4.3}$$

From the hypothesis (4.1) we deduce

$$0 < m - \beta \le \frac{\lambda h(\tau) - \beta\phi(\tau)}{\phi(\tau)} \le M - \beta$$

then

$$\frac{\lambda h(\tau) - \beta\phi(\tau)}{M-\beta} \le \phi(\tau) \le \frac{\lambda h(\tau) - \beta\phi(\tau)}{m-\beta}$$

by integrating on $[a,b]_{\mathbb{T}}$, we obtain

$$\begin{aligned}&\frac{1}{M-\beta}\left(\int_a^b \{\lambda h(\tau) - \beta\phi(\tau)\}^q \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}} \le \left(\int_a^b \phi^q(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}}\\ &\le \frac{1}{m-\beta}\left(\int_a^b \{\lambda h(\tau) - \beta\phi(\tau)\}^q \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}},\end{aligned}$$

consequently

$$\begin{aligned}&\frac{1}{M-\beta}\left(\int_a^b \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{q-p}{qp}}\left(\int_a^b \{\lambda h(\tau) - \beta\phi(\tau)\}^q \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}}\\ &\le \left(\int_a^b \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{q-p}{qp}}\left(\int_a^b \phi^q(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}}\\ &\le \frac{1}{m-\beta}\left(\int_a^b \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{q-p}{qp}}\left(\int_a^b \{\lambda h(\tau) - \beta\phi(\tau)\}^q \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}},\end{aligned}$$

now applying the inequality (3.1) on the left-hand side of the last obtained inequality, we deduce that

$$\begin{aligned}
&\frac{1}{M-\beta}\left(\int_a^b \{\lambda h(\tau)-\beta\phi(\tau)\}^p \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}} \\
&\leq \left(\int_a^b \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{q-p}{qp}} \left(\int_a^b \phi^q(\tau)\nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}} \\
&\leq \frac{1}{m-\beta}\left(\int_a^b \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{q-p}{qp}} \left(\int_a^b \{\lambda h(\tau)-\beta\phi(\tau)\}^q \nu(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}},
\end{aligned} \tag{4.4}$$

By the inequalities (4.3) and (4.4), we get our desired inequality (4.2). □

Remark 4.2. If we put $\alpha = 1$ (resp $\alpha = 0$), we get the Δ-integral inequality (resp ∇-integral inequality).

By taking $q = p$, we get the following Corollary

Corollary 4.3. Let $\alpha \in [0, 1]$, $\lambda > 0$, $0 < p < \infty$ and $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$. Let $h, \phi > 0$ be continuous functions on $[a, b]_\mathbb{T}$. If

$$0 < \beta < m \leq \frac{\lambda h(s)}{\phi(s)} \leq M \quad for\ all\ s \in [a, b]_\mathbb{T},$$

then

$$\begin{aligned}
&\frac{M+\lambda}{\lambda(M-\beta)}\left(\int_a^b \{\lambda h(s)-\beta\phi(s)\}^p \nu(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}} \\
&\leq \left(\int_a^b h^p(s)\nu(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b \phi^p(s)\nu(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}} \\
&\leq \frac{m+\lambda}{\lambda(m-\beta)}\left(\int_a^b \{\lambda h(s)-\beta\phi(s)\}^p \nu(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.5}$$

By taking $\nu = 1$, then $K_{(p,q)} := (b-a)^{\frac{q-p}{p}} < \infty$. Hence we get the following Corollary

Corollary 4.4. Let $\alpha \in [0, 1]$, $\lambda > 0$, $0 < p < \infty$ and $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b < \infty$. Let $h, \phi > 0$ be continuous functions on $[a, b]_\mathbb{T}$. If

$$0 < \beta < m \leq \frac{\lambda h(s)}{\phi(s)} \leq M \quad for\ all\ s \in [a, b]_\mathbb{T},$$

then

$$\begin{aligned}
&\frac{M+\lambda}{\lambda(M-\beta)}\left(\int_a^b \{\lambda h(\tau)-\beta\phi(\tau)\}^p \diamondsuit_\alpha\tau\right)^{\frac{1}{p}} \\
&\leq \left(\int_a^b h^p(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{p}} + \left((b-a)^{\frac{q-p}{p}} \int_a^b \phi^q(\tau)\diamondsuit_\alpha\tau\right)^{\frac{1}{q}} \\
&\leq \frac{m+\lambda}{\lambda(m-\beta)}\left((b-a)^{\frac{q-p}{p}} \int_a^b \{\lambda h(\tau)-\beta\phi(\tau)\}^q \diamondsuit_\alpha\tau\right)^{\frac{1}{q}}.
\end{aligned} \tag{4.6}$$

If we put $\mathbb{T} = \mathbb{R}$ in Theorem 4.1, we get

Corollary 4.5. Let $\lambda > 0$, $0 < p \leq q < \infty$ and $-\infty \leq a < b \leq +\infty$. Let $h, \phi > 0$ be continuous functions on $[a, b]$. If

$$0 < \beta < m \leq \frac{\lambda h(s)}{\phi(s)} \leq M \quad for\ all\ s \in [a, b], \tag{4.7}$$

then

$$\begin{aligned}
&\frac{M+\lambda}{\lambda(M-\beta)}\left(\int_a^b \{\lambda h(s)-\beta\phi(s)\}^p \nu(s)ds\right)^{\frac{1}{p}} \\
&\leq \left(\int_a^b f^p(s)\nu(s)ds\right)^{\frac{1}{p}} + \left(\int_a^b \nu(s)ds\right)^{\frac{q-p}{qp}} \left(\int_a^b \phi^q(s)\nu(s)ds\right)^{\frac{1}{q}} \qquad (4.8)\\
&\leq \frac{m+\lambda}{\lambda(m-\beta)}\left(\int_a^b \nu(s)ds\right)^{\frac{q-p}{qp}} \left(\int_a^b \{\lambda h(s)-\beta\phi(s)\}^q \nu(s)ds\right)^{\frac{1}{q}}.
\end{aligned}$$

By taking $\nu(s)=1$ and $-\infty<a<b<+\infty$ in the above Corollary, we get

Corollary 4.6. Let $\lambda>0$, $0<p\leq q<\infty$ and $h,\phi>0$ be continuous functions on $[a,\,b]$. If

$$0<\beta<m\leq \frac{\lambda h(s)}{\phi(s)}\leq M \quad for\ all\ s\in[a,b],$$

then

$$\begin{aligned}
&\frac{M+\lambda}{\lambda(M-\beta)}\left(\int_a^b \{\lambda h(s)-\beta\phi(s)\}^p ds\right)^{\frac{1}{p}} \\
&\leq \left(\int_a^b h^p(s)ds\right)^{\frac{1}{p}} + \left((b-a)^{\frac{q-p}{p}} \int_a^b \phi^q(s)ds\right)^{\frac{1}{q}} \qquad (4.9)\\
&\leq \frac{m+\lambda}{\lambda(m-\beta)}\left((b-a)^{\frac{q-p}{p}} \int_a^b \{\lambda h(s)-\beta\phi(s)\}^q ds\right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 4.7. The corollary 4.6 is a direct extension of the [3, theorem 2.1] with $0<p\leq q$.

If we put $\mathbb{T}=\mathbb{Z},\lambda=1,a=0,b=n$ and $\nu\equiv 1$ in Theorem 4.1, we get

Corollary 4.8. Let $\lambda>0$, $0<p\leq q<\infty$. $\{a_i\}$, $\{b_i\}$ for $i=0,1,2,...,n,$ $n\in\mathbb{N}^\star$ be positive sequences of real numbers. If

$$0<\beta<m\leq \frac{\lambda a_i}{b_i}\leq M \quad \text{for } i=0,1,2,...,n, \qquad (4.10)$$

then

$$\begin{aligned}
&\frac{M+\lambda}{\lambda(M-\beta)}\left(\sum_{i=0}^{n-1}\{\lambda a_i-\beta b_i\}^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=0}^{n-1}\{a_i\}^p\right)^{\frac{1}{p}} + \left(n^{\frac{q-p}{p}}\sum_{i=0}^{n-1} b_i^q\right)^{\frac{1}{q}} \qquad (4.11)\\
&\leq \frac{m+\lambda}{\lambda(m-\beta)}\left(n^{\frac{q-p}{p}}\sum_{i=0}^{n-1}\{\lambda a_i-\beta b_i\}^q\right)^{\frac{1}{q}}.
\end{aligned}$$

By taking $p=q$ in the above Corollary, we get

Corollary 4.9. Let $\lambda>0$, $0<p<\infty$. $\{a_i\}$, $\{b_i\}$ for $i=0,1,2,...,n,$ $n\in\mathbb{N}^\star$ be positive sequences of real numbers. If

$$0<\beta<m\leq \frac{\lambda a_i}{b_i}\leq M \quad \text{for } i=0,1,2,...,n, \qquad (4.12)$$

then

$$\begin{aligned}
&\frac{M+\lambda}{\lambda(M-\beta)}\left(\sum_{i=0}^{n-1}\{\lambda a_i-\beta b_i\}^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=0}^{n-1}\{a_i\}^p\right)^{\frac{1}{p}} + \left(\sum_{i=0}^{n-1} b_i^p\right)^{\frac{1}{p}} \qquad (4.13)\\
&\leq \frac{m+\lambda}{\lambda(m-\beta)}\left(\sum_{i=0}^{n-1}\{\lambda a_i-\beta b_i\}^p\right)^{\frac{1}{p}}.
\end{aligned}$$

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