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Characterizing n-multipliers on Banach algebras through zero products

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Abstract

Let A be a unital Banach algebra and X be a unital A-bimodule. In this paper, among other things, we characterize n-multipliers $T : A \longrightarrow X$ by applying zero products preserving bilinear maps. We also describe n-multipliers from C^*-algebra A into X through the action on zero products.

Keywords: n-multiplier, Bilinear maps, W^*-algebra, unital A-bimodule
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1 Introduction and Preliminaries

Let A be a Banach algebra and X be an A-bimodule. A linear map $T : A \longrightarrow X$ is called *left n-multiplier* [*right n-multiplier*] if for all $a_1, a_2, ..., a_n \in A$,

$$T(a_1a_2...a_n) = T(a_1a_2...a_{n-1})a_n, \quad [T(a_1a_2...a_n) = a_1T(a_2...a_n)],$$

and T is called an *n-multiplier* if it is both left and right n-multiplier.

The concept of n-multiplier was introduced and studied by Laali and Fozouni in [15]. A 2-multiplier is called simply a multiplier. One may refer to [14] and the monograph [16] for the additional fundamental results in the theory of multipliers.

Clearly, every left (right) multiplier is a left (right) n-multiplier, but the converse is not true in general. The next example illustrates this fact.

Example 1.1. *Let*

$$A = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad a, b, c \in \mathbb{C} \right\},$$

and define $T : A \longrightarrow A$ by

$$T\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

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Then, $T(x)y = xT(y) \neq T(xy) = 0$ *for all* $x, y \in A$, *, hence* T *is not left (right) multiplier, in general, but for all* $n \geq 3$ *and for every* $x_1, x_2, ..., x_n \in A$,

$$T(x_1x_2...x_n) = T(x_1x_2...x_{n-1})x_n = x_1T(x_2...x_n).$$

Therefore, T *is an* n*-multiplier for every* $n \geq 3$.

Suppose that A is a unital (Banach) algebra with unit e_A. An A-bimodule X is called *unital* if $e_Ax = xe_A = x$, for all $x \in X$.

The following characterization of n-multiplier presented by the author in [18].

Theorem 1.2. [18, Corollary 2.10] Suppose that A is a unital Banach algebra and X is a unital Banach A-bimodule. Let $T : A \longrightarrow X$ be a continuous linear map such that

$$a, b \in A, \quad ab = e_A \quad \Longrightarrow \quad T(ab) = aT(b). \tag{1.1}$$

Then T is a right n-multiplier.

The set of idempotents of given Banach algebra A is denoted by $\mathcal{I}(A)$ and let $\mathfrak{J}(A)$ be the subalgebra of A generated by idempotents. We say that the Banach algebra A is generated by idempotents, if $A = \overline{\mathfrak{J}(A)}$.

Recall that a C^*-algebra A is called a W^*-algebra (or von-Neumann algebra) if it is a dual space as a Banach space [8], [17].

Let A be a W^*-algebra, then the linear span of projections is norm dense in A, hence $A = \overline{\mathfrak{J}(A)}$. Moreover, it turned out in [2] that the group algebra $L^1(G)$ for a compact group G and topologically simple Banach algebras containing a non-trivial idempotent are generated by idempotents. For more examples of Banach algebra A with the property that $A = \overline{\mathfrak{J}(A)}$, see [2].

Let A be a Banach algebra and X be a Banach space. Then the continuous bilinear mapping $\phi : A \times A \longrightarrow X$ preserves zero products if

$$ab = 0 \quad \Longrightarrow \quad \phi(a, b) = 0, \quad a, b \in A. \tag{1.2}$$

Definition 1.3. [2] A Banach algebra A *has the property* $(\mathbb{B})$ if for every continuous bilinear mapping $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, the condition (1.2) implies that $\phi(ab, c) = \phi(a, bc)$, for all $a, b, c \in A$.

It follows from [2, Theorem 2.11] that C^*-algebras, group algebras and Banach algebras that generated by idempotents have the property $(\mathbb{B})$.

Characterizing (Jordan) homomorphisms, derivations, Jordan derivations on (Banach) algebras and C^*-algebras through the action on zero products have been studied by many authors, see for example [1, 3, 6, 9, 10, 11, 12, 13, 19] and the references therein.

In this paper we consider the subsequent conditions on a linear map T from a Banach algebra A into an A-bimodule X:

$(\mathbb{M}1)$ $a, b \in A, \quad ab = 0 \implies aT(b) = 0,$
$(\mathbb{M}2)$ $a, b \in A, \quad ab = ba = 0 \implies aT(b) + bT(a) = 0,$
$(\mathbb{M}3)$ $a, b \in A, \quad a \circ b = 0 \implies aT(b) + bT(a) = 0,$

where $a \circ b = ab + ba$ is a Jordan product in A.

We investigate whether these conditions characterizes n-multipliers on Banach algebras and C^*-algebras. We prove that Theorem 1.2 is remain valid for C^*-algebras if (1.1) replaced by any of the above conditions.

2 Characterizing n-multipliers on Banach algebras

In this section, we characterizes n-multipliers from unital Banach algebra A into unital A-bimodule X, that satisfy one of the conditions $(\mathbb{M}1)$-$(\mathbb{M}3)$.

Theorem 2.1. [7, Theorem 4.1] If ϕ is a bilinear mapping from $A \times A$ into a vector space X such that

$$a, b \in A, \quad ab = 0 \implies \phi(a, b) = 0,$$

then

$$\phi(a, x) = \phi(ax, e_A), \quad \textit{and} \quad \phi(x, a) = \phi(e_A, xa),$$

for all $a \in A$ and $x \in \mathfrak{J}(A)$.

Proposition 2.2. Suppose that $T : A \longrightarrow X$ is a linear mapping such that the condition ($\mathbb{M}1$) holds. Then $T(xa) = xT(a)$ for all $a \in A$ and $x \in \mathfrak{J}(A)$.

Proof . Define a bilinear mapping $\phi : A \times A \longrightarrow X$ by

$$\phi(a, b) = aT(b) - abT(e_A), \qquad a, b \in A.$$

Then $\phi(a, b) = 0$, whenever $ab = 0$. Applying Theorem 2.1, we obtain

$$pT(a) - paT(e_A) = \phi(p, a) = \phi(e_A, pa) = e_AT(pa) - paT(e_A), \qquad a \in A,\ p \in \mathcal{I}(A).$$

Therefore $T(pa) = pT(a)$ for each $a \in A$ and $p \in \mathcal{I}(A)$. Now from definition of $\mathfrak{J}(A)$ it follows that $T(xa) = xT(a)$ for all $a \in A$ and $x \in \mathfrak{J}(A)$. □

As a consequence of Proposition 2.2, we have the next result.

Corollary 2.3. Let $T : A \longrightarrow X$ be a [continuous] linear mapping such that the condition ($\mathbb{M}1$) holds. If $A = \mathfrak{J}(A)$ [$A = \overline{\mathfrak{J}(A)}$], then T is a right n-multiplier.

We say that $w \in A$ is a left (right) separating point of A-bimodule X if the condition $wx = 0$ [$xw = 0$] for all $x \in X$ implies that $x = 0$. An ideal I of A is called left (right) separating set if every $w \in I$ is a left (right) separating point of X.

Theorem 2.4. Let $T : A \longrightarrow X$ be a linear map satisfying ($\mathbb{M}1$). If X has a right separating set $I \subseteq \mathfrak{J}(A)$, then T is a right n-multiplier.

Proof . It follows from Proposition 2.2 that $T(wab) = wT(ab)$ and

$$T(wab) = T((wa)b) = waT(b), \quad a, b \in A,\ w \in I.$$

Thus, $w\big(T(ab) - aT(b)\big) = 0$ for all $a, b \in A$ and every $w \in I$. Since I is a right separating set of X, $T(ab) = aT(b)$ for all $a, b \in A$. Consequently, T is a right multiplier and hence it is a right n-multiplier. □

Theorem 2.5. [5, Lemma 2.2] If ϕ is a bilinear mapping from $A \times A$ into a vector space X such that

$$a, b \in A, \quad ab = ba = 0 \implies \phi(a, b) = 0,$$

then

$$\phi(a, x) + \phi(x, a) = \phi(ax, e_A) + \phi(e_A, xa),$$

for all $a \in A$ and $x \in \mathfrak{J}(A)$.

Our first main theorem is the following.

Theorem 2.6. Suppose that T is a linear mapping from A into X such that the condition ($\mathbb{M}2$) holds. Then $T(xa) = xT(a)$ for all $a \in A$ and every $x \in \mathfrak{J}(A)$.

Proof . Define a bilinear mapping $\phi : A \times A \longrightarrow X$ by

$$\phi(a,b) = aT(b) + bT(a) - abT(e_A) - baT(e_A),$$

for all $a, b \in A$. Then $ab = ba = 0$ implies that $\phi(a,b) = 0$. Hence by Theorem 2.5,

$$\phi(a,p) + \phi(p,a) = \phi(ap, e_A) + \phi(e_A, pa), \tag{2.1}$$

for all $a \in A$ and each $p \in \mathcal{I}(A)$. Define $\psi : A \longrightarrow X$ via $\psi(a) = T(a) - aT(e_A)$. Since $p(e_A - p) = (e_A - p)p = 0$, we have $\psi(p) = 0$. Indeed,

$$pT(e_A - p) + (e_A - p)T(p) = 0,$$

which implies that $pT(e_A) = T(p) = pT(p)$, for every $p \in \mathcal{I}(A)$. Now by (2.1) we obtain

$$\begin{aligned} \psi(ap) + \psi(pa) &= \phi(ap, e_A) + \phi(e_A, pa) \\ &= \phi(a,p) + \phi(p,a) \\ &= 2a\big(T(p) - pT(e_A)\big) + 2p\big(T(a) - aT(e_A)\big) \\ &= 2p\psi(a). \end{aligned}$$

Therefore

$$2p\psi(a) = \psi(ap) + \psi(pa). \tag{2.2}$$

Replacing a by ap and pa in (2.2), respectively, we get

$$2p\psi(ap) = \psi(ap) + \psi(pap), \tag{2.3}$$

and

$$2p\psi(pa) = \psi(pap) + \psi(pa). \tag{2.4}$$

Multiplying the relation (2.3) by p from the left hand side, gives

$$p\psi(ap) = p\psi(pap). \tag{2.5}$$

Similarly, from (2.4) we arrive at

$$p\psi(pa) = p\psi(pap). \tag{2.6}$$

Replacing a by $a - ap$ in (2.2), we get

$$2p\psi(a - ap) = \psi(pa - pap). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$p\psi(a) = p\psi(ap), \ \textit{and} \ \ \psi(pa) = \psi(pap). \tag{2.8}$$

By (2.4) and (2.8),

$$p\psi(pa) = \psi(pa) = \psi(pap). \tag{2.9}$$

Multiplying the relation (2.2) by p from the left hand side, we obtain

$$2p\psi(a) = p\psi(ap) + p\psi(pa). \tag{2.10}$$

From (2.8), (2.9) and (2.10), we arrive at

$$p\psi(a) = p\psi(pa) = \psi(pa),$$

for all $a \in A$ and every idempotent $p \in A$. This means that

$$p\big(T(a) - aT(e_A)\big) = T(pa) - paT(e_A).$$

Consequently, $T(pa) = pT(a)$ for all $a \in A$ and each $p \in \mathcal{I}(A)$. Now from definition of $\mathfrak{J}(A)$ we get $T(xa) = xT(a)$ for all $a \in A$ and $x \in \mathfrak{J}(A)$. This finishes the proof. □

Corollary 2.7. Let $T : A \longrightarrow X$ be a [continuous] linear mapping such that the condition ($\mathbb{M}2$) holds. If $A = \mathfrak{J}(A)$ [$A = \overline{\mathfrak{J}(A)}$], then T is a right n-multiplier.

Similar to the proof of Theorem 2.4, we have the next result.

Theorem 2.8. Suppose that $T : A \longrightarrow X$ is a linear map satisfying ($\mathbb{M}2$). If X has a right separating set $I \subseteq \mathfrak{J}(A)$, then T is a right n-multiplier.

Theorem 2.9. [4, Theorem 2.1] If ϕ is a bilinear mapping from $A \times A$ into a vector space X such that

$$a, b \in A, \quad a \circ b = 0 \implies \phi(a, b) = 0,$$

then

$$\phi(a, x) = \frac{1}{2}\big(\phi(ax, e_A) + \phi(xa, e_A)\big),$$

for all $a \in A$ and $x \in \mathfrak{J}(A)$.

Theorem 2.10. Let $T : A \longrightarrow X$ be a linear mapping such that the condition ($\mathbb{M}3$) holds. Then $T(xa) = xT(a)$ for all $a \in A$ and every $x \in \mathfrak{J}(A)$.

Proof . By applying Theorem 2.9 to the bilinear mapping $\phi : A \times A \longrightarrow X$ defined by

$$\phi(a, b) = aT(b) + bT(a) - (a \circ b)T(e_A), \quad a, b \in A,$$

we obtain

$$2\phi(a, p) = \phi(ap, e_A) + \phi(pa, e_A), \tag{2.11}$$

for all $a \in A$ and each $p \in \mathcal{I}(A)$. Define $\psi : A \longrightarrow X$ via $\psi(a) = T(a) - aT(e_A)$. As $p \circ (e_A - p) = 0$, we have $\psi(p) = 0$. Thus, from (2.11) we get

$$\begin{aligned} \psi(ap) + \psi(pa) &= \phi(ap, e_A) + \phi(pa, e_A) \\ &= 2\phi(a, p) \\ &= 2a\big(T(p) - pT(e_A)\big) + 2p\big(T(a) - aT(e_A)\big) \\ &= 2p\psi(a). \end{aligned}$$

Now the rest of proof is similar to the proof of Theorem 2.6. □

3 Characterizing n-multipliers on C^*-algebras

In this section, by using zero products preserving bilinear maps, we prove that each linear mapping T from unital C^*-algebra A into unital Banach A-bimodule X which satisfies one of the conditions ($\mathbb{M}1$)-($\mathbb{M}3$) is an n-multiplier.

Theorem 3.1. Let A be a unital C^*-algebra and let $T : A \longrightarrow X$ be a continuous linear map satisfying ($\mathbb{M}1$). Then T is a right n-multiplier.

Proof . Let us define a continuous bilinear mapping $\phi : A \times A \longrightarrow X$ by $\phi(a, b)aT(b)$. Then $\phi(a, b) = 0$ whenever $ab = 0$. Hence by [2, Theorem 2.11],

$$abT(c) = \phi(ab, c) = \phi(a, bc) = aT(bc),$$

for all $a, b, c \in A$. Taking $a = e_A$, we get $T(bc) = bT(c)$ for all $b, c \in A$. Therefore T is a right multiplier and hence it is a right n-multiplier. □

The following remark generalize [1, Lemma 2.1] for every commutative C^*-algebras.

Remark 3.2. Let A be a commutative C^*-algebra and $\phi : A \times A \longrightarrow X$ be a continuous bilinear mapping. Then by [3, Theorem 2.1], if ϕ preserving zero products, then there is a continuous linear mapping $f : A \longrightarrow X$ such that $\phi(a, b) = f(ab)$, for all $a, b \in A$. Thus,

$$\phi(a, b) = f(ab) = f(ba) = \phi(b, a), \quad a, b \in A.$$

On the other hand, ϕ is *symmetric*.

From Theorem 3.1, we get the next result.

Corollary 3.3. Let A be a commutative unital C^*-algebra. If $T : A \longrightarrow X$ is a continuous linear mapping such that the condition ($\mathbb{M}1$) holds, then $aT(b) = bT(a)$ for all $a, b \in A$.

Next we show that Theorem 3.1 is true if condition ($\mathbb{M}1$) replaced by ($\mathbb{M}2$). First we prove it for W^*-algebras. Note that every W^*-algebra is unital [8].

Theorem 3.4. Let A be a W^*-algebra and let $T : A \longrightarrow X$ is a continuous linear mapping such that the condition ($\mathbb{M}2$) holds. Then T is a right n-multiplier.

Proof . By Theorem 2.6, $T(pb) = pT(b)$ for all $b \in A$ and $p \in \mathcal{I}(A)$. Let A_{sa} denote the set of self-adjoint elements of A and let $x \in A_{sa}$. Then by Lemma 1.7.5 and Proposition 1.3.1 of [17], x is the limit of a sequence of linear combinations of projections in A, i.e., self-adjoint idempotents. Thus,

$$x = \lim_n \sum_{k=1}^{n} \lambda_k p_k,$$

and hence for all $b \in A$,

$$T(xb) = \lim_n T(\sum_{k=1}^{n} \lambda_k p_k b) = \lim_n \sum_{k=1}^{n} \lambda_k T(p_k b) = \lim_n \sum_{k=1}^{n} \lambda_k p_k T(b) = xT(b).$$

Now let $a \in A$ be arbitrary. Then $a = x + iy$ for $x, y \in A_{sa}$ and thus we get

$$\begin{aligned} T(ab) &= T\big((x + iy)b\big) \\ &= xT(b) + iyT(b) = aT(b). \end{aligned}$$

Consequently, $T(ab) = aT(b)$ for all $a, b \in A$ and hence T is a right n-multiplier. □

It is well-known that on the second dual space A^{**} of a Banach algebra A there are two multiplications, called the first and second Arens products which make A^{**} into a Banach algebra [8]. If these products coincide on A^{**}, then A is said to be Arens regular. It is shown [8] that every C^*-algebra A is Arens regular.

For each Banach A-bimodule X, the second dual X^{**} turns into a Banach A^{**}-bimodule where A^{**} equipped with the first Arens product. The module actions are defined by

$$\Phi \cdot u = w^* - \lim_i \lim_j a_i \cdot x_j, \quad u \cdot \Phi = w^* - \lim_j \lim_i x_j \cdot a_i, \quad \Phi \in A^{**},\ u \in X^{**},$$

where $\{a_i\}_{i\in I}$ and $\{x_i\}_{j\in I}$ are nets in A and X that converge, in w^*-topologies, to Φ and u, respectively. One may refer to the monograph of Dales [8] for a full account of Arens product and w^*-continuity of the above structures.

Since the second dual of each C^*-algebra is a W^*-algebra [8], hence by extending the continuous linear map $T : A \longrightarrow X$ to the second adjoint $T^{**} : A^{**} \longrightarrow X^{**}$ and applying Theorem 3.4, we get the following result.

Corollary 3.5. Let A be a unital C^*-algebra and let $T : A \longrightarrow X$ be a continuous linear mapping such that the condition ($\mathbb{M}2$) holds. Then T is a right n-multiplier.

It should be note that the condition ($\mathbb{M}3$) implies ($\mathbb{M}2$) and therefore Theorem 3.4 and Corollary 3.5 still works with condition ($\mathbb{M}2$) replaced by ($\mathbb{M}3$).

Example 3.6. *Let*

$$A = \left\{ \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix} : \quad z, w \in \mathbb{C} \right\}.$$

We make $X = \mathbb{C}$ an A-bimodule by defining

$$a\lambda = 0, \quad \lambda a = \lambda z, \quad \lambda \in \mathbb{C}, \ a \in A.$$

Define $T : A \longrightarrow X$ by $T(\begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix}) = w$. Then neither T is a left multiplier nor right multiplier. However, $T(ab) = T(b)a$ for all $a, b \in A$. This example leads us to define the following concept.

Definition 3.7. A linear operator T from Banach algebra A into an A-bimodule X is called *left anti n-multiplier* [*right anti n-multiplier*] if for all $a_1, a_2, ..., a_n \in A$.

$$T(a_1a_2...a_n) = a_nT(a_1a_2...a_{n-1}), \quad [T(a_1a_2...a_n) = T(a_2...a_n)a_1],$$

and T is called *anti n-multiplier* if it is both left and right anti n-multiplier.

Next we show that every anti n-multiplier from C^*-algebra A into an A-bimodule X is exact an n-multiplier. The idea of the proof can be found in [3].

Theorem 3.8. Let A be a C^*-algebra and X be an A-bimodule. Suppose that $T : A \longrightarrow X$ is a continuous right anti n-multiplier. Then T is a left n-multiplier.

Proof . By assumption

$$T(a_1a_2...a_n) = T(a_2...a_n)a_1,$$

for all $a_1, a_2, ..., a_n \in A$. If A is unital, then by taking $a_2 = ... = a_n = e_A$, we conclude that $T(a) = T(e_A)a$ for all $a \in A$. Therefore

$$T(a_1a_2...a_n) = T(e_A)a_1a_2...a_n = T(a_1a_2...a_{n-1})a_n, \quad a_1, a_2, ..., a_n \in A.$$

Hence T is a left n-multiplier. For nonunital case we extending $T : A \longrightarrow X$ to the second adjoint $T^{**} : A^{**} \longrightarrow X^{**}$ and based on the Arens regularity of A, the w^*-w^*-continuity of T^{**} and the separate weak continuity of the module operations on X^{**}, we get

$$T^{**}(a_1a_2...a_n) = T^{**}(a_2...a_n)a_1,$$

for all $a_1, a_2, ..., a_n \in A^{**}$. Setting $\xi = T^{**}(e_{A^{**}}) \in X^{**}$. Then it follows from the above equality with $a_2 = ... = a_n = e_{A^{**}}$ that

$$T^{**}(a) = \xi a,$$

for all $a \in A^{**}$. In particular, we have

$$T(a) = \xi a, \quad a \in A. \tag{3.1}$$

Note that $\xi a \in X$ for all $a \in A$. Of course, it suffices to prove it for each positive element $a \in A$. Suppose that $a \in A$ be a positive element and let $b \in A$ with $a = b^2$. According to (3.1),

$$\xi a = \xi b^2 = T(b^2) \in X.$$

Consequently, from (3.1) it follows that T is a left n-multiplier. □

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