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Monotone method for discrete fractional boundary value problems

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Abstract

In this paper, by using the Schauder fixed point theorem, we obtain the existence of positive solutions for discrete fractional boundary value problem. Also, we establish upper and lower solution for this problem. Our results extend some recent works in the literature.

Keywords: Positive solutions, Discrete fractional equations, Lower and upper solutions
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1 Introduction

In his 1695 letter to Leibniz, L'Hospital posed the question, what does it mean to take a half- derivative of a function? This question is said to have launched fractional calculus, which has since developed many ways of extending derivatives to be of any complex order. Basic background in fractional calculus can be found in [16]. Fractional calculus has recently developed into a relatively vibrant research area. It also provides an excellent tool to describe the hereditary properties of materials and processes. Many successful new applications of fractional calculus in various fields have also been reported recently. For example, Nieto and Pimentel [14] extended a second order thermostat model to the fractional model; Ding and Jiang [5] used waveform relaxation methods to study some fractional functional differential equation models. Also, fractional differential systems have been used in 3D printing and oil drilling [4]. For the basic theories of fractional calculus and some recent work in application, the reader is referred to references [10, 13, 15, 16, 17]. Discrete fractional calculus consists of studying fractional derivatives of functions defined on a discrete domain. Discrete fractional calculus has attracted slowly but steadily increasing attention in the past seven years or so . In particular, several recent papers by Atici and Eloe as well as other recent papers by the present authors have addressed some basic theory of both discrete fractional initial value problems and discrete fractional boundary value problems (DFBVPs). More specially, Atici and Eloe [3] have already analyzed a transform method in discrete fractional calculus. Goodrich [8] considered a discrete right-focal fractional boundary value problem. All of the fundamental background in discrete fractional calculus can be found in [9] which is written by Goodrich and Peterson. Other recent works have considered DFBVPs with a variety of boundary conditions, see [11, 12] and the references therein.

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The purpose of this paper is to establish a method of lower and upper solutions for the following discrete fractional boundary value problem, as shown

$$-\nabla^{\mu}_{a^*}x(t) = f(t, x(t-1))), \qquad t \in \mathbb{N}^{b-1}_{a+1} \tag{1.1}$$

$$\begin{aligned} &\alpha x(a-1) - \beta \nabla x(a) = 0, \\ &\gamma x(b) + \delta \nabla x(b) = 0 \end{aligned} \tag{1.2}$$

in the context of discrete nabla fractional calculus, where $1 < \mu \le 2,\ \alpha, \gamma, \beta, \delta > 0$ such that $\dfrac{\beta}{\alpha} > 1,\ b - a \ge 2,\ f : \mathbb{N}^{b-1}_{a+1} \longrightarrow \mathbb{R}$ and x are defined on $\mathbb{N}^{b}_{a-1}$.

In this paper we will consider the Green's function for this DFBVP (1.1)-(1.2) and give some properties and bounds of this Green's function in Section 3. Also in this section, using these bounds we establish upper and lower solutions and existence results for DFBVP (1.1)-(1.2).

2 Preliminaries

In this section, we collect some basic definitions and lemmas for manipulating discrete fractional operators. These can be found in the references [16].

For any real number β, let $N_\beta = \{\beta, \beta+1, \beta+2, ...\}$ and we define $t^{\overline{k}} = \dfrac{\Gamma(t+k)}{\Gamma(t)}$ for any $t, k \in \mathbb{R}$. If $n \in \mathbb{N}$ then $t^{\overline{n}} := t(t+1)...(t+n-1)$.

Remark 2.1. Let n and N be nonnegative integers. Then

$$\frac{\Gamma(-n)}{\Gamma(-N)} = (-1)^{N-n}\frac{N!}{n!}.$$

Also, if t is a nonpositive integer and t + r is not a nonpositive integer, then

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)} = 0.$$

Theorem 2.2. The following equality hold

$$\nabla(t+a)^{\overline{n}} = n(t+a)^{\overline{n-1}}, \quad t \in \mathbb{R}$$

for values of $n \in \mathbb{N}$ and $a \in \mathbb{R}$.

Definition 2.3. We define the nabla Taylor monomials, $H_n(t,a),\ n \in \mathbb{N}_0$ by $H_0(t,a) = 1,\ t \in \mathbb{N}_a$ and

$$H_n(t,a) = \frac{(t-a)^{\overline{n}}}{n!},\ t \in \mathbb{N}_{a-n+1}$$

for $n \in \mathbb{N}_0$.

Definition 2.4. Let $\mu \ne -1, -2, ...$ we define μ-th order nabla fractional Taylor monomial $H_\mu(t,a)$, by

$$H_\mu(t,a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} \tag{2.1}$$

whenever the right hand side of the equation (2.1) makes sense.

Definition 2.5. Let $f : \mathbb{N}_{a+1} \longrightarrow \mathbb{R}$ be given and $\mu \in \mathbb{R}^+$, then

$$\nabla^{-\mu}_{a} f(t) := \int_a^t H_{\mu-1}(t, \rho(s)) f(s) \nabla s$$

for $t \in \mathbb{N}_a$, where by convention $\nabla^{\mu}_{a}(a) = 0$.

Definition 2.6. Let $f : \mathbb{N}_{a+1} \longrightarrow \mathbb{R},\ \mu \in \mathbb{R}^+$ and $N-1 < \mu \leq N$. Then we define μ-th nabla fractional difference $\nabla_a^\mu f(t)$ by

$$\nabla_a^\mu f(t) := \nabla^N \nabla_a^{-(N-\mu)} f(t).$$

Definition 2.7. Assume $f : \mathbb{N}_{a-N+1} \longrightarrow \mathbb{R},\ \mu \in \mathbb{R}^+$ and $N-1 < \mu \leq N$. Then we define μ-th Caputo nabla fractional difference of f is defined by

$$\nabla_{a^*}^\mu f(t) := \nabla_a^{-(N-\mu)} \nabla^N f(t),\ t \in \mathbb{N}_{a+1}$$

where $N := \lceil \mu \rceil$.

The following propositions will be used in the theorem that follows stating the bounds of the Green's function.

Proposition 2.8. [1] Let f, g be real-valued functions on a set S, such that $f(t), g(t) \geq 0$ for all $t \in S$. Moreover, assume there exists $s_0, s_1 \in S$ where $\max_{s\in S} f(s) = f(s_0)$ and $\max_{s\in S} g(s) = g(s_1)$, i.e. f and g attain their maximum in S. Then for each fixed $t \in S$,

$$|f(t) - g(t)| \leq \max\{f(t), g(t)\} \leq \max\{\max_{t\in S} f(t), \max_{t\in S} g(t)\}.$$

Proposition 2.9. [6] Let $g(\tau, s) := (\tau - \rho(s))^{\overline{a}}$ where $a > 0,\ s \in \mathbb{N}_{a+1}^b$, and $\tau \in \mathbb{N}_{s-1}^b$.
(i) $g(t, s) \geq 0$,
(ii) g is decreasing function of s,
(iii) g is increasing function of τ.

Theorem 2.10. Let $a, \mu \in \mathbb{R},\ 1 < \mu \leq 2,\ N := \lceil \mu \rceil = 2$ and $f : \mathbb{N}_a^b \longrightarrow \mathbb{R}$. Then a general solution to the nabla Caputo fractional equation

$$-\nabla_{a^*}^\mu x(t) = f(t), \quad t \in \mathbb{N}_{a+1} \tag{2.2}$$

is given by

$$\begin{aligned} x(t) &= c_0(t-a)^{\overline{0}} + c_1(t-a)^{\overline{1}} - \nabla_a^{-\mu} f(t) \\ &= c_0 + c_1(t-a)^1 - \nabla_a^{-\mu} f(t) \end{aligned} \tag{2.3}$$

for $c_0,\ c_1 \in \mathbb{R}$ and $t \in \mathbb{N}_{a-1}$.

Proof. First, we will show that $x(t)$ given by (2.3), is a solution to (2.2) on $\mathbb{N}_{a-1}$. For $t \in \mathbb{N}_{a-1}$, consider

$$-\nabla_{a^*}^\mu x(t) = -\nabla_{a^*}^\mu [c_0 + c_1(t-a)^1] + \nabla_{a^*}^\mu \nabla_a^{-\mu} f(t)$$

where we have made use of the linearity of $-\nabla_{a^*}^\mu$. Now,

$$\nabla_{a^*}^\mu (t-a) = \nabla_a^{-(2-\mu)} \nabla^2 (t-a) = \nabla_a^{-(2-\mu)} \nabla 1 = \nabla_a^{-(2-\mu)} 0 = 0,$$

for $t \in \mathbb{N}_{a-1}$. Hence

$$-\nabla_{a^*}^\mu [c_0 + c_1(t-a)] + \nabla_{a^*}^\mu \nabla_a^{-\mu} f(t) = f(t),$$

which shows that $x(t)$ is a solution to (2.2). Next, we will show that any solution, $y(t)$ of (2.2) is of the form (2.3). We will show that we can express y in the form (2.3) for fixed given constants $c_0,\ c_1 \in \mathbb{R}$.
First, define $A_k := \nabla^k y(a),\ for\ k \in \{0, 1\}$. Then note that $y(t)$ is a solution to the IVP

$$\begin{aligned} -\nabla_{a^*}^\mu y(t) &= f(t), \quad t \in \mathbb{N}_{a+1} \\ \nabla^k y(a) &= A_k. \end{aligned} \tag{2.4}$$

Using Theorem 3.12 in [5], the unique solution $y(t)$ to the IVP (2.4) is

$$\begin{aligned} y(t) &= H_0(t,a)A_0 + H_1(t,a)A_1 - \nabla_a^{-\mu} f(t) \\ &= A_0 + (t-a)A_1. \end{aligned}$$

Then, taking $c_0 := A_0\ c_1 := A_1$, we shown that y is of the form (2.3). □

□

The following fixed point theorems are fundamental and important to the proof of our main results.

Theorem 2.11. Consider the DFBVP (1.1)-(1.2) with $1 < \mu \leq 2$, $N = 2$ and $f : \mathbb{N}_{a+1}^{b-1} \longrightarrow \mathbb{R}$. Then $x : \mathbb{N}_{a-1} \longrightarrow \mathbb{R}$ is a solution of the DFBVP (1.1)-(1.2) if and only if $x(t)$ satisfies the integral equation

$$x(t) = \int_a^b G_\mu(t,s) f(s) \nabla s, \tag{2.5}$$

for $t \in \mathbb{N}_{a-1}^b$, where $G_\mu(t,s)$ is the Green's function for the homogeneous equation

$$-\nabla_{a^*}^\mu x(t) = 0, \quad t \in \mathbb{N}_{a+1}$$

with the boundary conditions (1.2) and is given by

$$G_\mu(t,s) := \begin{cases} u(t,s), & t \leq \rho(s) \\ v(t,s), & \rho(s) \leq t \end{cases} \tag{2.6}$$

where $u(t,s) = \dfrac{\alpha(t-a-1)+\beta}{d}\Big(\gamma H_{\mu-1}\big(b,\rho(s)\big) + \delta H_{\mu-2}\big(b,\rho(s)\big)\Big)$, $v(t,s) = u(t,s) + H_{\mu-1}(t,\rho(s))$ and $d = \gamma\big(\alpha(b-a+1)+\beta\big)+\delta\alpha$.
Note that $G_\mu : \mathbb{N}_{a-1}^b \longrightarrow \mathbb{R}$.

Proof. Assume $x(t)$ is a solution of the DFBVP (1.1)-(1.2). By Theorem 2.10, we have

$$x(t) = c_0 + c_1(t-a)^{\overline{1}} - \nabla_a^{-\mu} f(t). \tag{2.7}$$

Using the first boundary value condition, we get

$$\begin{aligned} \alpha x(a-1) - \beta \nabla x(a) &= \alpha\big(c_0 + c_1(a-1-a)^{\overline{1}} - \nabla_a^{-\mu} f(a-1)\big) - \beta\big(c_1 - \nabla\nabla_a^{-\mu} f(a)\big) \\ &= \alpha\big(c_0 - c_1 - \nabla_a^{-\mu} f(a-1)\big) - \beta\big(c_1 - \nabla\nabla_{a-1}^{-\mu} f(a)\big) \\ &= \alpha(c_0 - c_1) - \beta c_1 - \Big(\nabla_a^{-\mu} f(a) - \nabla\nabla_{a-1}^{-\mu} f(a)\Big), \end{aligned}$$

For $t \in \mathbb{N}_a^{b-1}$ we know that $-\nabla_{a^*}^\mu x(t) = f(t)$. So we have
$\alpha\nabla_a^{-\mu} f(a-1) + \beta\nabla\nabla_a^{-\mu} f(a) = \alpha x(a-1) - \beta\nabla x(a)$. Using the first boundary condition we get $\alpha\nabla_a^{-\mu} f(a-1) - \beta\nabla\nabla_a^{-\mu_a} f(a) = 0$. So,

$$\alpha(c_0 - c_1) - \beta c_1 = 0$$

and

$$c_0 = \frac{\beta+\alpha}{\alpha} c_1. \tag{2.8}$$

Next, from the second boundary condition and (2.8) we get

$$\gamma x(b) + \delta\nabla x(b) = \gamma\Big(c_0 + c_1(b-a)^{\overline{1}} - \nabla_a^{-\mu} f(b)\Big) + \delta\Big(c_1 - \nabla\nabla_a^{-\mu} f(b)\Big) = 0$$

and also

$$\begin{aligned} \gamma\Big(\frac{\beta+\alpha}{\alpha} c_1 + c_1(b-a) - \nabla_a^{-\mu} f(b)\Big) + \delta\Big(c_1 - \nabla_a^{1-\mu} f(b)\Big) &= 0 \\ \gamma\Big(\frac{\beta+\alpha}{\alpha} c_1 + c_1(b-a)\Big) + \delta c_1 - \Big(\gamma\nabla_a^{-\mu} f(b) + \delta\nabla_a^{1-\mu} f(b)\Big) &= 0 \\ c_1\Big(\gamma\big(\frac{\beta+\alpha}{\alpha} + b - a\big) + \delta\Big) &= \gamma\nabla_a^{-\mu} f(b) + \delta\nabla_a^{1-\mu} f(b). \end{aligned}$$

Then it follows that

$$c_1 = \frac{\alpha\Big(\gamma\nabla_a^{-\mu} f(b) + \delta\nabla_a^{1-\mu} f(b)\Big)}{d}. \tag{2.9}$$

It is easy to see that $d > 0$ and now from (2.8), we have

$$\begin{aligned} c_0 &= \frac{\beta+\alpha}{\alpha} c_1 \\ &= \frac{(\beta+\alpha)}{d} \Big(\gamma \nabla_a^{-\mu} f(b) + \delta \nabla_a^{1-\mu} f(b)\Big). \end{aligned}$$

Hence from (2.7) and (2.9), we have

$$\begin{aligned} x(t) &= c_0 + c_1 (t-a)^{\overline{1}} - \nabla_a^{-\mu} f(t) \\ &= \frac{\beta+\alpha}{d}\big(\gamma \nabla_a^{-\mu} f(b) + \delta \nabla_a^{1-\mu} f(b)\big) + \frac{\alpha}{d}\big(\gamma \nabla_a^{-\mu} f(b) + \delta \nabla_a^{1-\mu} f(b)\big)(t-a) - \nabla_a^{-\mu} f(t) \\ &= \frac{\beta+\alpha}{d}\Big(\gamma \int_a^b H_{\mu-1}(b,\rho(s)) f(s) \nabla s + \delta \int_a^b H_{\mu-2}(b,\rho(s)) f(s)\nabla s\Big) \\ &\quad + \frac{\alpha}{d}\Big(\gamma \int_a^b H_{\mu-1}(b,\rho(s)) f(s) \nabla s + \delta \int_a^b H_{\mu-2}(b,\rho(s)) f(s)\nabla s\Big)(t-a) \\ &\quad - \int_a^t H_{\mu-1}(t,\rho(s)) f(s)\nabla s \\ &= \frac{\alpha(t-a+1)+\beta}{d} \int_a^b \big(\gamma H_{\mu-1}(b,\rho(s)) + \delta H_{\mu-2}(b,\rho(s))\big) f(s)\nabla s - \int_a^t H_{\mu-1}(t,\rho(s)) f(s) \nabla s \\ &= \int_a^b G_\mu(t,s) f(s) \nabla s. \end{aligned}$$

Hence for $s \in \mathbb{N}_{a+1}^t$ and $t \in \mathbb{N}_s^b$,

$$\begin{aligned} G_\mu(t,s) = v(t,s) &:= u(t,s) - H_{\mu-1}(t,\rho(s)) \\ &= \frac{\alpha(t-a+1)+\beta}{d}\Big(\gamma H_{\mu-1}\big(b,\rho(s)\big) + \delta H_{\mu-2}\big(b,\rho(s)\big)\Big) - \frac{(t-\rho(s))^{\mu-1}}{\Gamma(\mu)}. \end{aligned}$$

Also, for $s \in \mathbb{N}_{t+1}^b$ and $t \in \mathbb{N}_a^{s-1}$ or $s \in \mathbb{N}_{a+1}^b$,

$$G_\mu(t,s) = u(t,s) := \frac{\alpha(t-a-1)+\beta}{d}\Big(\gamma H_{\mu-1}\big(b,\rho(s)\big) + \delta H_{\mu-2}\big(b,\rho(s)\big)\Big).$$

Noting that $u(t,s) = v(t,s)$ when $t = \rho(s)$, we can also write the Green's function as $v(t,s)$ for $\rho(s) \le t$ and $u(t,s)$ for $t \le \rho(s)$ for $(t,s) \in \mathbb{N}_{a-1}^b \times \mathbb{N}_{a+1}^b$. Thus, we have $x(t) = \int_a^b G_\mu(t,s) f(s) \nabla s$, where $G_\mu(t,s)$ is given by (2.6) are determined by the boundary value condition (1.2) is shown by the above steps, it follows that $\int_a^b G_\mu(t,s) f(s)\nabla s$ is a solution to the DFBVP (1.1)-(1.2). □□

The next theorem will give a bound of the Green's function for the DFVBP (1.1)-(1.2).

Theorem 2.12. Let G_μ be defined as (2.6). Then

$$|G_\mu(t,s)| \le D H_{\mu-1}(b,a), \tag{2.10}$$

where $D := \dfrac{(\gamma+\delta)\big(\alpha(b-a+1)+\beta\big)}{d}$ for $(t,s) \in \mathbb{N}_{a-1}^b \times \mathbb{N}_{a+1}^b$.

Proof. First, consider $(t,s) \in \mathbb{N}_s^b \times \mathbb{N}_{a+1}^t$. Then note that from (2.6),

$$|G_\mu(t,s)| = \left| \frac{\gamma(b-\rho(s))^{\overline{\mu-1}} + \delta(\mu-1)(b-\rho(s))^{\overline{\mu-2}}}{d\Gamma(\mu)}\big(\alpha(t-a+1)+\beta\big) - \frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)} \right|. \tag{2.11}$$

Now by Proposition 2.9, $(t-\rho(s))^{\overline{\mu-1}} \ge 0$, $(t-\rho(s))^{\overline{\mu-2}} \ge 0$ for $t \in \mathbb{N}_s^b$ and also $(b-\rho(s))^{\mu-1} \ge 0$, $(b-\rho(s))^{\mu-2} \ge 0$. Note that for each $i \in \{0,1\}$, $(t-a)^{\overline{i}} = \dfrac{\Gamma(t-a+i)}{\Gamma(t-a)} \ge 0$ since $(t,s) \in \mathbb{N}_s^b \times \mathbb{N}_{a+1}^t$ implies $t \ge a+1$. Since $d > 0$ and

Proposition 2.8, we have

$$|G_\mu(t,s)| \leq \max\left\{\max_{t\in\mathbb{N}_a^b}\left[\frac{\gamma(b-\rho(s))^{\overline{\mu-1}}+\delta(\mu-1)(b-\rho(s))^{\overline{\mu-2}}}{d\Gamma(\mu)}\big(\alpha(t-a+1)+\beta\big)\right], \max_{t\in\mathbb{N}_a^b}\left[\frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}\right]\right\}.$$

Since $\dfrac{(b-\rho(s))^{\overline{\mu-1}}}{\mu-1} > (b-\rho(s))^{\overline{\mu-2}}$ and $D>1$, we have

$$\begin{aligned}|G_\mu(t,s)| &\leq \max\left\{\max_{t\in\mathbb{N}_a^b}\left[\frac{\gamma(b-\rho(s))^{\overline{\mu-1}}+\delta(b-\rho(s))^{\overline{\mu-1}}}{d\Gamma(\mu)}\big(\alpha(t-a+1)+\beta\big)\right], \max_{t\in\mathbb{N}_a^b}\left[\frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}\right]\right\}\\ &\leq \max\left\{\max_{t\in\mathbb{N}_a^b}\left[\frac{(\gamma+\delta)\big(\alpha(t-a+1)+\beta\big)}{d}\frac{\big(b-\rho(s)^{\overline{\mu-1}}\big)}{\Gamma(\mu)}\right], \max_{t\in\mathbb{N}_a^b}\left[\frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}\right]\right\}\\ &\leq \frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}\max\left\{\frac{(\gamma+\delta)\big(\alpha(b-a+1)+\beta\big)}{d},1\right\}\\ &= \frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}\max\{D,1\}\\ &= H_{\mu-1}(b,\rho(s))\max\{D,1\}\\ &= DH_{\mu-1}(b,a).\end{aligned}$$

Next, consider $(t,s)\in\mathbb{N}_{a-N+1}^{s-1}\times\mathbb{N}_{\max\{a+1,t+1\}}^b$. Then from (2.6), we get

$$\begin{aligned}|G_\mu(t,s)| &= \left|\frac{\gamma(b-\rho(s))^{\overline{\mu-1}}+\delta(\mu-1)(b-\rho(s))^{\overline{\mu-2}}}{d\Gamma(\mu)}\big(\alpha(t-a+1)+\beta\big)\right|\\ &\leq \frac{1}{\Gamma(\mu)}\left|\frac{\gamma(b-a)^{\overline{\mu-1}}+\delta(b-a)^{\overline{\mu-1}}}{d}\big(\alpha(t-a+1)+\beta\big)\right|\\ &= \frac{1}{\Gamma(\mu)}\left|\frac{\Big(\gamma(b-a)^{\overline{\mu-1}}+\delta(b-a)^{\overline{\mu-1}}\Big)\Big(\alpha(b-a+1)+\beta\Big)}{d}\right|\\ &\leq D\frac{(b-a)^{\overline{\mu-1}}}{\Gamma(\mu)}\\ &= DH_{\mu-1}(b,a).\end{aligned}$$

Hence we have $|G_\mu(t,s)|\leq DH_{\mu-1}(b,a)$ for all $(t,s)\in\mathbb{N}_{a-1}^b\times\mathbb{N}_{a+1}^b$. □□

Corollary 2.13. The Green's function, $G_\mu(t,s):\mathbb{N}_{a-1}^b\times\mathbb{N}_{a+1}^b\longrightarrow\mathbb{R}$ for the DFBVP (1.1)-(1.2) satisfies the inequality

$$\int_a^b |G_\mu(t,s)|\,\nabla s \leq DH_{\mu-1}(b,a)(b-a).$$

3 Upper and Lower Solutions

In this section, we present a method of lower and upper solutions for the DFBVP (1.1)-(1.2) and prove the existence of positive solutions for this problem.

Theorem 3.1 ([9], Theorem(3.123)). Assume that $N-1<\nu<N,\ f:\mathbb{N}_{a-N+1}\longrightarrow\mathbb{R},\ \nabla_{a^*}f(t)\geq 0$ for $t\in\mathbb{N}_{a+1}$ and $\nabla^{N-1}f(t)\geq 0$ for $t\in\mathbb{N}_a$.

For $N=2$, we get following;

Theorem 3.2 ([9], Theorem(3.116)). Assume that $f:\mathbb{N}_{a-1}^b\longrightarrow\mathbb{R},\ \nabla_{a^*}^{\mu}f(t)\geq 0$ for each $t\in\mathbb{N}_{a+1}$ and $\nabla f(a)\geq q0$ with $1<\mu\geq 2$. Then $\nabla f(t)\geq 0$, for $t\in\mathbb{N}_a$.

Theorem 3.3. (Schauder-Tychonov Fixed Point Theorem) Let X be a Banach space. Assume that K is a closed, bounded, convex subset of X. If $T : K \longrightarrow K$ is compact, then T has a fixed point in K.

Theorem 3.4. Assume that f is continuous on $[a-1, b] \times \mathbb{R}$. If $M > 0$ and $DH_{\mu-1}(b, a)(b-a) \leq \dfrac{M}{Q}$ where $Q > 0$ satisfies

$$Q \geq \max\{|f(t, x)| : t \in [a-1, b],\ |x| \leq M\},$$

then the DFBVP (1.1)-(1.2) has a solution.

Proof. Define B to the Banach space of all continuous functions on $\mathbb{N}_a^b$ equipped with the norm $\|.\|$ defined by

$$\|x\| := \max_{t\in[a-1,b]} |x(t)| \quad for\ all\ x \in B.$$

Let

$$K := \{x \in B : \|x\| \leq M\}.$$

It can be shown that K is a closed, bounded and convex subset of B. Define $T : K \longrightarrow B$ by

$$Tx(t) := \int_a^b G_\mu(t, s) f(s, x(s-1))\nabla s$$

for $t \in [a-1, b]$. It is easy to shown that $T : K \longrightarrow B$ is continuous. Let $x \in K$ and consider

$$\begin{aligned}
|Tx(t)| &= \left| \int_a^b G_\mu(t, s) f(s, x(s-1))\nabla s \right| \\
&\leq \int_a^b |G_\mu(t, s) f(s, x(s-1))| \nabla s \\
&\leq Q \int_a^b |G_\mu(t, s)| \nabla s \\
&\leq QDH_{\mu-1}(b, a)(b-a) \\
&\leq M,
\end{aligned}$$

for all $t \in [a, b]$. This implies that $\|Tx\| \leq M$. Hence $T : K \longrightarrow K$. Using the Arzela-Ascoli theorem it can be shown that $T : K \longrightarrow K$ is a compact operator. Hence T has a fixed point x in K by the Schauder-Tychonov theorem. This implies that x is a solution of the boundary value problem (1.1) and (1.2). □□

Corollary 3.5. If f is continuous and bounded on $[a-1, b] \times \mathbb{R}$, then the DFBVP (1.1)-(1.2) has a solution.

We define the set

$$H := \{y : \nabla^\mu y\ is\ continuous\ on\ \mathbb{N}_{a-1}\}.$$

For any $u, v \in H$, we define the sector $[u, v]$ by

$$[u, v] = \{w \in H : u \leq w \leq v\}.$$

Definition 3.6. We say that $u \in H$ is called a lower solution of DFBVP (1.1)-(1.2) on $\mathbb{N}_{a-1}^b$ provided

$$\begin{aligned}
-\nabla_{a^*}^\mu u(t) &\geq f(t, u(t-1)), \quad t \in \mathbb{N}_{a+1}^{b-1} \\
\alpha u(a-1) - \beta\nabla u(a) &\leq 0, \\
\gamma u(b) + \delta\nabla u(b) &\leq 0.
\end{aligned}$$

Similarly, $v \in H$ is called upper solution of (1.1)-(1.2) on $\mathbb{N}_{a-1}^b$ provided

$$\begin{aligned}
-\nabla_{a^*}^\mu v(t) &\leq f(t, v(t-1)), \quad t \in \mathbb{N}_{a+1}^{b-1} \\
\alpha v(a-1) - \beta\nabla v(a) &\geq 0, \\
\gamma v(b) + \delta\nabla v(b) &\geq 0.
\end{aligned}$$

We will prove that when the lower and upper solutions are given in the well order, i.e. $u \leq v$, DFBVP (1.1)-(1.2) has a solution admit lying between both functions.

Theorem 3.7. Let $f : \mathbb{N}_{a-1}^{b} \longrightarrow \mathbb{R}$ be continuous functions. Assume that u and v are respectively lower and upper solutions for the DFBVP (1.1)-(1.2) such that $u \leq v$ on $\mathbb{N}_{a-1}^{b}$. Then the problem (1.1)-(1.2) has a solution $x \in [u, v]$ on $\mathbb{N}_{a-1}^{b}$.

Proof. Consider the following DFBVP

$$-\nabla_{a^*}^{\mu} x(t) = F(t, x(t-1)), \qquad t \in \mathbb{N}_{a+1}^{b-1} \tag{3.1}$$

$$\begin{aligned} \alpha x(a-1) - \beta \nabla x(a) = 0, \\ \gamma x(b) + \delta \nabla x(b) = 0 \end{aligned} \tag{3.2}$$

where

$$F(t,\zeta) = \begin{cases} f(t, v(t-1)) + \dfrac{\zeta - v(t)}{1+|x(c)|}, & \zeta \geq v(t), \\ f(t,\zeta) & u(t) \leq \zeta \leq v(t), \\ f(t, u(t-1)) + \dfrac{\zeta - u(t)}{1+|x(c)|}, & \zeta \leq u(t), \end{cases}$$

for $t \in \mathbb{N}_{a-1}^{b}$.

Clearly, the function F is bounded for $t \in [a-1, b]$ and $\zeta \in \mathbb{R}$, and is continuous in ζ. Thus, by Corollary 3.5, there exists a solution $x(t)$ of the the DFBVP (3.1)-(3.2).

We claim that $x(t) \leq v(t)$ for $t \in \mathbb{N}_{a-1}^{b}$. If not, from the boundary conditions we know that $x(t) - v(t)$ has a positive maximum solutions at some $c \in \mathbb{N}_{a}^{b-1}$. Since

$$\begin{aligned} -\nabla_{a^*}^{\mu} x(c) &= F(c, x(c-1)) \\ &= f(c, v(c-1)) + \frac{x(c) - v(c)}{1+|x(c)|} \\ &> f(c, v(c-1)) \\ &\geq -\nabla_{a}^{\mu} v(c), \end{aligned}$$

we have $\nabla_{a^*}^{\mu}\big(x - v\big)(c) \leq 0$. Also we know $\nabla(x - v)(a) \leq 0$, using $(x - v)(t)$ has a positive max of $x = a$. So from Theorem 3.1 we get

$$\nabla(x - v)(c) \leq 0, \qquad c \in \mathbb{N}_{a}^{b-1},$$

which is contradiction with maximum of $t = c$. It follows that $x(t) \leq v(t)$ on $\mathbb{N}_{a+1}^{b-1}$.

Similarly, we have $u(t) \leq x(t)$ on $\mathbb{N}_{a+1}^{b-1}$.

Also, from the first boundary condition we get $\alpha x(a-1) - \beta \nabla x(a) = \alpha x(a-1) - \beta\big(x(a) - x(a-1)\big) = 0$ and from Definition 3.1 we get $\alpha u(a-1) - \beta \nabla u(a) = \alpha u(a-1) - \beta\big(u(a) - u(a-1)\big) \leq 0$. Thus, we have

$$(\alpha + \beta)(u - x)(a-1) \leq \beta(u - x)(a) \leq 0.$$

That is to say $(u - x)(a-1) \leq 0$, so $u(a-1) \leq x(a-1)$.

On the other hand, from the second boundary condition we get $\gamma x(b) + \delta \nabla x(b) = 0$ and from Definition 3.1 we get $\gamma u(b) + \delta \nabla u(b) \leq 0$. Thus, we have

$$(\gamma + \delta)(u - x)(b) - \delta(u - x)(b-1) \leq 0.$$

That is to say $(u - x)(b) \leq 0$, so $u(b) \geq x(b)$.

Similarly, we get $x(a-1) \leq v(a-1)$ and $x(b) \leq v(b)$. So, we have $u(t) \leq x(t) \leq v(t)$ for $t \in \mathbb{N}_{a-1}^{b}$. □□

Example 3.8. Consider the following DFBVP

$$\begin{aligned} -\nabla_{1^*}^{\mu} x(t) = -e^{-x(t-1)} + x(t-1),\ t \in \mathbb{N}_{2}^{9} \\ x(0) - 2\nabla x(1) = 0, \\ x(10) + \nabla x(10) = 0. \end{aligned}$$

It is easy to check that $u(t) = 0$ is a lower solution and $v(t) = 1$ is an upper solution of the problem. Thus, by Theorem 3.4, problem has a solution $x(t) \in [0, 1]$ on $t \in [0, 10]$.

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