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On the differentiability of norms in Banach spaces

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Abstract

The purpose of this paper is to show some particularities that the differentiability sets generated from the norms have in the Banach spaces. In this sense, it will be shown that the Gaussian measure of the Fréchet differentiability set of the norm of the space $\ell^{\infty}(\mathbb{R})$ of real bounded sequences is zero and that in the case of the space $BV[a,b]$ of bounded variation functions its norm is not Fréchet derivable in any element of this space.

Keywords: Gâteaux differentiable, Fréchet differentiable, locally Lipschitz function, Gauss null set
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1 Introduction

The present paper contains results on the study of the differentiability sets generated from norms on separable and non-separable Banach spaces. Here we will consider two definitions of differentiability in Banach spaces. If X, Y are two Banach spaces, we will say that:

i. A function $f : X \to Y$ is *Gâteaux differentiable (G-differentiable)* $a \in X$, if there is a bounded linear operator u from X to Y, called the Gâteaux derivative of f at a and is represented as $\partial_G f(a)$, such that for every $h \in X$,

$$\lim_{t\to 0} \frac{f(a+th)-f(a)}{t} = u\left(h\right).$$

iii. A function $f : X \to Y$ is *Fréchet differentiable* (F-differentiable) in $a \in X$, if there is a continuous bounded linear operator u, called the Fréchet derivative of f at a and denoted by $\partial_F f(a)$, which satisfies

$$f\left(a+h\right)-f\left(a\right)-u\left(h\right)=r\left(h\right), \ \textit{where} \ \lim_{h\to 0}\frac{\left\|r\left(h\right)\right\|_Y}{\left\|h\right\|_X}=0.$$

For these types of derivatives, there are reasonably satisfactory results on the existence of Gˆateaux derivatives of the Lipschitz functions, while the results on the existence or not of Fréchet derivatives are rare and generally very difficult to prove. It is important then to highlight that if f is a norm and X has finite dimension then the notions of Gâteaux differentiability and Fréchet differentiability coincide. But the situation is completely different if the dimension of X is infinite. In this sense, the infinite case has been divided into two parts, when X is of separable infinite dimension and when X is of non-septable infinite dimension. In the case of separable spaces, Phelps [8] was able to establish the following generalization of Rademacher's Theorem [11] to this type of spaces.

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Theorem 1.1 (Phelps). Let X be separable and Y have the RNP. Then every locally Lipschitz function from an open set Ω in X into Y is Gâteaux differentiable outside a Gauss null set.

On the other hand, considering that a norm is an example of a Lipschitz function, the study of the differentiability of norms in Banach spaces, has been a very active field in the last decades. Particularly, in the Potapov and Sukachev's work [10] answer to a question in the theory of Schatten- Von Neumann ideals of compact operators is whether their norms have same differentiability properties as the norms of their commutative counterparts. In this same period of time, Lindenstrauss et al. [6] prove that a real valued Lipschitz function on an Asplund space has points of Fréchet differentiability and Dore and Maleva [2] have studied on the calculation of a F-differentiability set of Lipschitz functions on Banach spaces.

Moreover, the study of the differentiability of norms in Banach spaces, has applications to the study of diffusion equations, elliptic equations in infinite dimension and stochastic control. Particularly Goldys and Gozzi [5], use the Phelps Theorem to study the Hamilton-Jacobi-Bellman equation related to the optimal control of a stochastic semilinear equation over a Hilbert H space, obtaining optimal feedback for controlled stochastic delay equations. It has also had an impact on the investigation of systems research second-order quasi-linear ellipticals with non-linear boundary conditions; as shown by Shi and Wang [15] in their work on the abstract global bifurcation in elliptical systems, through the use of the properties of the admissible norms proposed by Restrepo [12].

All of the above attracts attention to research on the properties of these types of derivatives defined in functions over spaces in infinite dimension and even more in describing which and what properties the sets of differentiability associated with these functions have.

In this direction this paper, first we will show that if two spaces X, Y are isomorphic then X is G-Asplund if and only if Y is G -Asplund, using this to show that the space $\mathcal{C}(T)$ of the continuous functions on the real compact and non-countable metric space topological space T, is a G-Asplund space. We will explicitly find that the measure for the complement of the F-differentiability set with the norm of the supreme in $\ell^\infty(\mathbb{R})$ is zero. In addition, for a measure space $(\Omega,\mathcal{A},\mu)$, we will show that for a norm defined in $L^\infty(\Omega)$ space, there is no Fréchet derivative at any element belonging to this Banach space. Finally, we will prove that the seminorm $\alpha(f) = |f|([a,b])$ and the norm $\varphi(f) = |f(a)| + |f|([a,b])$ defined on space $BV[a,b]$ is not F-differentiable at any element of these space.

2 Preliminaries

Schirotzek [14], using the definition and properties of bornologies, shows that to be F-differentiable implies to be G-differentiable. The previous implication does not occur in the opposite direction, in the following example this situation can be observed.

Example 1. Let $g : L^1[0,\pi] \to \mathbb{R}$ the function defined by

$$x \to g(x) := \int_0^\pi sen(x(t))dt.$$

This function is G-differentiable but it is never F-differentiable in $x \in L^1[0,\pi]$ (see Gieraltowska [4]).

Let's assume that a set is called set is called G_δ *set* if it can be expressed as a countable intersection of open sets. This type of set it is fundamental in the definition of the spaces *F-Asplund or G-Asplund*, since these respectively are Banach spaces in which all convex and continuous function defined in an open and convex subset is F-differentiable or G-differentiable in a dense and G_δ set. The following theorem that appears in Phelps [9] gives us an idea of how to identify some examples of G-Asplund spaces.

Theorem 2.1 (Mazur). All separable Banach space is a G-Asplund space.

Continuing with the separability hypothesis, Phelps [9] demonstrates the following equivalence taking into account the topological duals of a Banach space.

Theorem 2.2. Let X be a separable Banach space. X is a F -Asplund space if and only if its dual topological X' is separable.

On the other hand, a Banach space X has the *Radon-Nikodým Property* (RNP) if for any finite measure space $(\Omega, \mathcal{F}, \mu)$, any σ-aditive of bounded variation and μ-continuous function $\alpha : \mathcal{F} \to X$ admits a integral representation of the form

$$\alpha(A) = \int_A g(\omega) d\mu(\omega),$$

for all $A \in \mathcal{F}$ and $g \in L^1(\mu, X)$.

Example 2. Every Hilbert space has the RNP.

In the works of Namioka, Phelps [7] and Stegall [16], the following equivalence that relates the two previous concepts is demonstrated.

Theorem 2.3. X is an F-Asplund space if and only if X' has the RNP.

If X is a Banach space and X' is its topological dual. A *Gaussian measure* μ in a Banach space X is a defined probabilistic measure on the borelians of X, such that $x'(\mu)$ is a Gaussian measure in $\mathbb{R}$ for all $x' \in X'$ non-zero. In addition, it is said that a Gaussian measure μ is *not degenerate*, if it has finite variance not null.

Example 3. Let Dirac's δ_α measure concentrated on a value α, it satisfies $\hat{\delta}_\alpha(x) = e^{i\alpha x}$. Then $\sigma = 0$, of which δ_α is a degenerated Gaussian measure.

Given the above we can say that a borelian set N in a Banach space X is called *Gauss null* if $\mu(N) = 0$ for any non-degenerate Gaussian measure μ on X.

Example 4. If X is a separable Banach space, then the set of elements $x \in X$ where a φ norm defined over X is not G-differentiable, it is by Theorem 1.1, a null Gauss set.

3 Some properties of some differentiability sets

Let $\mathcal{C}(T)$ the Banach space of the continuous functions on the compact and Hausdorff topological space T in $\mathbb{R}$, with the norm defined by $\varphi(x) = \sup_{t\in T} |x(t)|$. In Deville [1] and Fabian [3] it is shown that the norm φ of $\mathcal{C}(T)$ is G-differentiable at $x \in \mathcal{C}(T)$ if and only if it reaches the supreme at a single point $t_0 \in T$ and is F-differentiable at $x \in \mathcal{C}(T)$ if and only if $\varphi(x)$ reaches the supreme at a single isolated point $t_0 \in T$. It is also proven that

$$\partial_F \varphi(x)(h) = sig(x(t_0))h(t_0).$$

Now, a lemma will be presented which will be useful in the demonstration of the property that has the previous differentiability sets.

Lemma 3.1. Let X and Y spaces of Banach isomorphic. X is F-Asplund (G-Asplund) if and only if Y is F-Asplund (G-Asplund).

Proof . Since X, Y are isomorphic, then there is a bicontinuous and bijective function $\varphi : X \to Y$ such that:

$$\varphi(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \varphi(x_1) + \lambda_2 \varphi(x_2) \quad \text{and} \quad \varphi^{-1}(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 \varphi^{-1}(y_1) + \lambda_2 \varphi^{-1}(y_2),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Now, the equivalence corresponding to spaces F- Asplund will be demonstrated. If Y is F-Asplund, then for any convex function and g continues on an open and convex Ω' it is satisfied that for $\tilde{u} \in \mathcal{L}(Y, \mathbb{R})$

$$\lim_{h' \to 0} \frac{\|g(b + h') - g(b) - \tilde{u}(h')\|}{\|h'\|} = 0,$$

where b belongs to a dense set and G_δ contained in Ω', then there is a $a \in \Omega$ with $\varphi(\Omega) = \Omega'$ such that $\varphi(a) = b$, in the same way there is a single $h \in X$ such that $\varphi(h) = h'$, $h' \in Y$ and also

$$\lim_{h'\to 0}\frac{\|g(b+h')-g(b)-\tilde{u}(h')\|}{\|h'\|}=\lim_{\varphi(h)\to 0}\frac{\|g(\varphi(a)+\varphi(h))-g(\varphi(a))-\tilde{u}(\varphi(h))\|}{\|\varphi(h)\|}$$
$$=\lim_{h\to 0}\frac{\|g(\varphi(a+h))-g(\varphi(a))-\tilde{u}(\varphi(h))\|}{\|h\|}=\lim_{h\to 0}\frac{\|f(a+h)-f(a)-u(h)\|}{\|h\|},$$

where $f=g\circ\varphi$ is clearly a continuous convex function on the open and convex set Ω and $u=\tilde{u}\circ\varphi\in\mathcal{L}(X,\mathbb{R})$, such that:

$$\lim_{h\to 0}\frac{\|f(a+h)-f(a)-u(h)\|}{\|h\|}=0,$$

then f is F-derivable on $a\in\Omega$.

On the other hand, if X is F-Asplund then for any convex and continuous function f defined on an open and convex Ω and a on a dense subset and G_δ of Ω it is satisfied that

$$\lim_{h\to 0}\frac{\|f(a+h)-f(a)-u(h)\|}{\|h\|}=0,$$

where $h\in\Omega$ and in turn the following equality is valid

$$\lim_{h\to 0}\frac{\|f(a+h)-f(a)-u(h)\|}{\|h\|}=\lim_{h'\to 0}\frac{\|f(\varphi^{-1}(b+h'))-f(\varphi^{-1}(b))-u(\varphi^{-1}(h'))\|}{\|h'\|}$$
$$=\lim_{h'\to 0}\frac{\|g(b+h')-g(b)-\tilde{u}(h')\|}{\|h'\|}.$$

That is, for $g=f\circ\varphi^{-1}$ and $\tilde{u}=u\circ\varphi^{-1}\in\mathcal{L}(X,\mathbb{R})$ it is true that g is a convex and continuous function on the open and convex Ω', which is F-derivable on $b\in\Omega'$.

This demonstrates that having a topological isomorphism between X and Y Banach spaces implies that X is F-Asplund if and only if Y is F-Asplund. The argument for the corresponding demonstration for the G-Asplund part follows similar guidelines to what was done previously and is therefore omitted. □

Since it is known that the topological dual space of the Banach space $\mathcal{C}(T)$ is not a separable space. Then is clear from the Theorem 2.2 that the space $\mathcal{C}(T)$ is not a space F-Asplund. Therefore, the following result is limited to showing that $\mathcal{C}(T)$ is G -Asplund.

Theorem 3.2. Let T be a compact and non-countable metric space. If $\varphi(x)=\sup_{t\in T}|x(t)|$ is the norm in $\mathcal{C}(T)$, then $\mathcal{C}(T)$ is G-Asplund.

Proof . Since T is a compact and non-countable metric space, then from the Milyutin Theorem that appears in Semadeni [13], it is concluded that $\mathcal{C}(T)$ is isomorphic to $\mathcal{C}[0,1]$. On the other hand, it is known that $\mathcal{C}[0,1]$ is separable and for Theorem 2.1 is G-Asplund. Therefore, the Lemma 3.1 it is concluded that $\mathcal{C}(T)$, with T is a compact metric space and non-countable is G-Asplund. □

Remark 3.3. The set of G-differentiability of the norm φ defined for the Banach space $\mathcal{C}([a,b])$ is not an open set. In fact, for $[a,b]=[0,2]$, $0<\epsilon<1/4$ and the functions defined as:

$$f(x)=\begin{cases} x, & \text{if } 0\le x<1,\\ 2-x, & \text{if } 1\le x\le 2.\end{cases}\qquad g(x)=\begin{cases}\frac{x}{1-\epsilon}, & \text{if } 0\le x<1-\epsilon,\\ 1, & \text{if } 1-\epsilon\le x\le 1+\epsilon,\\ \frac{2-x}{1-\epsilon}, & \text{if } 1+\epsilon<x\le 2.\end{cases}$$

It is verified that $f\in B$, because this function is continuous and takes its supreme only at the point $x=1$. Also, it is clear that $\varphi(f-g)<\epsilon$ and finally $g\notin B$, since g reaches the supreme on the interval $[1-\epsilon,1+\epsilon]$.

Explicitly finding sets of differentiability of norms is not limited only to the cases of separable Banach spaces. For example, the book by Deville and Godefroy [1] shows that for $\ell^\infty(\mathbb{R})$ the Banach space of the bounded sequences with the norm $\varphi(x) = \sup_n |x_n|$, the set of F-differentiability of the norm φ is defined as the set of sequences $x \in \ell^\infty(\mathbb{R})$ for which there are $p \in \mathbb{N}$ and $\epsilon > 0$, such that $|x_k| < |x_p| - \epsilon$, for all $k \neq p$. If $\mathcal{B} \subseteq \ell^\infty(\mathbb{R})$ is the set of F-differentiability of the norm φ. Then we state and prove the following result.

Theorem 3.4. The complement of $\mathcal{B}$ is a set of null Gaussian measure.

Proof . First of all, the existence of Gaussian measures in $\ell^\infty(\mathbb{R})$ is guaranteed, since Vakhania states in Section 2.4.3 of [17] that μ Gaussian measures exist over $\mathbb{R}^{\mathbb{N}}$, which satisfies $\mu(\ell^\infty(\mathbb{R})) = 1$. On the other hand, for $x = (x_1, \cdots, x_t, \cdots) \in \mathcal{B}$ and $1 \leq p \leq t \in \mathbb{N}$. If $g_t^p : \mathcal{B} \to \mathbb{R}^t$ is defined by

$$g_t^p(x_1, \cdots, x_t, \cdots) = (x_1, \cdots, x_p, \cdots, x_t), \text{ with } x_p \text{ in the } p\text{-th coordinate of } \mathbb{R}^t.$$

Then g_t^p is continuous. In fact, for an element y of the open ball $B(x, \delta)$, with $x, y \in \mathcal{B}$ and $\|(x_1, \cdots, x_t)\|_{\mathbb{R}^t} = |x_1| + \cdots + |x_t|$, is satisfied:

$$\begin{aligned} \|g_t^p(x) - g_t^p(y)\|_{\mathbb{R}^t} &= \|(x_1 - y_1, \cdots, x_p - y_p, \cdots, x_t - y_t)\|_{\mathbb{R}^t} \\ &= |x_1 - y_1| + \cdots + |x_p - y_p| + \cdots + |x_t - y_t| \\ &\leq t \sup_n |x_n - y_n| = t\|x - y\|_{\ell^\infty(\mathbb{R})}. \end{aligned}$$

Which implies that for all $\epsilon > 0$, there is a $\delta = \frac{\epsilon}{t}$ such that $\|x - y\|_{\ell^\infty(\mathbb{R})} < \delta$ implies $\|g_t^p(x) - g_t^p(y)\|_{\mathbb{R}^t} < \epsilon$. Therefore, for each Gaussian measure μ_t in each $\mathbb{R}^t$, $A \subseteq \mathcal{B}$ and for all $1 \leq p \leq t \in \mathbb{N}$ It is true that $\mu(A) = \mu_t(g_t^p(A))$.

Second, if we take $t = 2$ and for all $\epsilon > 0$ we define

$$B_2 = \bigcup_{p=1}^{2} \{g_2^p(x) : |x_k| < |x_p| - \epsilon,\ 1 \leq k \leq 2 \text{ and } k \neq p\},$$

it is clear that

$$B_2 = \mathbb{R}^2 - \{(x_1, x_2) : x_2 = x_1,\ x_2 = -x_1\}.$$

Then $\mu_2(B_2) = 1$ and therefore $B_2^c = \{(x_1, x_2) : x_2 = x_1,\ x_2 = -x_1\}$ satisfies

$$\mu_2(B_2^c) = 0.$$

Now, if the previous result is taken true until $t = n$. That is, for all $2 \leq t \leq n$, $\mu_t(B_t) = 1$, to

$$B_t = \bigcup_{p=1}^{t} \{g_t^p(x) : |x_k| < |x_p| - \epsilon,\ 1 \leq k \leq t \text{ and } k \neq p\}.$$

Later

$$B_{n+1} = \bigcup_{p=1}^{n+1} \{g_{n+1}^p(x) : |x_k| < |x_p| - \epsilon,\ 1 \leq k \leq n+1 \text{ and } k \neq p\} \supseteq B_{n-1} \times B_2.$$

Getting that for all $x_p \in \mathbb{R}$, inequality is satisfied

$$\mu_{n+1}(B_{n+1}) \geq \mu_{n+2}(B_{n-1} \times B_2) = \mu_{n-1}(B_{n-1}) \times \mu_2(B_2) = 1.$$

Consequently $\mu_{n+1}(B_{n+1}) = 1$ and $\mu_{n+1}(B_{n+1}^c) = 0$, of which for all $t \geq 2$, is true that

$$\mu_t(B_t^c) = 0.$$

In addition, if we take into account the definition of $\mathcal{B}$ and that $B_t \times \ell^{\infty-\{1,\cdots,t\}}(\mathbb{R})$ the set of sequences in $\ell^\infty(\mathbb{R})$ with the first t components in B_t, then

$$B_t \times \ell^{\infty-\{1,\cdots,t\}}(\mathbb{R}) \subseteq \mathcal{B}$$

is satisfied, which is equivalent to $\mathcal{B}^c \subseteq \left[B_t \times \ell^{\infty-\{1,\cdots,t\}}(\mathbb{R})\right]^c$. Hence, from what is shown at the beginning of this demonstration it is true that

$$\mu(\mathcal{B}^c) \leq \mu\left(\left[B_t \times \ell^{\infty-\{1,\cdots,t\}}(\mathbb{R})\right]^c\right) = \mu_t\left(g_t^p\left(\left[B_t \times \ell^{\infty-\{1,\cdots,t\}}(\mathbb{R})\right]^c\right)\right) \leq \mu_t(B_t^c).$$

Then, $\mu(\mathcal{B}^c) \leq \mu_t(B_t^c)$ for all $t \geq 2$. That is, $\mu(\mathcal{B}^c) = 0$. $\square$

The previous theorem, makes us think about the possibility of extending this result to a Banach space that generalizes to $\ell^\infty(\mathbb{R})$ and in this sense we will use $L^\infty(\Omega)$. But as will be shown later, it cannot be concluded that the measure of the complement of the set of G-differentiality of the norm associated with $L^\infty(\Omega)$ is zero.

Let $(\Omega,\mathcal{A},\mu)$ be a measure space, with μ a nonatomic measure. We will consider $L^\infty(\Omega)$, as the Banach space of the class of measurable functions and essentially bounded $g : \Omega \to \mathbb{R}$. That is, every representative f of g meets condition

$$\inf\{M \in \mathbb{R} : |f(x)| \leq M, c.t.p.\} < \infty$$

. In the following theorem we will show that unlike the space $\ell^\infty(\mathbb{R})$, the norm defined for $L^\infty(\Omega)$ is not F-differentiable in any element of space.

Theorem 3.5. In $L^\infty(\Omega)$ the norm $\varphi(f) = \inf\{M \in \mathbb{R} : |f(x)| \leq M,\ a.s.\}$ is not F-differentiable in any element of space.

Proof . Here we will use the implication in which if the norm φ is not G-differentiable in $g \in L^\infty(\Omega)$ then φ is not F-differentiable in $g \in L^\infty(\Omega)$. Let $g \in L^\infty(\Omega)$ and f be a representative of g. If $\varphi(f) = L > 0$ and $h \in L^\infty(\Omega)$ we define as $h(x) = 1$ for all $x \in \Omega$. Then for $|t| < L$:

$$\varphi(f + th) = \begin{cases} L + t, & \text{if } t > 0, \\ L, & \text{if } t < 0. \end{cases}$$

Which implies that for $|t| < L$ we get

$$\frac{\varphi(f+th)-\varphi(f)}{t} = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases}$$

From which we conclude $\partial_+\varphi(f, h) = 1 \neq 0 = \partial_-\varphi(f, h)$.

On the other hand, if g has a representative f such that $\varphi(f) = 0$. Then it is clear from the definition of the norm φ that $f = 0$ a.s. which implies that for all $h \in L^\infty(\mathbb{R})$, it is true:

$$\lim_{t\to 0^+} \frac{\varphi(f + th) - \varphi(f)}{t} = \varphi(h) \neq -\varphi(h) = \lim_{t\to 0^-} \frac{\varphi(f + th) - \varphi(f)}{t},$$

showing in this case the non-existence of the G-derived from the φ norm. Therefore, the φ norm is not G-differentiable in any $g \in L^\infty(\Omega)$. $\square$

Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable space, E a Banach space with norm $\|\cdot\|$, $[a, b] \subset \mathbb{R}$, $P_n = \{t_k : 0 \leq k \leq n\} \subset [a, b]$ a finite collection of points that satisfy inequality $t_0 < t_1 < \cdots < t_n$ and $\alpha : \mathcal{B}(\mathbb{R}) \to E$ a φ-additive function. The variation of α in $[a, b] \in \mathcal{B}(\mathbb{R})$ is the number

$$|\alpha|([a, b]) = \sup\left\{\sum_{k=1}^{n} \|\alpha(t_k) - \alpha(t_{k-1})\|\right\},$$

and α is of bounded variation if $|\alpha| < \infty$. The application $|\alpha| : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ inherits the φ-addivity or σ-addivity of the α function and $\|\alpha([a, b])\| \le |\alpha|([a, b])$. In addition, a bounded variation function $\alpha : [a, b] \to \mathbb{C}$, is normal if it is continuous on the left at all points. That is, if

$$\lim_{y \to x^-} \alpha(y) = \alpha(x), \quad \text{for all } x > a.$$

Taking into account that the space of the normal functions of bounded variation $NBV[a, b]$ on $[a, b]$ is the dual space of Banach space $\mathcal{C}[a, b]$ we will show that a dual space does not inherit properties of G-differentiability that the original space possesses.

Now, we will first show some of the results obtained for the seminorm and norm of Banach space not separable from the bounded variation functions $BV[a, b]$ in the interval $[a, b]$. In the following theorems we will use the contrapositive of the statement written at the beginning of the preliminaries of this document. That is, if a function is not G-differentiable in a x value this implies that that same function is not F-differentiable in that same value x.

Theorem 3.6. If $BV[a, b]$ is the Banach space of real value and variation bounded functions with seminorm $\alpha(f) = |f|([a, b])$. Then α is not F-differentiable at any element of $BV[a, b]$.

Proof . Let $f \in BV[a, b]$ constant and $h \in BV[a, b]$ be a non-constant function. Then, it is clear that $|f|([a, b]) = 0$, $|h|([a, b]) \ne 0$ and

$$\frac{\alpha(f + th) - \alpha(f)}{t} = \frac{|f + th|([a, b]) - |f|([a, b])}{t} = \frac{|t|}{t}|h|([a, b]).$$

Therefore, there is no G-derived from α over f, since

$$\lim_{t \to 0^-} \frac{\alpha(f + th) - \alpha(f)}{t} = -|h|([a, b]) \ne |h|([a, b]) = \lim_{t \to 0^+} \frac{\alpha(f + th) - \alpha(f)}{t}.$$

Now, let f be non-constant and $h = |f|([a, b])I_{[a, \frac{a+b}{2}]} + 0I_{(\frac{a+b}{2}, b]}$. Then $h \in BV[a, b]$, and for $t > 0$ it is satisfied

$$\begin{aligned} \frac{\alpha(f + th) - \alpha(f)}{t} &= \frac{|f + th|([a, b]) - |f|([a, b])}{t} \le \frac{|f|([a, b]) + |t||h|([a, b]) - |f|([a, b])}{t} \\ &= \frac{|f|([a, b]) + t|f|([a, b]) - |f|([a, b])}{t} = |f|([a, b]) < \infty. \end{aligned}$$

If $t < 0$

$$\begin{aligned} \frac{\alpha(f + th) - \alpha(f)}{t} &= \frac{|f + th|([a, b]) - |f|([a, b])}{t} > \frac{t|h|([a, b]) - |f|([a, b])}{t} \\ &= \frac{t|f|([a, b]) - |f|([a, b])}{t} = \left(\frac{t - 1}{t}\right)|f|([a, b]), \end{aligned}$$

where the order of the previous inequality is strict, since f is not a constant function. Then $|f + th|([a, b]) > 0$ and for $t < 0$ it is clear that $t|h|([a, b]) < 0$. Therefore, we conclude that

$$\lim_{t \to 0^+} \frac{\alpha(f + th) - \alpha(f)}{t} < \lim_{t \to 0^-} \frac{\alpha(f + th) - \alpha(f)}{t} = \infty,$$

where the limit of the right side is obtained from $\lim_{t \to 0^-} \frac{t-1}{t} = \lim_{t \to 0^-} 1 - \frac{1}{t} = +\infty$. Then α is not G-differentiable for every function f not constant. All of which implies that the seminorm α is not G-differentiable in any $f \in BV[a, b]$ and therefore is not F-differentiable in any $f \in BV[a, b]$. □

Remark 3.7. The function $\alpha(f) = |f|([a, b])$ is Lipschitz continues, since

$$|\alpha(f) - \alpha(g)| = ||f|([a, b]) - |g|([a, b])| \le |f - g|([a, b]).$$

Then $\alpha : BV[a,b] \to \mathbb{R}$ is a example of Lipschitz function defined from a non-separable space to another with the RNP, which is not G-derivable in any element belonging to $BV[a,b]$. Which demonstrates that the Theorem 1.1 cannot be extended to locally Lipschitzian functions with a non-separable domain.

Theorem 3.8. If $BV[a,b]$ is the Banach space of the real variation bounded functions with the norm $\varphi(f) = |f(a)| + |f|([a,b])$. Then φ is not F-differentiable at any element of $BV[a,b]$.

Proof . If $f \in BV[a,b]$ equals a constant $k \in \mathbb{R}$ and $h = 0I_{[a,\frac{a+b}{2}]} + 1I_{(\frac{a+b}{2},b]}$ then it is satisfied

$$\begin{aligned}\frac{\varphi(f+th)-\varphi(f)}{t} &= \frac{|(k+th)(a)| + |k+th|([a,b]) - (|k| + |k|([a,b])}{t}\\ &= \frac{|k+th(a)| + |t||h|([a,b]) - |k|}{t}\\ &= \frac{|k+t\cdot 0| + |t|\cdot 1 - |k|}{t} = \frac{|t|}{t}.\end{aligned}$$

Then there is no G-derived from φ over f, since

$$\lim_{t\to 0^-} \frac{\varphi(f+th)-\varphi(f)}{t} = -1 \neq 1 = \lim_{t\to 0^+} \frac{\varphi(f+th)-\varphi(f)}{t}.$$

Now, if $f \in BV[a,b]$ is not a constant and $h = 0I_{[a,\frac{a+b}{2}]} + |f|([a,b])I_{(\frac{a+b}{2},b]}$, we obtain:

$$\begin{aligned}\frac{\varphi(f+th)-\varphi(f)}{t} &= \frac{|(f+th)(a)| + |f+th|([a,b]) - (|f(a)| + |f|([a,b])}{t}\\ &= \frac{|f(a)| + |f+th|([a,b]) - (|f(a)| + |f|([a,b])}{t}.\end{aligned}$$

If $t > 0$, then

$$\frac{\varphi(f+th)-\varphi(f)}{t} \leq \frac{|f(a)| + (1+t)|f|([a,b]) - |f(a)| - |f|([a,b])}{t} = |f|([a,b]).$$

If $t < 0$, then

$$\frac{\varphi(f+th)-\varphi(f)}{t} > \frac{|f(a)| + t|f|([a,b]) - |f(a)| - |f|([a,b])}{t} = \left(\frac{t-1}{t}\right)|f|([a,b]),$$

where the order of the previous inequality is strict, since f is not a constant function. Then $|f+th|([a,b]) > 0$, and for $t < 0$ it is evident that $t|h|([a,b]) < 0$. Therefore, for $h = 0I_{[a,\frac{a+b}{2}]} + |f|([a,b])I_{(\frac{a+b}{2},b]}$, we obtain

$$\lim_{t\to 0^+} \frac{\varphi(f+th)-\varphi(f)}{t} < \lim_{t\to 0^-} \frac{\varphi(f+th)-\varphi(f)}{t} = +\infty,$$

where the limit on the right side is obtained from $\lim_{t\to 0^-} \frac{t-1}{t} = \lim_{t\to 0^-} 1 - \frac{1}{t} = +\infty$. Which implies that φ is not G-differentiable for every function f not constant. Finally, we can be concluded that the φ norm is not G-differentiable in any $f \in BV[a,b]$ and therefore is not F-differentiable in any $f \in BV[a,b]$. □

Finally, taking into account that $NBV[a,b]$ is a subspace of $BV[a,b]$ and that this subspace inherits the norm of space $BV[a,b]$. Then, the following corollary is a direct product of the previous results.

Corollary 3.9. If $NBV[a,b]$ is the Banach space of the normal real variation bounded functions with the norm $\varphi(f) = |f(a)| + |f|([a,b])$. Then φ is not F-differentiable at any element of $NBV[a,b]$.

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