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CUBIC-QUARTIC FUNCTIONAL EQUATIONS IN FUZZY NORMED SPACES

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ABSTRACT. In this paper, we investigate the generalized Hyers–Ulam stability of the functional equation

$$4(f(3x+y)+f(3x-y)) = -12(f(x+y)+f(x-y))$$
$$+12(f(2x+y)+f(2x-y)) - 8f(y) - 192f(x) + f(2y) + 30f(2x).$$

in fuzzy normed spaces..

1. INTRODUCTION AND PRELIMINARIES

In 1984, Katsaras [12] defined a fuzzy norm on a linear space and at the same year Wu and Fang [35] also introduced a notion of fuzzy normed space. In [5], Biswas defined and studied fuzzy inner product spaces. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [4, 8, 14, 30, 31, 32, 34]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [13]. In 2003, Bag and Samanta [2] modified the definition of Cheng and Mordeson [6] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms. Following [2], we give the notion of a fuzzy norm.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:
(N_1) $N(x, a) = 0$ for $a \leq 0$;
(N_2) $x = 0$ if and only if $N(x, a) = 1$ for all $a > 0$;
(N_3) $N(ax, b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;
(N_4) $N(x + y, a + b) \geq min\{N(x, a), N(y, b)\}$;
(N_5) $N(x, .)$ is non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x, a) = 1$;
(N_6) For $x \neq 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.
The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, a)$ as the

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truth value of the statement the norm of x is less than or equal to the real number a.

Example 1.1. Let $(X, \|.\|)$ be a normed linear space. Then

$$N(x,a) = \begin{cases} \frac{a}{a+\|x\|}, & a > 0\ , x \in X, \\ 0, & a \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.2. Let (X, N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N\text{-}lim_{n\to\infty}\, x_n = x$.

Definition 1.3. A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $a > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \epsilon$.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

The stability problem of functional equations originated from a question of Ulam [33] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [10] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}-$linear. In 1950, Aoki [1] generalized Hyers' theorem for approximately additive mappings. In 1978, Th. M. Rassias [28] proved the following theorem.

Theorem 1.4. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$*. If* $p < 0$ *then inequality* (1.1) *holds for all* $x, y \neq 0$*, and* (1.2) *for* $x \neq 0$*. Also, if the function* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *into* E' *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in E$*, then* T *is* $\mathbb{R}-$*linear.*

In 1991, Gajda [9] answered the question for the case $p > 1$, which was raised by Th.M. Rassias. This new concept is known as *generalized Hyers–Ulam stability* of functional equations. On the other hand, J.M. Rassias [23], generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms (see also [22, 24, 25]).

Jun and Kim [11] introduced the following functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.3}$$

and they established the general solution and the generalized Hyers–Ulam stability for the functional equation (1.3). The function $f(x) = x^3$ satisfies the functional equation (1.3), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic mapping. Jun and Kim proved that a mapping f between real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique mapping $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. The stability of quartic functional equation was introduced by J. M. Rassias [26, 27], and was employed by W. Park [21] and others such that:

$$f(x+2y)+f(x-2y)=4(f(x+y)+f(x-y))+24f(y)-6f(x). \tag{1.4}$$

We deal with the following functional equation deriving from quartic and cubic functions:

$$\begin{aligned}4(f(3x+y)+f(3x-y)) = &-12(f(x+y)+f(x-y))+12(f(2x+y)+f(2x-y))\\ &-8f(y)-192f(x)+f(2y)+30f(2x).\end{aligned} \tag{1.5}$$

It is easy to see that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^4+bx^3$ is a solution of the functional equation (1.5). M. Eshaghi Gordji, A. Ebadian and S. Zolfaghari [7] investigated the general solution and the generalized Hyers–Ulam stability and Ulam–Gavruta–Rassias stability of the functional equation (1.5).

The generalized Hyers–Ulam stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [15]-[20] and [29].

In the present paper, we investigate the generalized Hyers–Ulam stability for functional equation (1.5) in fuzzy normed spaces.

2. Main results

Throughout this section, assume that X, (Z, N') and (Y, N) are linear space, fuzzy normed space and fuzzy Banach space, respectively. For convenience, we use the following abbreviation for a given mapping $f : X \to Y$:

$$\begin{aligned}D_f(x,y) = {}&4[f(3x+y)+f(3x-y)]-12[f(2x+y)+f(2x-y)]+12[f(x+y)+f(x-y)]\\ &-f(2y)+8f(y)-30f(2x)+192f(x)\end{aligned}$$

for all $x, y \in X$.

We now investigate the generalized Hyers–Ulam stability problem for functional equation (1.5).

Theorem 2.1. *Let $\beta \in \{1, -1\}$ be fixed and let $\varphi_1 : X \times X \to Z$ be a mapping such that for some $\alpha > 0$ with $(\frac{\alpha}{16})^\beta < 1$*

$$N^{'}(\varphi_1(0, 2^\beta x), a) \geq N^{'}(\alpha^\beta \varphi_1(0, x), a) \tag{2.1}$$

for all $x \in X$ and all $a > 0$, and $\lim_{n\to\infty} N^{'}(\varphi_1(2^{\beta n}x, 2^{\beta n}y), 16^{\beta n}a) = 1$ for all $x, y \in X$ and all $a > 0$. Suppose that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_f(x, y), a) \geq N^{'}(\varphi_1(x, y), a) \tag{2.2}$$

for all $a > 0$ and all $x, y \in X$. Then the limit

$$Q(x) = N\text{-}\lim_{n\to\infty} \frac{1}{16^{\beta n}} f(2^{\beta n}x)$$

exists for all $x \in X$ and the mapping $Q : X \to Y$ is the unique quartic mapping satisfying

$$N(f(x) - Q(x), a) \geq N^{'}(\varphi_1(0, x), a|16 - \alpha|) \tag{2.3}$$

for all $x \in X$ and all $a > 0$.

Proof. Let $\beta = 1$. Letting $x = 0$ in (2.2), we get

$$N(f(2y) - 16f(y), a) \geq N^{'}(\varphi_1(0, y), a) \tag{2.4}$$

for all $y \in X$ and all $a > 0$. Replacing y by x in (2.4), we get

$$N(f(2x) - 16f(x), a) \geq N^{'}(\varphi_1(0, x), a) \tag{2.5}$$

for all $x \in X$ and all $a > 0$. Replacing x by $2^n x$ in (2.5), we obtain

$$N(\frac{f(2^{n+1}x)}{16} - f(2^n x), \frac{a}{16}) \geq N^{'}(\varphi_1(0, 2^n x), a) \tag{2.6}$$

for all $x \in X$ and all $a > 0$. Using (2.1), we get

$$N(\frac{f(2^{n+1}x)}{16} - f(2^n x), \frac{a}{16}) \geq N^{'}(\varphi_1(0, x), \frac{a}{\alpha^n}) \tag{2.7}$$

for all $x \in X$ and all $a > 0$. Replacing a by $\alpha^n a$ in (2.7), we get

$$N(\frac{f(2^{n+1}x)}{16^{n+1}} - \frac{f(2^n x)}{16^n}, \frac{a\alpha^n}{16(16^n)}) \geq N^{'}(\varphi_1(0, x), a) \tag{2.8}$$

for all $x \in X$ and all $a > 0$. It follows from $\frac{f(2^n x)}{16^n} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{16^{i+1}} - \frac{f(2^i x)}{16^i}$ and (2.8) that

$$\begin{aligned} N(\frac{f(2^n x)}{16^n} - f(x), \sum_{i=0}^{n-1} \frac{a\alpha^i}{16(16^i)}) &\geq min\{N(\frac{f(2^{i+1}x)}{16^{i+1}} - \frac{f(2^i x)}{16^i}, \frac{a\alpha^i}{16(16^i)}) : i = 0, 1, ..., n-1\} \\ &\geq N^{'}(\varphi_1(0, x), a) \end{aligned} \tag{2.9}$$

for all $x \in X$ and all $a > 0$. Replacing x with $2^m x$ in (2.9), we obtain

$$N(\frac{f(2^{n+m}x)}{16^{n+m}} - \frac{f(2^m x)}{16^m}, \sum_{i=0}^{n-1} \frac{a\alpha^i}{16(16^{i+m})}) \geq N^{'}(\varphi_1(0, 2^m x), a) \geq N^{'}(\varphi_1(0, x), \frac{a}{\alpha^m}),$$

and so

$$N(\frac{f(2^{n+m}x)}{16^{n+m}}-\frac{f(2^m x)}{16^m}, \sum_{i=m}^{n+m-1}\frac{a\alpha^i}{16(16^i)}) \geq N^{'}(\varphi_1(0,x), a)$$

for all $x \in X$, all $a > 0$ and all $m, n \geq 0$. Hence

$$N(\frac{f(2^{n+m}x)}{16^{n+m}} - \frac{f(2^m x)}{16^m}, a) \geq N^{'}(\varphi_1(0,x), \frac{a}{\sum_{i=m}^{n+m-1}\frac{\alpha^i}{16(16^i)}}) \tag{2.10}$$

for all $x \in X$, all $a > 0$ and all $m, n \geq 0$. Since $0 < \alpha < 16$ and $\sum_{i=0}^{\infty}(\frac{\alpha}{16})^i < \infty$, the Cauchy criterion for convergence and (N_5) imply that $\{\frac{f(2^n x)}{16^n}\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $Q(x) \in Y$. So one can define the mapping $Q : X \to Y$ by $Q(x) := N\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{16^n}$ for all $x \in X$.

The evenness of f implies that Q is even. Letting $m = 0$ in (2.10), we get

$$N(\frac{f(2^n x)}{16^n} - f(x), a) \geq N^{'}(\varphi_1(0,x), \frac{a}{\sum_{i=0}^{n-1}\frac{\alpha^i}{16(16^i)}}) \tag{2.11}$$

for all $x \in X$ and all $a > 0$. Taking the limit as $n \to \infty$ and using (N_6) we get

$$N(f(x) - Q(x), a) \geq N^{'}(\varphi_1(0,x), a(16-\alpha))$$

for all $x \in X$ and $a > 0$.

Now we claim that Q is quartic. Replacing x, y by $2^n x, 2^n y$ in (2.2), respectively, we get

$$N(\frac{1}{16^n}D_f(2^n x, 2^n y), a) \geq N^{'}(\varphi_1(2^n x, 2^n y), 16^n a)$$

for all $x, y \in X$ and all $a > 0$. Since $\lim_{n\to\infty} N^{'}(\varphi_1(2^n x, 2^n y), 16^n a) = 1$ and then by Corollary 2.2 of [7] we get that the mapping $Q : X \to Y$ is quartic.

To prove the uniqueness of Q, let $Q^{'} : X \to Y$ be another quartic mapping satisfying (2.3). Fix $x \in X$. Clearly $Q(2^n x) = 16^n Q(x)$ and $Q^{'}(2^n x) = 16^n Q^{'}(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. It follows from (2.3) that

$$\begin{aligned} N(Q(x) - Q^{'}(x), a) &= N(\frac{Q(2^n x)}{16^n} - \frac{Q^{'}(2^n x)}{16^n}, a) \\ &\geq min\{N(\frac{Q(2^n x)}{16^n} - \frac{f(2^n x)}{16^n}, \frac{a}{2}), N(\frac{f(2^n x)}{16^n} - \frac{Q^{'}(2^n x)}{16^n}, \frac{a}{2})\} \\ &\geq N^{'}(\varphi_1(0, 2^n x), \frac{16^n a(16-\alpha)}{2}) \geq N^{'}(\varphi_1(0,x), \frac{16^n a(16-\alpha)}{2\alpha^n}) \end{aligned}$$

for all $x \in X$ and all $a > 0$.

Since $\lim_{n\to\infty} \frac{a(16-\alpha)(16^n)}{2\alpha^n} = \infty$, we obtain $\lim_{n\to\infty} N^{'}(\varphi_1(0,x), \frac{16^n a(16-\alpha)}{2\alpha^n}) = 1$. Thus $N(Q(x) - Q^{'}(x), a) = 1$ for all $x \in X$ and all $a > 0$, and so $Q(x) = Q^{'}(x)$.

For $\beta = -1$, we can prove the result by a similar method. □

Theorem 2.2. *Let $\beta \in \{1, -1\}$ be fixed and let $\varphi_2 : X \times X \to Z$ be a mapping such that for some $\alpha > 0$ with $(\frac{\alpha}{8})^\beta < 1$*

$$N^{'}(\varphi_2(0, 2^\beta x), a) \geq N^{'}(\alpha^\beta \varphi_2(0,x), a) \tag{2.12}$$

for all $x \in X$ *and all* $a > 0$, *and* $\lim_{n\to\infty} N'(\varphi_2(2^{\beta n}x, 2^{\beta n}y), 8^{\beta n}a) = 1$ *for all* $x, y \in X$ *and all* $a > 0$. *Suppose that an odd mapping* $f : X \to Y$ *satisfies the inequality*

$$N(D_f(x,y),a) \geq N'(\varphi_2(x,y),a) \tag{2.13}$$

for all $a > 0$ *and all* $x, y \in X$. *Then the limit*

$$C(x) = N\text{-}\lim_{n\to\infty} \frac{1}{8^{\beta n}} f(2^{\beta n}x)$$

exists for all $x \in X$ *and the mapping* $C : X \to Y$ *is a unique cubic mapping satisfying*

$$N(f(x) - C(x), a) \geq N'(\varphi_2(0,x), a|8-\alpha|) \tag{2.14}$$

for all $x \in X$ *and all* $a > 0$.

Proof. Let $\beta = 1$. Letting $x = 0$ in (2.13), we get

$$N(f(2y) - 8f(y), a) \geq N'(\varphi_2(0,y), a) \tag{2.15}$$

for all $y \in X$ and all $a > 0$. Replacing y by x in (2.15), we get

$$N(f(2x) - 8f(x), a) \geq N'(\varphi_2(0,x), a) \tag{2.16}$$

for all $x \in X$ and all $a > 0$. Replacing x by $2^n x$ in (2.16), we obtain

$$N(\frac{f(2^{n+1}x)}{8} - f(2^n x), \frac{a}{8}) \geq N'(\varphi_2(0, 2^n x), a) \tag{2.17}$$

for all $x \in X$ and all $a > 0$. Using (2.13), we get

$$N(\frac{f(2^{n+1}x)}{8} - f(2^n x), \frac{a}{8}) \geq N'(\varphi_2(0,x), \frac{a}{\alpha^n}) \tag{2.18}$$

for all $x \in X$ and all $a > 0$. Replacing a by $\alpha^n a$ in (2.18), we get

$$N(\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}, \frac{a\alpha^n}{8(8^n)}) \geq N'(\varphi_2(0,x), a) \tag{2.19}$$

for all $x \in X$ and all $a > 0$. It follows from $\frac{f(2^n x)}{8^n} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{8^{i+1}} - \frac{f(2^i x)}{8^i}$ and (2.19) that

$$N(\frac{f(2^n x)}{8^n} - f(x), \sum_{i=0}^{n-1} \frac{a\alpha^i}{8(8^i)}) \geq min \bigcup_{i=0}^{n-1} \{N(\frac{f(2^{i+1}x)}{8^{i+1}} - \frac{f(2^i x)}{8^i}, \frac{a\alpha^i}{8(8^i)})\} \geq N'(\varphi_2(0,x), a) \tag{2.20}$$

for all $x \in X$ and all $a > 0$. Replacing x with $2^m x$ in (2.20), we obtain

$$N(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, \sum_{i=0}^{n-1} \frac{a\alpha^i}{8(8^{i+m})}) \geq N'(\varphi_2(0, 2^m x), a) \geq N'(\varphi_2(0,x), \frac{a}{\alpha^m}),$$

and so

$$N(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, \sum_{i=m}^{n+m-1} \frac{a\alpha^i}{8(8^i)}) \geq N'(\varphi_2(0,x), a)$$

for all $x \in X$, all $a > 0$ and all $m, n \geq 0$. Hence

$$N(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, a) \geq N^{'}(\varphi_2(0,x), \frac{a}{\sum_{i=m}^{n+m-1}\frac{\alpha^i}{8(8^i)}}) \tag{2.21}$$

for all $x \in X$, all $a > 0$ and all $m, n \geq 0$. Since $0 < \alpha < 8$ and $\sum_{i=0}^{\infty}(\frac{\alpha}{8})^i < \infty$, the Cauchy criterion for convergence and (N_5) imply that $\{\frac{f(2^n x)}{8^n}\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $C(x) \in Y$. So one can define the mapping $C : X \to Y$ by $C(x) := N\text{-}\lim_{n\to\infty}\frac{f(2^n x)}{8^n}$ for all $x \in X$.

Since f is odd, C is odd. Letting $m = 0$ in (2.21), we get

$$N(\frac{f(2^n x)}{8^n} - f(x), a) \geq N^{'}(\varphi_2(0,x), \frac{a}{\sum_{i=0}^{n-1}\frac{\alpha^i}{8(8^i)}}) \tag{2.22}$$

for all $x \in X$ and all $a > 0$. Taking the limit as $n \to \infty$ and using (N_6), we get

$$N(f(x) - C(x), a) \geq N^{'}(\varphi_2(0,x), a(8-\alpha))$$

for all $x \in X$ and all $a > 0$.

Now we claim that C is cubic. Replacing x, y by $2^n x, 2^n y$ in (2.13), respectively, we get

$$N(\frac{1}{8^n}D_f(2^n x, 2^n y), a) \geq N^{'}(\varphi_2(2^n x, 2^n y), 8^n a)$$

for all $x, y \in X$ and all $a > 0$. Since $\lim_{n\to\infty} N^{'}(\varphi_2(2^n x, 2^n y), 8^n a) = 1$, by Corollary 2.2 of [7], we get that the mapping $C : X \to Y$ is cubic.

To prove the uniqueness of C, let $C^{'} : X \to Y$ be another cubic mapping satisfying (2.14). Fix $x \in X$. Clearly $C(2^n x) = 8^n C(x)$ and $C^{'}(2^n x) = 8^n C^{'}(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. It follows from (2.14) that

$$\begin{aligned} N(C(x) - C^{'}(x), a) &= N(\frac{C(2^n x)}{8^n} - \frac{C^{'}(2^n x)}{8^n}, a) \\ &\geq min\{N(\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}, \frac{a}{2}), N(\frac{f(2^n x)}{8^n} - \frac{C^{'}(2^n x)}{8^n}, \frac{a}{2})\} \\ &\geq N^{'}(\varphi_2(0, 2^n x), \frac{8^n a(8-\alpha)}{2}) \geq N^{'}(\varphi_2(0,x), \frac{8^n a(8-\alpha)}{2\alpha^n}) \end{aligned}$$

for all $x \in X$ and all $a > 0$.

Since $\lim_{n\to\infty}\frac{a(8-\alpha)(8^n)}{2\alpha^n} = \infty$, we obtain $\lim_{n\to\infty} N^{'}(\varphi_2(0,x), \frac{8^n a(8-\alpha)}{2\alpha^n}) = 1$. Thus $N(C(x) - C^{'}(x), a) = 1$ for all $x \in X$ and all $a > 0$, and so $C(x) = C^{'}(x)$.

For $\beta = -1$, we can prove the result by a similar method. □

We now prove our main theorem in section.

Theorem 2.3. *Let $\beta \in \{1, -1\}$ be fixed and let $\varphi : X \times X \to Z$ be a mapping such that for some $\alpha > 0$ with $\alpha^\beta < (4(-\beta + 3))^\beta$*

$$N^{'}(\varphi(0, 2^\beta x), a) \geq N^{'}(\alpha^\beta \varphi(0,x), a) \tag{2.23}$$

for all $x \in X$ and all $a > 0$, and

$$\lim_{n\to\infty} N^{'}(\varphi(2^{\beta n}x, 2^{\beta n}y), [(|\beta| + \beta)2^{4\beta n-1} + (|\beta| - \beta)2^{3\beta n}]a) = 1$$

for all $x, y \in X$ *and all* $a > 0$*. Suppose that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$N(D_f(x,y), a) \geq N'(\varphi(x,y), a) \tag{2.24}$$

for all $a > 0$ *and all* $x, y \in X$*. Then there exist a unique quartic mapping* $Q : X \to Y$ *and a unique cubic mapping* $C : X \to Y$ *such that*

$$N(f(x) - Q(x) - C(x), a) \geq N''(x, a) \tag{2.25}$$

for all $x \in X$ *and all* $a > 0$*, where*

$$N''(x,a) := min\{N'(\varphi(0,x), \frac{a(16-\alpha)}{2}), N'(\varphi(0,x), \frac{a(8-\alpha)}{2})\} \quad .$$

Proof. Assume $\beta = 1$. Then we have $\alpha < 8$. Let $f_e(x) = \frac{f(x)+f(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$ and

$$\begin{aligned} N(D_{f_e}(x,y), a) &= N(\frac{1}{2}[D_f(x,y) + D_f(-x,-y)], a) \\ &\geq min\{N(D_f(x,y), a), N(D_f(-x,-y), a)\} \end{aligned}$$

for all $x \in X$ and all $a > 0$. Hence, by Theorem 2.1, there exists a unique quartic mapping $Q : X \to Y$ satisfying

$$N(f_e(x) - Q(x), a) \geq N'(\varphi(0,x), a(16-\alpha)) \tag{2.26}$$

for all $x \in X$ and all $a > 0$.

Let $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$ and

$$\begin{aligned} N(D_{f_o}(x,y), a) &= N(\frac{1}{2}[D_f(x,y) - D_f(-x,-y)], a) \\ &\geq min\{N(D_f(x,y), a), N(D_f(-x,-y), a)\} \end{aligned}$$

for all $x \in X$ and all $a > 0$. Hence, by Theorem 2.2, there exists a unique cubic mapping $C : X \to Y$ satisfying

$$N(f_o(x) - C(x), a) \geq N'(\varphi(0,x), a(8-\alpha)) \tag{2.27}$$

for all $x \in X$ and all $a > 0$. Hence (2.25) follows from (2.26) and (2.27). If $\beta = 1$, then we have $\alpha > 16$. The rest of proof is similar to the case $\beta = 1$. □

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