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New classes of certain analytic functions

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Abstract

Considering a function $f(z)$ which is the extremal function for $p-$valently starlike of order α in the open unit disk, two new classes $S_p^*(m,\alpha)$ and $K_p(m,\alpha)$ are introduced. The object of the present paper is to discuss some interesting problems of functions $f(z)$ concerned with $S_p^*(m,\alpha)$ and $K_p(m,\alpha)$.

Keywords: Appell's symbol, Analytic function, p-valently starlike of order α, p-valently convex of order α
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1 Introduction

Let a be a complex number and $k \in \mathbb{N} = \{1, 2, \ldots\}$. With such a and k, we define

$$(a,k) = a(a+1)(a+2)\ldots(a+k-1) \tag{1.1}$$

$$(a,0) = 1 \tag{1.2}$$

and

$$(a,-k) = \frac{1}{(a-1)(a-2)\ldots(a-k)} \quad ; \quad (a \neq 1, 2, \ldots, k). \tag{1.3}$$

This symbol (a,k) is said to be Appell's symbol (cf.Carlson [3]).

Let $\mathcal{A}_p(n)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad ; \quad n \in \mathbb{N} \tag{1.4}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : 0 \leq |z| < 1\}$ where $p \in \mathbb{N}$. If $f(z) \in \mathcal{A}_p(n)$ satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}), \tag{1.5}$$

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then we say that $f(z)$ is $p-$valently starlike of order $\alpha\,(0 \le \alpha < p)$ in $\mathbb{U}$.

Furthermore, if $f(z) \in \mathcal{A}_p(n)$ satisfies

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}), \tag{1.6}$$

then we say $f(z)$ is $p-$valently convex of order $\alpha\,(0 \le \alpha < p)$ in $\mathbb{U}$.

It is clear that $f \in \mathcal{A}_p(n)$ is $p-$valently convex of order α in $\mathbb{U}$ if and only if $\frac{zf'(z)}{p}$ is $p-$valently starlike of order α in $\mathbb{U}$, and $f(z)$ is $p-$valently starlike of order α in $\mathbb{U}$ if and only if $\int_0^z \frac{pf(t)}{t}dt$ is $p-$valently convex of order α in $\mathbb{U}$ (cf. Hayami and Owa [12]).

Let us consider a function $f(z)$ given by

$$f(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}} \qquad (z \in \mathbb{U}) \tag{1.7}$$

with $p \in \mathbb{N}$ and $0 \le \alpha < p$. Then $f(z)$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(\frac{p+(p-2\alpha)z}{1-z}\right) > \alpha \qquad (z \in \mathbb{U}). \tag{1.8}$$

Therefore, the function $f(z)$ given by (1.7) is the extremal function for $p-$valently starlike of order α in $\mathbb{U}$. If $p = 1$ in (1.7), then $f(z)$ becomes

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \qquad (z \in \mathbb{U}) \tag{1.9}$$

and $w = f(z)$ maps $\mathbb{U}$ onto the domain such that $\mathrm{Re}w > \alpha$ $(0 \le \alpha < 1)$(cf. Duren[4], Goodmann [8]). Further, the function $f(z)$ given by (1.7) is written by

$$f(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}} = z^p + \sum_{k=1}^{\infty} \frac{(2p-2\alpha,k)}{k!} z^{p+k}, \qquad (z \in \mathbb{U}) \tag{1.10}$$

where $(2p-2\alpha,k)$ is Appell's symbol. For such function $f(z)$, we consider

$$g(z) = \frac{z^p}{(1-\sqrt{z})^{2(p-\alpha)}} = z^p + \sum_{k=1}^{\infty} \frac{(2p-2\alpha,k)}{k!} z^{p+\frac{k}{2}}, \qquad (z \in \mathbb{U}). \tag{1.11}$$

This function $g(z)$ satisfies

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) = \mathrm{Re}\left(\frac{p-\alpha\sqrt{z}}{1-\sqrt{z}}\right) > \frac{p+\alpha}{2} > \alpha \qquad (z \in \mathbb{U}). \tag{1.12}$$

This means that $g(z)$ is $p-$valently starlike of order $\frac{p+\alpha}{2}$ in $\mathbb{U}$. If we consider a function $g(z)$ such that

$$g(z) = \frac{zh'(z)}{p} = \frac{z^p}{(1-\sqrt{z})^{2(p-\alpha)}}, \tag{1.13}$$

$h(z)$ satisfies

$$\mathrm{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) = \mathrm{Re}\left(\frac{p-\alpha\sqrt{z}}{1-\sqrt{z}}\right) > \frac{p+\alpha}{2} > \alpha \qquad (z \in \mathbb{U}). \tag{1.14}$$

Thus $h(z)$ is $p-$valently convex of order $\frac{p+\alpha}{2}$ in $\mathbb{U}$.

With the above mention, let $\mathcal{A}_p(n,m)$ be the class of functions

$$f(z) = f(0) + z^p + \sum_{k=n}^{\infty} a_{p+\frac{k}{m}} z^{p+\frac{k}{m}} \qquad ; \qquad m \in \mathbb{N} \tag{1.15}$$

which are analytic in $\mathbb{U}_0 = \{z \in \mathbb{C} : 0 < |z| < 1\}$. If $f(z) \in \mathcal{A}_p(n,m)$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)-f(0)}\right) > \alpha \qquad (z \in \mathbb{U}_0) \tag{1.16}$$

for some real $\alpha\,(0 \le \alpha < p)$, then we say that $f(z) \in S_p^*(m,\alpha)$.

To define the class $\mathcal{K}_p(m,\alpha)$, we use

$$1+\frac{zf''(z)}{f'(z)} = z\frac{d}{dz}\left(\log\left(\frac{zf'(z)}{f(z)-f(0)}\right)\right)+\frac{zf'(z)}{f(z)-f(0)}. \tag{1.17}$$

If $f(z) \in \mathcal{A}_p(n,m)$ satisfies

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}_0) \tag{1.18}$$

for some real $\alpha\,(0 \le \alpha < p)$, then we write that $f(z) \in \mathcal{K}_p(m,\alpha)$.

Classes of starlike and convex p-valent analytic functions were previously introduced and studied by many authors regarding different aspects. Differential subordinations in the class of analytic and p-valent functions in the unit disc were studied in [10]. Properties of certain classes of meromorphic functions are obtained in [6, 7] and in very recent papers such as [5]. Recent investigations on p-valent functions can be seen in [1, 9, 12, 13] and starlikeness and convexity for p-valent analytic functions are considered in [2] and [11]. p-valently analytic functions still inspire studies with interesting outcomes. Therefore, in this study, we discuss some interesting problems of functions $f(z)$ concerned with $S_p^*(m,\alpha)$ and $K_p(m,\alpha)$ which are introduced considering a function $f(z)$ which is the extremal function for $p-$valently starlike of order α in the open unit disk.

2 Main results

We first derive the following theorem.

Theorem 2.1. If $f(z) \in \mathcal{A}_p(n,m)$ satisfies

$$\sum_{k=n}^{\infty}\left(p+\frac{k}{m}-\alpha\right)\left|a_{p+\frac{k}{m}}\right| \le p-\alpha \tag{2.1}$$

for some real $\alpha\,(0 \le \alpha < p)$, then $f(z) \in S_p^*(m,\alpha)$. The equality in (2.1) is attained for $f(z)$ given by

$$f(z) = f(0)+z^p+\sum_{k=n}^{\infty}\frac{mn(p-\alpha)\varepsilon}{k(k+1)(mp+k-m\alpha)}z^{p+\frac{k}{m}} \tag{2.2}$$

where $|\varepsilon| = 1$.

Proof . We note that if $f(z) \in \mathcal{A}_p(n,m)$ satisfies

$$\left|\frac{zf'(z)}{f(z)-f(0)}-p\right| < p-\alpha \qquad (z \in \mathbb{U}_0) \tag{2.3}$$

for some real α $(0 \le \alpha < p)$, then $f(z) \in \mathcal{S}_p^*(m,\alpha)$. Since

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)-f(0)}-p\right| &= \left|\frac{\sum_{k=n}^{\infty}\frac{k}{m}a_{p+\frac{k}{m}}z^{\frac{k}{m}}}{1+\sum_{k=n}^{\infty}a_{p+\frac{k}{m}}z^{\frac{k}{m}}}\right| \\ &< \frac{\sum_{k=n}^{\infty}\frac{k}{m}\left|a_{p+\frac{k}{m}}\right|}{1-\sum_{k=n}^{\infty}\left|a_{p+\frac{k}{m}}\right|} \qquad (z \in \mathbb{U}_0),\end{aligned} \tag{2.4}$$

and (2.1) implies

$$\sum_{k=n}^{\infty}\left|a_{p+\frac{k}{m}}\right| \le \frac{p-\alpha}{p+\frac{n}{m}-\alpha} < 1,$$

we consider

$$\frac{\sum_{k=n}^{\infty}\frac{k}{m}\left|a_{p+\frac{k}{m}}\right|}{1-\sum_{k=n}^{\infty}\left|a_{p+\frac{k}{m}}\right|} \le p-\alpha.$$

If the above inequality holds true, then $f(z) \in \mathcal{S}_p^*(m,\alpha)$. This means that if $f(z)$ satisfies

$$\sum_{k=n}^{\infty} \frac{k}{m} \left|a_{p+\frac{k}{m}}\right| \leq (p-\alpha)\left(1 - \sum_{k=n}^{\infty} \left|a_{p+\frac{k}{m}}\right|\right), \tag{2.5}$$

that is that the inequality (2.1)is satisfied, then $f(z) \in \mathcal{S}_p^*(m,\alpha)$.

Next we consider a function $f(z) \in \mathcal{A}_p(n,m)$ which satisfies

$$\begin{aligned} \sum_{k=n}^{\infty} \left(p + \frac{k}{m} - \alpha\right) \left|a_{p+\frac{k}{m}}\right| = p - \alpha & \\ &= (p-\alpha)n\left\{\left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \dots\right\} \\ &= (p-\alpha)\sum_{k=n}^{\infty}\left(\frac{n}{k} - \frac{n}{k+1}\right) \\ &= (p-\alpha)\sum_{k=n}^{\infty}\frac{n}{k(k+1)}. \end{aligned} \tag{2.6}$$

This is implied by

$$\left(p + \frac{k}{m} - \alpha\right)\left|a_{p+\frac{k}{m}}\right| = \frac{(p-\alpha)n}{k(k+1)} \tag{2.7}$$

for all $k \geq n$. Taking $a_{p+\frac{k}{m}}$ such that

$$a_{p+\frac{k}{m}} = \frac{mn(p-\alpha)\varepsilon}{k(k+1)(mp+k-m\alpha)} \qquad (|\varepsilon| = 1), \tag{2.8}$$

we know that $f(z) \in \mathcal{S}_p^*(m,\alpha)$. $\square$

Letting $m = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. If $f(z) \in \mathcal{A}_p(n,1)$ satisfies

$$\sum_{k=n}^{\infty} (p+k-\alpha)\left|a_{p+k}\right| \leq p - \alpha \tag{2.9}$$

for some real α $(0 \leq \alpha < p)$, then $f(z) \in \mathcal{S}_p^*(1,\alpha)$. The equality in (2.9) is attained for $f(z)$ given by

$$f(z) = f(0) + z^p + \sum_{k=n}^{\infty} \frac{n(p-\alpha)\varepsilon}{k(k+1)(p+k-\alpha)} z^{p+k}, \tag{2.10}$$

where $|\varepsilon| = 1$.

Next, we derive the following theorem.

Theorem 2.3. If $f(z) \in \mathcal{A}_p(n,m)$ satisfies

$$\sum_{k=n}^{\infty} \left(p + \frac{k}{m}\right)\left(p + \frac{k}{m} - \alpha\right)\left|a_{p+\frac{k}{m}}\right| \leq p(p-\alpha) \tag{2.11}$$

for some real α $(0 \leq \alpha < p)$, then $f(z) \in \mathcal{K}_p(m,\alpha)$. The equality in (2.11) is attained for

$$f(z) = f(0) + z^p + \sum_{k=n}^{\infty} \frac{m^2 np(p-\alpha)\varepsilon}{k(k+1)(mp+k)(mp+k-m\alpha)} z^{p+\frac{k}{m}}, \tag{2.12}$$

where $|\varepsilon| = 1$.

Proof . Note that $f(z) \in \mathcal{K}_p(m,\alpha)$ if and only if $\frac{zf'(z)}{p} \in \mathcal{S}_p^*(m,\alpha)$. Noting that

$$\frac{zf'(z)}{p} = z^p + \sum_{k=n}^{\infty} \frac{p+\frac{k}{m}}{p} a_{p+\frac{k}{m}} z^{p+\frac{k}{m}} \in \mathcal{S}_p^*(m,\alpha), \tag{2.13}$$

we know that if $f(z)$ satisfies the inequality (2.11), then $f(z) \in \mathcal{K}_p(m,\alpha)$. Also, the function $f(z)$ given by (2.12) satisfies the equality in (2.11). □

Making $m = 1$ in Theorem 2.3, we have the following corollary.

Corollary 2.4. If $f(z) \in \mathcal{A}_p(n,1)$ satisfies

$$\sum_{k=n}^{\infty} (p+k)(p+k-\alpha)\,|a_{p+k}| \leq p(p-\alpha) \tag{2.14}$$

for some real α $(0 \leq \alpha < p)$, then $f(z) \in \mathcal{K}_p(1,\alpha)$. The equality in (2.14) is attained for

$$f(z) = f(0) + z^p + \sum_{k=n}^{\infty} \frac{np(p-\alpha)\varepsilon}{k(k+1)(p+k)(p+k-\alpha)} z^{p+k}, \tag{2.15}$$

where $|\varepsilon| = 1$.

Next we consider a function $f(z) \in \mathcal{A}_p(1,m)$ given by

$$\begin{aligned} f(z) &= f(0) + \frac{z^p}{\left(1 - z^{\frac{1}{m}}\right)^{2(p-\alpha)}} \\ &= f(0) + z^p + \sum_{k=1}^{\infty} \frac{(2p-2\alpha,k)}{k!} z^{p+\frac{k}{m}} \qquad (z \in \mathbb{U}_0), \end{aligned} \tag{2.16}$$

where $(2p-2\alpha,k)$ is Appell's symbol. For such function $f(z)$, we have the following theorem.

Theorem 2.5. If $f(z) \in \mathcal{A}_p(1,m)$ is given by (2.16), then $f(z)$ belongs to the class $\mathcal{S}_p^*\left(m, \frac{(m-1)p+\alpha}{m}\right)$.

Proof . It follows from (2.16) that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)-f(0)}\right) = \mathrm{Re}\left(p + \frac{2(p-\alpha)z^{\frac{1}{m}}}{m(1-z^{\frac{1}{m}})}\right) > \frac{(m-1)p+\alpha}{m} \qquad (z \in \mathbb{U}_0). \tag{2.17}$$

Noting that

$$0 \leq \frac{(m-1)p+\alpha}{m} < \alpha < p, \tag{2.18}$$

we prove the theorem. □

Remark 2.6. Using the function $f(z)$ given by (2.16), we see that if $p - \alpha > \frac{1}{2}$

$$\begin{aligned} \sum_{k=1}^{\infty}\left(p+\frac{k}{m}-\alpha\right)\left|a_{p+\frac{k}{m}}\right| &= \sum_{k=1}^{\infty}\left(p+\frac{k}{m}-\alpha\right)\frac{(2p-2\alpha,k)}{k!} \\ &> \sum_{k=1}^{\infty}(p-\alpha)\frac{(2p-2\alpha,k)}{k!} \\ &> p-\alpha. \end{aligned} \tag{2.19}$$

Thus, $f(z)$ doesn't satisfy the inequality (2.1) of Theorem 2.1.

Theorem 2.7. If a function $f(z)$ is given by

$$\begin{aligned} f(z) &= f(0) + \frac{z}{\left(1 - z^{\frac{1}{m}}\right)^{2(1-\alpha)}} \\ &= f(0) + z + \sum_{k=1}^{\infty} \frac{(2-2\alpha, k)}{k!} z^{1+\frac{k}{m}} \qquad (z \in \mathbb{U}_0), \end{aligned} \tag{2.20}$$

for some real α $(0 \leq \alpha \leq \frac{1}{2})$ and $m \in \mathbb{N}$, then $f(z)$ is convex of order $\frac{(1-2\alpha)(m+\alpha-1)}{2m(1-\alpha)}$ in $\mathbb{U}_0$.

Proof . Note that $f(z)$ satisfies

$$\frac{zf'(z)}{f(z)-f(0)} = \frac{m + (2-2\alpha-m)z^{\frac{1}{m}}}{m(1-z^{\frac{1}{m}})}. \tag{2.21}$$

It follows from (2.21) that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{m+2\alpha-2}{m} + \frac{3-2\alpha}{m(1-z^{\frac{1}{m}})} - \frac{1}{m-(m+2\alpha-2)z^{\frac{1}{m}}}. \tag{2.22}$$

Letting $z = e^{i\theta}$ $(0 < \theta < 2\pi)$ in (2.22), we see that

$$\begin{aligned} \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &= \frac{m+2\alpha-2}{m} + \frac{3-2\alpha}{2m} \\ &+ \frac{m-(m+2\alpha-2)cos\theta}{m^2+(m+2\alpha-2)^2-2m(m+2\alpha-2)cos\theta} \\ &> \frac{2m+2\alpha-1}{2m} - \frac{1}{2(1-\alpha)} \\ &= \frac{(1-2\alpha)(m+\alpha-1)}{2m(1-\alpha)} \geq 0 \qquad (0 \leq \alpha \leq \frac{1}{2}). \end{aligned} \tag{2.23}$$

Therefore, we say that $f(z)$ is convex of order $\frac{(1-2\alpha)(m+\alpha-1)}{2m(1-\alpha)}$ in $\mathbb{U}_0$. □

Next we derive the following theorem.

Theorem 2.8. If a function $f(z)$ is given by

$$\begin{aligned} f(z) &= f(0) + \frac{z^p}{1-z^{\frac{1}{m}}} \\ &= f(0) + z^p + \sum_{k=1}^{\infty} z^{p+\frac{k}{m}} \qquad (z \in \mathbb{U}_0) \end{aligned} \tag{2.24}$$

for $m \in \mathbb{N}$, then $f(z)$ is $p-$valently starlike of order $\frac{2mp-1}{2m}$ in $\mathbb{U}_0$, and $p-$valently convex in $\mathbb{U}_0$.

Proof . It is easy to see that $f(z)$ satisfies

$$\frac{zf'(z)}{f(z)-f(0)} = p + \frac{z^{\frac{1}{m}}}{m(1-z^{\frac{1}{m}})} = \frac{mp-1}{m} + \frac{1}{m(1-z^{\frac{1}{m}})} \tag{2.25}$$

and

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= \frac{mp-1}{m} + \frac{1}{m(1-z^{\frac{1}{m}})} - \frac{(mp-1)z^{\frac{1}{m}}}{m(mp-(mp-1)z^{\frac{1}{m}})} + \frac{z^{\frac{1}{m}}}{m(1-z^{\frac{1}{m}})} \\ &= \frac{mp-1}{m} + \frac{2}{m(1-z^{\frac{1}{m}})} - \frac{p}{mp-(mp-1)z^{\frac{1}{m}}}. \end{aligned} \tag{2.26}$$

Thus we know that

$$\text{Re}\left(\frac{zf'(z)}{f(z)-f(0)}\right) > \frac{mp-1}{m} + \frac{1}{2m} = \frac{2mp-1}{2m} \qquad (z \in \mathbb{U}_0) \tag{2.27}$$

and

$$\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > p - \text{Re}\left(\frac{p}{mp-(mp-1)z^{\frac{1}{m}}}\right) \geq 0 \qquad (z \in \mathbb{U}_0). \tag{2.28}$$

□

With Theorem 2.7 and Theorem 2.8, we give the following problem.

Problem 2.9. For a function $f(z)$ given by

$$f(z) = f(0) + \frac{z^p}{\left(1-z^{\frac{1}{m}}\right)^{2(p-\alpha)}} \qquad (z \in \mathbb{U}_0), \tag{2.29}$$

consider starlikeness and convexity of $f(z)$.

Finally, we have the following theorem.

Theorem 2.10. If a function $f(z)$ is given by

$$\begin{aligned} f(z) &= f(0) + \frac{z\left(m-(m-1)z^{\frac{1}{m}}\right)}{m\left(1-z^{\frac{1}{m}}\right)^2} \\ &= f(0) + z + \sum_{k=1}^{\infty}\left(1+\frac{k}{m}\right)z^{1+\frac{k}{m}} \qquad (z \in \mathbb{U}_0) \end{aligned} \tag{2.30}$$

for $m \in \mathbb{N}$, then $f(z)$ is starlike in $\mathbb{U}_0$.

Proof . Since $f(z)$ satisfies

$$\frac{zf'(z)}{f(z)-f(0)} = 1 + \frac{z^{\frac{1}{m}}}{m(1-z^{\frac{1}{m}})} + \frac{1}{m(1-z^{\frac{1}{m}})} - \frac{1}{m-(m-1)z^{\frac{1}{m}}} \tag{2.31}$$

and

$$\frac{zf'(z)}{f(z)-f(0)} = 1 \tag{2.32}$$

for $z = 0$, we consider a point z given by

$$z^{\frac{1}{m}} = e^{i\frac{\theta}{m}} = e^{i\varphi} \qquad (\varphi = \frac{\theta}{m}). \tag{2.33}$$

Then $f(z)$ satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)-f(0)}\right) = \frac{2m-1}{2m}\left(1 - \frac{1-cos\varphi}{(2m^2-2m+1)-(2m-1)^2cos\varphi+2m(m-1)cos^2\varphi}\right). \tag{2.34}$$

Letting $t = cos\varphi$ $(-1 \leq t \leq 1)$, we consider

$$g(t) = \frac{1-t}{(2m^2-2m+1)-(2m-1)^2t+2m(m-1)t^2}. \tag{2.35}$$

Since $g'(t) \geq 0$, we say that

$$g(t) \leq \lim_{t\to 0} g(t) = 1. \tag{2.36}$$

This means that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(0)}\right) > 0 \qquad (z \in \mathbb{U}_0). \tag{2.37}$$

□

With the above theorem, we give the following problem.

Problem 2.11. For a function $f(z)$ given by

$$\begin{aligned} g(z) &= g(0) + \frac{z^p(m-(m-1)z^{\frac{1}{m}})}{m(1-z^{\frac{1}{m}})^2} \\ &= g(0) + z^p + \sum_{k=1}^{\infty}\left(1+\frac{k}{m}\right) z^{p+\frac{k}{m}} \qquad (z \in \mathbb{U}_0) \end{aligned} \tag{2.38}$$

and

$$f(z) - f(0) = z^{p-1}(g(z) - g(0))$$

consider starlikeness and convexity of $f(z)$.

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