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A unified class of analytic functions associated with Erdély–Kober integral operator

Kommula Amarender Reddy, Gangadharan Murugusundaramoorthy*

School of Advanced Sciences, Vellore Institute of Technology, Vellore-632014, TN, India

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Abstract

Making use of convolution product, we introduce a unified class of analytic functions with negative coefficients. Also, we obtain the coefficient bounds, extreme points and radius of starlikeness for functions belonging to the generalized class $\mathcal{TP}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $\mathcal{P}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$ are determined. Relevant connections of the results with various known results are also considered.

Keywords: Analytic, univalent, starlikeness, convexity, Hadamard product (or convolution), uniformly convex, uniformly starlike functions, Erdély–Kober integral operator.
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1 Introduction

The theory of analytic function is an ancient subject, yet it ruins an active field of current research. As a privileged topic regarding inequalities in complex analysis, there have been lots of studies based on the classes of analytic functions. The interplay of geometry and analysis is the most attractive aspect of complex function theory. This fast progress has been concerned mainly with such relations between analytic structure and geometric behaviour. Motivated by this approach, in the present study, we have introduced a new subclass of analytic functions concerning Erdély–Kober operator. Many authors have examined the properties of subclasses of analytic functions and shown their results have several applications in engineering, hydrodynamics and signal theory. One of the significant problems in geometric function theory are the extremal problems. Extremal problems play a central role in geometric function theory, discovery of coefficient bounds, sharp estimates, and an extremal function. The theory of analytic univalent functions is a influential tool in the study of many problems related to the time evolution of the free boundary of a viscous fluid for planar flows in Hele-Shaw cells under injection. The results we obtained here may have prospective application in other branches of mathematics, both pure and applied.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

*Corresponding author
Email addresses: amarenderkommula@gmail.com (Kommula Amarender Reddy), gmsmoorthy@yahoo.com (Gangadharan Murugusundaramoorthy)

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which are analytic and univalent in the open disc $\mathfrak{U} = \{z : z \in \mathcal{C},\ |z| < 1\}$. A function $f \in A$ is said to be starlike of order α $(0 \leq \alpha < 1)$, if and only if $\mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathfrak{U})$. This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in A$ that are starlike in $\mathfrak{U}$ with respect to the origin. A function $f \in A$ is said to be convex of order α $(0 \leq \alpha < 1)$ if and only if $\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathfrak{U})$. This class is denoted by $\mathcal{C}(\alpha)$. Further, $\mathcal{C} = \mathcal{C}(0)$, the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{C}(\alpha)$ if and only if $zf' \in \mathcal{S}^*(\alpha)$[1].

For functions $f \in A$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, then the Hadamard product (or convolution) of f, g is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathfrak{U}. \tag{1.2}$$

Now we recall the Erdély–Kober type ([12] Ch. 5) integral operator definition which shall be used throughout the paper as below:

Definition 1.1. Erdély–Kober fractional-order derivative Let for $\vartheta > 0, a, c \in \mathbb{C}$, be such that $\mathfrak{Re}(c-a) \geq 0$, an Erdély–Kober type integral operator

$$I_{\vartheta}^{a,c} : \mathcal{A} \to \mathcal{A}$$

be defined for $\mathfrak{Re}(c-a) > 0$ and $\mathfrak{Re}(a) > -\vartheta$ by

$$\mathfrak{I}_{\vartheta}^{a,c} f(z) = \frac{\Gamma(c+\vartheta)}{\Gamma(a+\vartheta)} \frac{1}{\Gamma(c-a)} \int_0^1 (1-t)^{c-a-1} t^{a-1} f(zt^{\vartheta}) dt, \vartheta > 0. \tag{1.3}$$

For $\vartheta > 0, \mathfrak{Re}(c-a) \geq 0$, $\mathfrak{Re}(a) > -\vartheta$ and $f \in A$ of the form (1.1) we have

$$\begin{aligned} \mathfrak{I}_{\vartheta}^{a,c} f(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma(c+\vartheta)\Gamma(a+n\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+n\vartheta)} a_n z^n \quad (z \in \mathfrak{U}) & (1.4) \\ &= z + \sum_{n=2}^{\infty} \Phi_{\vartheta}^{a,c}(n) a_n z^n \quad (z \in \mathfrak{U}) & (1.5) \end{aligned}$$

where

$$\Phi_{\vartheta}^{a,c}(n) = \frac{\Gamma(c+\vartheta)\Gamma(a+n\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+n\vartheta)} \quad \text{and} \quad \Phi_{\vartheta}^{a,c}(2) = \frac{\Gamma(c+\vartheta)\Gamma(a+2\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+2\vartheta)}. \tag{1.6}$$

Note that

$$\mathfrak{I}_{\vartheta}^{a,a} f(z) = f(z)$$

Remark 1.2. By fixing the parameters a, c, ϑ as mentioned below, the operator $\mathfrak{I}_{\vartheta}^{a,c}$ includes various operators studied in the literature as cited below:

1. For $a = \kappa; c = \varsigma + \kappa \quad and \quad \vartheta = 1$, we obtain the operator $\mathcal{Q}_{\kappa}^{\varsigma} f(z) (\varsigma \geq 0; \kappa > 1)$ studied by Jung et al.[11];
2. For $a = \varsigma - 1; c = \kappa - 1 \quad and \quad \vartheta = 1$, we obtain the operator $\mathfrak{L}_{\varsigma,\kappa} f(z) (\varsigma; \kappa \in \mathbb{C} \in \mathbb{Z}_0; \mathbb{Z}_0 = \{0; -1; -2; \cdots\}$ studied by Carlson and Shafer[6];
3. For $a = \rho - 1; c = \ell \quad and \quad \vartheta = 1$, we obtain the operator $\mathfrak{I}_{\rho,\ell} (\rho > 0; \ell > 1)$ studied by Choi et al. [7];
4. For $a = \varsigma; c = 0 \quad and \quad \vartheta = 1$, we obtain the operator $\mathfrak{D}^{\varsigma} (\varsigma > -1)$ studied by Ruscheweyh[22];
5. For $a = 1; c = n \quad and \quad \mu = 1$, we obtain the operator $\mathfrak{I}_n (n > \mathbb{N}_0)$ studied in[15, 16];
6. For $a = \kappa; c = \kappa + 1 \quad and \quad \vartheta = 1$; we obtain the integral operator $\mathfrak{I}_{\kappa,1}$ which studied by Bernardi[4];

7. For $a=1; c=2 \quad and \quad \vartheta=1$, we obtain the integral operator $\mathfrak{I}_{1,1}=I$ studied by Libera[13] and Livingston[14].

Motivated by earlier works in this paper, by making use of the operator $\mathfrak{I}_{\vartheta}^{a,c}$ we introduced a new subclass of analytic functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $\tau \geq 0$, $-1 \leq \alpha < 1$ and $\beta \geq 0$, we let $\mathcal{P}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the inequality

$$\mathfrak{Re}\left\{\frac{z(\mathfrak{I}_{\vartheta}^{a,c}f)'(z)+\tau z^2(\mathfrak{I}_{\vartheta}^{a,c}f)''(z)}{(1-\tau)(\mathfrak{I}_{\vartheta}^{a,c}f)(z)+\tau z(\mathfrak{I}_{\vartheta}^{a,c}f)'(z)}-\alpha\right\} > \beta\left|\frac{z(\mathfrak{I}_{\vartheta}^{a,c}f)'(z)+\tau z^2(\mathfrak{I}_{\vartheta}^{a,c}f)''(z)}{(1-\tau)(\mathfrak{I}_{\vartheta}^{a,c}f)(z)+\tau z(\mathfrak{I}_{\vartheta}^{a,c}f)'(z)}-1\right| \tag{1.7}$$

where $z \in \mathfrak{U}$, $\mathfrak{I}_{\vartheta}^{a,c}f(z)$ is given by (1.4) . We further let $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta) = P_{\vartheta,\tau}^{a,c}(\alpha,\beta) \cap T$, where

$$T \ := \ \left\{f \in A : f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n > 0; \quad z \in \mathfrak{U}\right\} \tag{1.8}$$

is a subclass of $\mathcal{A}$ introduced and studied by Silverman [24].

In particular, for $\tau = 1$ and $\tau = 0$, the class $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$ provides the following two new subclasses of $\beta-$ starlike functions and $\beta-$uniformly convex functions as given below.

Example 1.3. For $\tau = 0$, $-1 \leq \alpha < 1$ and $\beta \geq 0$, we let $\mathcal{S}_{\vartheta}^{a,c}(\alpha,\beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the inequality

$$\mathfrak{Re}\left(\frac{z(\mathfrak{I}_{\vartheta}^{a,c}f)'(z)}{(\mathfrak{I}_{\vartheta}^{a,c}f)(z)}-\alpha\right) > \beta\left|\frac{z(\mathfrak{I}_{\vartheta}^{a,c}f)'(z)}{(\mathfrak{I}_{\vartheta}^{a,c}f)(z)}-1\right| \tag{1.9}$$

where $z \in \mathfrak{U}$, $\mathfrak{I}_{\vartheta}^{a,c}f(z)$ is given by (1.4) .

Example 1.4. Fixing $\tau = 1$, $-1 \leq \alpha < 1$ and $\beta \geq 0$, we let $\mathcal{C}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the inequality

$$\mathfrak{Re}\left(1+\frac{z(\mathfrak{I}_{\vartheta}^{a,c}f)''(z)}{(\mathfrak{I}_{\vartheta}^{a,c}f)'(z)}-\alpha\right) > \beta\left|\frac{z(\mathfrak{I}_{\vartheta}^{a,c}f)''(z)}{(\mathfrak{I}_{\vartheta}^{a,c}f)'(z)}\right| \tag{1.10}$$

where $z \in \mathfrak{U}$, $\mathfrak{I}_{\vartheta}^{a,c}f(z)$ is given by (1.4) .

Example 1.5. For $\tau = 0$, $-1 \leq \alpha < 1$ and $\beta = 0$, we let $\mathfrak{S}_{\vartheta}^{a,c}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the inequality

$$\mathfrak{Re}\left(\frac{z(\mathfrak{I}_{\vartheta}^{a,c}f)'(z)}{(\mathfrak{I}_{\vartheta}^{a,c}f)(z)}\right) > \alpha. \tag{1.11}$$

By fixing $\tau = 1$, $-1 \leq \alpha < 1$ and $\beta = 0$, we let $\mathfrak{C}_{\vartheta}^{a,c}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the inequality

$$\mathfrak{Re}\left(1+\frac{z(\mathfrak{I}_{\vartheta}^{a,c}f)''(z)}{(\mathfrak{I}_{\vartheta}^{a,c}f)'(z)}\right) > \alpha \tag{1.12}$$

where $z \in \mathfrak{U}$, $\mathfrak{I}_{\vartheta}^{a,c}f(z)$ is given by (1.4).

By suitably specializing the values of a, c, α, β and τ the class $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$ yields to the various new subclasses associated with the operator listed in Remark 1.2, and also the classes introduced and studied in [2, 5, 24, 28, 29].

Example 1.6. If $c = a$ and $\tau = 1$, then

$$\mathcal{TP}_{\vartheta,1}^{a,a}(\alpha,\beta) \equiv \mathfrak{UCT}(\alpha,\beta) := \left\{f \in T : \mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \mathfrak{U}\right\}. \tag{1.13}$$

A function in $\mathfrak{UCT}(\alpha,\beta)$ is called $\beta-$uniformly convex of order α, $0 \leq \alpha < 1$. This class was introduced in [5].

Example 1.7. If $c = a$, then

$$\mathcal{TP}^{a,a}_{\vartheta,0}(\alpha,\beta) \equiv \mathfrak{TS}_p(\alpha,\beta) := \left\{ f \in T : \mathfrak{Re}\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta \left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in \mathfrak{U} \right\}. \tag{1.14}$$

A function in $\mathfrak{TS}_p(\alpha,\beta)$ is called $\beta-$uniformly starlike of order α, $0 \le \alpha < 1$. This class was introduced in [5].Indeed it follows from (1.14) and (1.13) that

$$f \in \mathfrak{UCT}(\alpha,\beta) \Leftrightarrow zf' \in \mathfrak{TS}_p(\alpha,\beta). \tag{1.15}$$

We observe that $\mathfrak{UCT}(\alpha,0) \equiv \mathcal{C}(\alpha)$ and $\mathfrak{UCT}(0,0) \equiv \mathcal{C}^*$ further note that $\mathfrak{TS}_p(\alpha,0) = \mathcal{T}^*(\alpha)$ and $\mathfrak{TS}_p(0,0) = \mathcal{T}^*$ the classes were first introduced and studied by Silverman [24]. We remark that the classes of uniformly convex and uniformly starlike functions were introduced by Goodman [9, 10], and later generalized by and others [5, 18, 19, 20, 21, 28, 29].

The main object of this paper is to study the coefficient bounds, extreme points and radius of starlikeness for functions belong to the generalized class $\mathcal{TP}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $\mathcal{P}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$ are determined.

2 Coefficient Bounds

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $\mathcal{P}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$ and $\mathcal{TP}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$.

Theorem 2.1. A function $f(z)$ of the form (1.1) is in $\mathcal{P}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$ if

$$\sum_{n=2}^{\infty}(1+\tau(n-1))[n(1+\beta)-(\alpha+\beta)]|a_n||\Phi^{a,c}_{\vartheta}(n)| \le 1-\alpha, \tag{2.1}$$

$0 \le \tau \le 1$, $-1 \le \alpha < 1$, $\beta \ge 0$.

Proof . It suffices to show that

$$\beta\left|\frac{z(\mathfrak{I}^{a,c}_{\vartheta}f)'(z)+\tau z^2(\mathfrak{I}^{a,c}_{\vartheta}f)''(z)}{(1-\tau)(\mathfrak{I}^{a,c}_{\vartheta}f)(z)+\tau z(\mathfrak{I}^{a,c}_{\vartheta}f)'(z)} - 1\right| - |\mathfrak{Re}\left(\frac{z(\mathfrak{I}^{a,c}_{\vartheta}f)'(z)+\tau z^2(\mathfrak{I}^{a,c}_{\vartheta}f)''(z)}{(1-\tau)(\mathfrak{I}^{a,c}_{\vartheta}f)(z)+\tau z(\mathfrak{I}^{a,c}_{\vartheta}f)'(z)} - 1\right) \le 1-\alpha.$$

We have

$$\begin{aligned}
&\beta\left|\frac{z(\mathfrak{I}^{a,c}_{\vartheta}f)'(z)+\tau z^2(\mathfrak{I}^{a,c}_{\vartheta}f)''(z)}{(1-\tau)(\mathfrak{I}^{a,c}_{\vartheta}f)(z)+\tau z(\mathfrak{I}^{a,c}_{\vartheta}f)'(z)} - 1\right| - \mathfrak{Re}\left(\frac{z(\mathfrak{I}^{a,c}_{\vartheta}f)'(z)+\tau z^2(\mathfrak{I}^{a,c}_{\vartheta}f)''(z)}{(1-\tau)(\mathfrak{I}^{a,c}_{\vartheta}f)(z)+\tau z(\mathfrak{I}^{a,c}_{\vartheta}f)'(z)} - 1\right)\\
\le\ & (1+\beta)\left|\frac{z(\mathfrak{I}^{a,c}_{\vartheta}f)'(z)+\tau z^2(\mathfrak{I}^{a,c}_{\vartheta}f)''(z)}{(1-\tau)(\mathfrak{I}^{a,c}_{\vartheta}f)(z)+\tau z(\mathfrak{I}^{a,c}_{\vartheta}f)'(z)} - 1\right|\\
=\ & \frac{(1+\beta)\sum\limits_{n=2}^{\infty}(n-1)[1+\tau(n-1)]|a_n||\Phi^{a,c}_{\vartheta}(n)|}{1-\sum\limits_{n=2}^{\infty}[1+\tau(n-1)]|a_n||\Phi^{a,c}_{\vartheta}(n)|}.
\end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}(1+\tau(n-1))[n(1+\beta)-(\alpha+\beta)]|a_n||\Phi^{a,c}_{\vartheta}(n)| \le 1-\alpha$$

and hence the proof is complete. $\square$

Theorem 2.2. A necessary and sufficient condition for $f(z)$ of the form (1.8) to be in the class $\mathcal{TP}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$, $-1 \leq \alpha < 1, 0 \leq \tau \leq 1, \beta \geq 0$ is that

$$\sum_{n=2}^{\infty}(1+\tau(n-1))[n(1+\beta)-(\alpha+\beta)]\ a_n\Phi^{a,c}_{\vartheta}(n) \leq 1-\alpha. \tag{2.2}$$

Proof . In view of Theorem 2.1, we need only to prove the necessity. If $f \in \mathcal{P}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$ and z is real then

$$\frac{1-\sum\limits_{n=2}^{\infty} n[1+\tau(n-1)]a_n\Phi^{a,c}_{\vartheta}(n)z^{n-1}}{1-\sum\limits_{n=2}^{\infty}[1+\tau(n-1)]a_n\Phi^{a,c}_{\vartheta}(n)z^{n-1}} - \alpha \geq \beta\left|\frac{\sum\limits_{n=2}^{\infty}(n-1)[1+\tau(n-1)]|a_n||\Phi^{a,c}_{\vartheta}(n)|}{1-\sum\limits_{n=2}^{\infty}[1+\tau(n-1)]|a_n||\Phi^{a,c}_{\vartheta}(n)|}\right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}(1+\tau(n-1))[n(1+\beta)-(\alpha+\beta)]\ a_n\Phi^{a,c}_{\vartheta}(n) \leq 1-\alpha.$$

□

Corollary 2.3. If $f \in \mathcal{TP}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$, then

$$a_n \leq \frac{1-\alpha}{[n(\beta+1)-(\alpha+\beta)](1+\tau(n-1))\Phi^{a,c}_{\vartheta}(n)}, \quad (n \geq 2) \tag{2.3}$$

where $0 \leq \tau \leq 1$, $-1 \leq \alpha < 1$ and $\beta \geq 0$. Equality in (2.3) holds for the function

$$f(z) = z - \frac{1-\alpha}{[n(\beta+1)-(\alpha+\beta)](1+\tau(n-1))\Phi^{a,c}_{\vartheta}(n)}z^n. \tag{2.4}$$

By fixing $\tau = 0$ in Theorem 2.2 we get the following result:

Corollary 2.4. A function $f \in T$ of the form (1.8) is in the class $\mathcal{S}^{a,c}_{\vartheta}(\alpha,\beta)$, $-1 \leq \alpha < 1, \beta \geq 0$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\Phi^{a,c}_{\vartheta}(n)\ a_n \leq 1-\alpha.$$

By fixing $\tau = 1$ in Theorem 2.2, we get the following result:

Corollary 2.5. A function $f \in T$ of the form (1.8) is in the class $\mathcal{C}^{a,c}_{\vartheta}(\alpha,\beta)$, $-1 \leq \alpha < 1, \beta \geq 0$ if and only if

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)]\Phi^{a,c}_{\vartheta}(n)\ a_n \leq 1-\alpha.$$

Corollary 2.6. A function $f \in T$ of the form (1.8) is in $\mathfrak{S}^{a,c}_{\vartheta}(\alpha)$, if and only if

$$\sum_{n=2}^{\infty}(n-\alpha)\Phi^{a,c}_{\vartheta}(n)a_n \leq 1-\alpha.$$

Corollary 2.7. A function $f \in T$ of the form (1.8) is in $\mathfrak{C}^{a,c}_{\vartheta}(\alpha)$, if and only if

$$\sum_{n=2}^{\infty}n(n-\alpha)\Phi^{a,c}_{\vartheta}(n)a_n \leq 1-\alpha.$$

By fixing $a = c, \tau = 0$ and $a = c, \tau = 1$ in above Corollaries we get the results given in [5]. Further by taking $\beta = 0$ and $\tau = 0$ (or $\tau = 1$) with $a = c$, Theorem 2.2 gives the results given in [24]. Similarly many known results can be obtained as particular cases of the Theorem 2.2, so we omit stating the particular cases.

3 Closure Properties

For any compact family F of univalent functions, the maximum or minimum value on F of the real part of any continuous linear functional defined over the set of analytic functions occurs at one of the extreme points of the closed convex hull of F. Consequently, the determination of the extreme points of a family F enables us to solve many extremal problems for F. The extreme points of the closed convex hull for convex, starlike, close-to-convex,and typically real functions were firstly studied in [3] . Since then, the extreme points for many additional classes have been determined. In view of Theorem 2.2, in this section we made an attempt to find extreme points for the function class $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$.

Theorem 3.1. Let

$$\begin{aligned} f_1(z) &= z \quad \text{and} \\ f_n(z) &= z - \frac{1-\alpha}{[n(\beta+1)-(\alpha+\beta)](1+\tau(n-1))\Phi_\vartheta^{a,c}(n)} z^n, \qquad n \geq 2. \end{aligned} \tag{3.1}$$

Then $f \in \mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$, if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z), \qquad \omega_n \geq 0, \quad \sum_{n=1}^{\infty} \omega_n = 1. \tag{3.2}$$

Proof . Suppose $f(z)$ can be written as in (3.2). Then

$$f(z) = z - \sum_{n=2}^{\infty} \omega_n \frac{1-\alpha}{[n(\beta+1)-(\alpha+\beta)](1+\tau(n-1))\Phi_\vartheta^{a,c}(n)} z^n.$$

Now,

$$\sum_{n=2}^{\infty} \omega_n \frac{[n(\beta+1)-(\alpha+\beta)](1+\tau(n-1))\Phi_\vartheta^{a,c}(n)(1-\alpha)}{(1-\alpha)[n(\beta+1)-(\alpha+\beta)](1+\tau(n-1))\Phi_\vartheta^{a,c}(n)} = \sum_{n=2}^{\infty} \omega_n = 1 - \omega_1 \leq 1.$$

Thus $f \in \mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$. Conversely, let us have $f \in \mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$. Then by using (2.3), we set

$$\omega_n = \frac{[n(\beta+1)-(\alpha+\beta)](1+\tau(n-1))\Phi_\vartheta^{a,c}(n)}{1-\alpha} a_n \quad (n \geq 2)$$

and $\omega_1 = 1 - \sum_{n=2}^{\infty} \omega_n$. Then $f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z)$. This completes the proof. □

Theorem 3.2. The class $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$ is a convex set.

Proof . Let the function

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \quad j = 1, 2 \tag{3.3}$$

be in the class $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = \eta f_1(z) + (1-\eta) f_2(z), \quad (0 \leq \eta \leq 1)$$

is in the class $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$ then we have

$$h(z) = z - \sum_{n=2}^{\infty} [\eta a_{n,1} + (1-\eta) a_{n,2}] z^n.$$

By a simple computation with the aid of Theorem 2.2 , we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}(1+\tau(n-1))[n(\beta+1)-(\alpha+\beta)]\eta\Phi_{\vartheta}^{a,c}(n)a_{n,1}\\
&\qquad+\sum_{n=2}^{\infty}(1+\tau(n-1))[n(\beta+1)-(\alpha+\beta)](1-\eta)\Phi_{\vartheta}^{a,c}(n)a_{n,2}\\
&\qquad\leq \eta(1-\alpha)+(1-\eta)(1-\alpha)\\
&\qquad=1-\alpha
\end{aligned}$$

which implies that $h \in \mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$. Hence $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$ is convex. □

Next we obtain the radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$.

Theorem 3.3. Let the function $f(z)$ defined by (1.8)belong to the class $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$. Then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_1$, where

$$r_1 := \inf_{n\geq 2}\left[\frac{(1-\delta)[n(\beta+1)-(\alpha+\beta)](1+\tau(n-1))\Phi_{\vartheta}^{a,c}(n)}{n(1-\alpha)}\right]^{\frac{1}{n-1}}. \tag{3.4}$$

The result is sharp, with extremal function $f(z)$ given by (3.1).

Proof . Given $f \in T$, and f is close-to-convex of order δ, we have

$$|f'(z)-1| < 1-\delta. \tag{3.5}$$

For the left hand side of (3.5) we have

$$|f'(z)-1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n}{1-\delta}a_n|z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}\frac{(1+\tau(n-1))[n(\beta+1)-(\alpha+\beta)]\Phi_{\vartheta}^{a,c}(n)}{(1-\alpha)}a_n \leq 1.$$

We can say (3.5) is true if

$$\frac{n}{1-\delta}|z|^{n-1} \leq \frac{(1+\tau(n-1))[n(\beta+1)-(\alpha+\beta)]\Phi_{\vartheta}^{a,c}(n)}{(1-\alpha)}a_n,$$

or, equivalently,

$$|z|^{n-1} = \left[\frac{(1-\delta)(1+\tau(n-1))[n(\beta+1)-(\alpha+\beta)]\Phi_{\vartheta}^{a,c}(n)}{n(1-\alpha)}\right]$$

which completes the proof. □

Theorem 3.4. If $f \in \mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$, then

(i) f is starlike of order $\delta(0 \leq \delta < 1)$ in the disc $|z| < r_2$; that is, $\mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > \delta$, $(|z| < r_2\ ;\ 0 \leq \delta < 1)$, where

$$r_2 = \inf_{n\geq 2}\left[\left(\frac{1-\delta}{n-\delta}\right)\frac{(1+\tau(n-1))[n(\beta+1)-(\alpha+\beta)]\Phi_{\vartheta}^{a,c}(n)}{(1-\alpha)}\right]^{\frac{1}{n-1}} \tag{3.6}$$

and

(ii) f is convex of order δ $(0 \leq \delta < 1)$ in the unit disc $|z| < r_3$, that is $\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta$, $(|z| < r_3; 0 \leq \delta < 1)$, where

$$r_3 = \inf_{n \geq 2} \left[\left(\frac{1-\delta}{n(n-\delta)} \right) \frac{(1+\tau(n-1))[n(\beta+1)-(\alpha+\beta)]\Phi_{\vartheta}^{a,c}(n)}{(1-\alpha)} \right]^{\frac{1}{n-1}} . \tag{3.7}$$

Each of these results are sharp for the extremal function $f(z)$ given by (3.1).

Proof .(1) Given $f \in T$, and f is starlike of order δ, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta. \tag{3.8}$$

For the left hand side of (3.8) we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum\limits_{n=2}^{\infty} (n-1)a_n \ |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n \ |z|^{n-1}}.$$

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n \ |z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha, \beta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{(1+\tau(n-1))[n(\beta+1)-(\alpha+\beta)]}{(1-\alpha)} a_n \Phi_{\vartheta}^{a,c}(n) \leq 1.$$

We can say (3.8) is true if

$$\frac{n-\delta}{1-\delta} |z|^{n-1} < \frac{(1+\tau(n-1))[n(\beta+1)-(\alpha+\beta)]\Phi_{\vartheta}^{a,c}(n)}{(1-\alpha)}.$$

or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\delta}{n-\delta} \right) \frac{(1+\tau(n-1))[n(\beta+1)-(\alpha+\beta)]\Phi_{\vartheta}^{a,c}(n)}{(1-\alpha)} \right]$$

which yields the starlikeness of the family.

(2) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar to the proof of (i). □

4 Partial Sums

In 1997, Silverman [27] and Silvia [23] have studied results on partial sums of analytic functions given by (1.1) to its sequence of partial sums

$$f_k(z) = z + \sum_{n=2}^{k} a_n z^n$$

when the coefficients $\{a_n\}$ of f are sufficiently small they determine sharp lower bounds for

$$\mathfrak{Re}\left(\frac{f(z)}{f_k(z)}\right), \ \mathfrak{Re}\left(\frac{f_j(z)}{f(z)}\right), \ \mathfrak{Re}\left(\frac{f'(z)}{f'_k(z)}\right) \ \text{ and } \ \mathfrak{Re}\left(\frac{f'_k(z)}{f'(z)}\right). \tag{4.1}$$

In this section, motivated by Silverman [27] and Silvia [23] we will examine the ratio (4.1) for functions of the form (1.1) and satisfy the condition (2.1).

Theorem 4.1. Let $f(z) \in \mathcal{P}^{a,c}_{\vartheta,\tau}(\alpha,\beta)$. Define the partial sums $f_1(z)$ and $f_k(z)$, by

$$f_1(z) = z; \text{ and } f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \qquad (k \in \mathbb{N} \setminus \{1\}). \tag{4.2}$$

Suppose that $\sum\limits_{n=2}^{\infty} d_n |a_n| \le 1$, where

$$d_n := \frac{(1+\tau(n-1))[n(\alpha+\beta)-(\alpha+\beta)]\Phi^{a,c}_{\vartheta}(n)}{(1-\alpha)}. \tag{4.3}$$

Then $f \in P^{a,c}_{\vartheta,\tau}(\alpha,\beta)$. Furthermore,

$$\mathfrak{Re}\left(\frac{f(z)}{f_k(z)}\right) > 1 - \frac{1}{d_{k+1}}; \ , \qquad (z \in \mathfrak{U}, k \in \mathbb{N} \setminus \{1\}) \tag{4.4}$$

and

$$\mathfrak{Re}\left(\frac{f_k(z)}{f(z)}\right) > \frac{d_{k+1}}{1+d_{k+1}}. \tag{4.5}$$

Proof . For the coefficients d_n given by (4.3) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{4.6}$$

Therefore we have

$$\sum_{n=2}^{k} |a_n| + d_{k+1} \sum_{n=k+1}^{\infty} |a_n| \le \sum_{n=2}^{\infty} d_n |a_n| \le 1 \tag{4.7}$$

by using the hypothesis (4.3). By setting

$$\begin{aligned} g_1(z) &= d_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{d_{k+1}}\right)\right\} \\ &= 1 + \frac{d_{k+1} \sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} a_n z^{n-1}} \end{aligned} \tag{4.8}$$

and applying (4.7), we find that

$$\begin{aligned} \left|\frac{g_1(z)-1}{g_1(z)+1}\right| &\le \frac{d_{k+1} \sum\limits_{n=k+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=2}^{n} |a_n| - d_{k+1} \sum\limits_{n=k+1}^{\infty} |a_n|} \\ &\le 1, \ z \in \mathfrak{U}, \end{aligned} \tag{4.9}$$

which readily yields the assertion (4.4) of Theorem 4.1. In order to see that

$$f(z) = z + \frac{z^{k+1}}{d_{k+1}} \tag{4.10}$$

gives sharp result, we observe that for $z = re^{i\pi/k}$ that $\frac{f(z)}{f_k(z)} = 1 + \frac{z^k}{d_{k+1}} \to 1 - \frac{1}{d_{k+1}}$ as $z \to 1^-$. Similarly, if we take

$$\begin{aligned} g_2(z) &= (1+d_{k+1})\left\{\frac{f_k(z)}{f(z)} - \frac{d_{k+1}}{1+d_{k+1}}\right\} \\ &= 1 - \frac{(1+d_{n+1}) \sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} a_n z^{n-1}} \end{aligned} \tag{4.11}$$

and making use of (4.7), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_{k+1})\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{k}|a_n|-(1-d_{k+1})\sum\limits_{n=k+1}^{\infty}|a_n|} \tag{4.12}$$

which leads us immediately to the assertion (4.5) of Theorem 4.1. The bound in (4.5) is sharp for each $k \in \mathbb{N} \setminus \{1\}$ with the extremal function $f(z)$ given by (4.10). The proof of the Theorem 4.1, is thus complete. □

Theorem 4.2. If $f(z)$ of the form (1.1) satisfies the condition (2.1). Then

$$\mathfrak{Re}\left\{\frac{f'(z)}{f_k'(z)}\right\} \geq 1-\frac{k+1}{d_{k+1}}. \tag{4.13}$$

Proof . By setting

$$\begin{aligned} g(z) &= d_{k+1}\left\{\frac{f'(z)}{f_k'(z)}-\left(1-\frac{k+1}{d_{k+1}}\right)\right\} \\ &= \frac{1+\frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty}na_nz^{n-1}+\sum\limits_{n=2}^{\infty}na_nz^{n-1}}{1+\sum\limits_{n=2}^{k}na_nz^{n-1}} \\ &= 1+\frac{\frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty}na_nz^{n-1}}{1+\sum\limits_{n=2}^{k}na_nz^{n-1}}. \end{aligned} \tag{4.14}$$

Hence,

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty}n|a_n|}{2-2\sum\limits_{n=2}^{k}n|a_n|-\frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty}n|a_n|}. \tag{4.15}$$

Now $\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1$ if

$$\sum_{n=2}^{k}n|a_n|+\frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty}n|a_n| \leq 1. \tag{4.16}$$

Since the left hand side of (4.16) is bounded above by $\sum\limits_{n=2}^{k}d_n|a_n|$ if

$$\sum_{n=2}^{k}(d_n-n)|a_n|+\sum_{n=k+1}^{\infty}d_n-\frac{d_{k+1}}{k+1}n|a_n| \geq 0 \tag{4.17}$$

and the proof is complete. The result is sharp for the extremal function $f(z)=z+\frac{z^{k+1}}{c_{k+1}}$. □

Theorem 4.3. If $f(z)$ of the form (1.1) satisfies the condition (2.1) then

$$\mathfrak{Re}\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{d_{k+1}}{k+1+d_{k+1}}. \tag{4.18}$$

Proof . By setting

$$\begin{aligned} g(z) &= [(k+1)+d_{k+1}]\left\{\frac{f_k'(z)}{f'(z)} - \frac{d_{k+1}}{k+1+d_{k+1}}\right\} \\ &= 1 - \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n a_n z^{n-1}}{1+\sum\limits_{n=2}^{k} n a_n z^{n-1}} \end{aligned}$$

and making use of (4.17), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{k} n|a_n| - \left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|} \leq 1,$$

which leads us immediately to the assertion of the Theorem 4.3. □

5 Integral Means Inequalities

In 1925, Littlewood [17] proved the following subordination theorem.

Lemma 5.1. If the functions f and g are analytic in $\mathfrak{U}$ with $g \prec f$, then for $\rho > 0$, and $0 < r < 1$,

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\rho} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\rho} d\theta. \tag{5.1}$$

In [24], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [25] and settled in [26], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\rho} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\rho} d\theta,$$

for all $f \in T$, $\rho > 0$ and $0 < r < 1$. In [26], he also proved his conjecture for the subclasses $T^*(\alpha) = \mathcal{S}^*(\alpha) \cap T$ and $\mathcal{K}(\alpha) = \mathcal{C}(\alpha) \cap T$ of T. Using Lemma 5.1, Theorem 2.2 and Corollary 2.3, in the following theorem we obtain integral means inequalities for the functions in the family $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$.

Theorem 5.2. Suppose $f \in \mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$, $\rho > 0$, $0 \leq \alpha < 1$, $\beta \geq 0$ and $f_2(z)$ is defined by

$$f_2(z) = z - \frac{1-\alpha}{(2-\alpha)(1+\tau)\Phi_{\vartheta}^{a,c}(2)} z^2.$$

where $\Phi_{\vartheta}^{a,c}(2)$ is given by (1.6). Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^{\rho} d\theta \leq \int_0^{2\pi} |f_2(z)|^{\rho} d\theta. \tag{5.2}$$

Proof . For $f \in T$, (5.2) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^{\rho} d\theta \leq \int_0^{2\pi} \left|1 - \frac{1-\alpha}{(2-\alpha)(1+\tau)\Phi_{\vartheta}^{a,c}(2)} z\right|^{\rho} d\theta.$$

By Lemma 5.1, it suffices to show that

$$1-\sum_{n=2}^{\infty}|a_n|z^{n-1}\prec 1-\frac{1-\alpha}{(2-\alpha)(1+\tau)\Phi_{\vartheta}^{a,c}(2)}z.$$

Setting

$$1-\sum_{n=2}^{\infty}|a_n|z^{n-1}=1-\frac{1-\alpha}{(2-\alpha)(1+\tau)\Phi_{\vartheta}^{a,c}(2)}w(z) \tag{5.3}$$

and using (2.2), we obtain

$$\begin{aligned}|w(z)| &= \left|\sum_{n=2}^{\infty}\frac{(1+\tau(n-1))[n(1+\beta)-(\alpha+\beta)}{1-\alpha}\ a_n\Phi_{\vartheta}^{a,c}(n)z^{n-1}\right|\\ &\le |z|\sum_{n=2}^{\infty}\frac{(1+\tau(n-1))[n(1+\beta)-(\alpha+\beta)}{1-\alpha}\Phi_{\vartheta}^{a,c}(n)|a_n|\\ &\le |z|\end{aligned}$$

where $\Phi_{\vartheta}^{a,c}(n)$ is given by (1.6). Which completes the proof by Theorem 5.2. □

By taking appropriate choices of the parameters we obtain the integral means inequalities for several known as well as new subclasses given in Examples 1.3 to 1.7 in Section 1 and Theorem 5.2.

6 Conclusion

By suitably specializing the values of a, c, α, β and τ the class $\mathcal{TP}_{\vartheta,\tau}^{a,c}(\alpha,\beta)$ yields to the various new subclasses associated with the linear operator listed in Remark 1.2, and we can deduce the above results easily by proceeding as in Theorems 2.1 to 5.2 for several known as well as new subclasses given in Examples 1.3 to 1.5 in Section 1.

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