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Common fixed point theorems under the (CLR_g) property with applications

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Abstract

In this paper, we study some new common fixed point theorems for a pair of weakly compatible mappings satisfying the (CLR_g) property in modular metric spaces. We also generalize and improve several results available from the existing literature. As applications of our results, we also provide some applications on the existence and uniqueness of the solution to Volterra type integral equations.

Keywords: (CLR_g) property, weakly compatible mappings, fixed point, modular metric, integral type equation, Volterra integral equation
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1 Introduction and Preliminaries

The metric fixed point theory encompasses the branch of fixed point theory in which metric conditions play a fundamental role. It has many applications in science and engineering that can be reduced to fixed point problems. Most of the theorems ensure the existence of solutions to integral equations, differential equations or other operator equations. They are also used in new areas of mathematical applications like control theory, mathematical economics, game theory, fluid flow etc. Due to this fact, many authors have focused on this theory and have proved some attractive fixed point theories. The Banach contraction mapping is one of the most well-known and most important of these theories. Because of its importance and simplicity, many authors have offered various generalizations for single-valued and multi-valued maps. In 1976, Jungck [14] introduced a common fixed point theorem in which they generalized Banach's fixed point theorem using commutativity conditions. Then, in 1998, Jungck and Rhoades [15] introduced the concept of weakly compatible mappings. This new concept of weak compatible between a set-valued mapping and a single-valued mapping is used as a tool for proving some common fixed point theorems on metric spaces. In 2006, Chistyakov[9] gave the notion of metric modular on an arbitrary set and then in 2008 [10], he introduced the concept of modular metric space derived from F-modular. Afterwards, in 2010 modular metric space as a generalization of metric spaces was defined [11][12]. Ever since, there has been tremendous development and growth on this subject (see [24]-[21]). In 2011, Sintunavarat and Kumam [23] introduced a new concept called $(CLRg)$ property ((CLR_g) common limit in the range of g). The benefit of $(CLRg)$ property provides that one does not require the closeness of range subspaces. Many authors have proved fixed point theorems using this property (see [18]-[7]).

Before going towards our findings, we need the following definitions, lemmas and notions.

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Definition 1. Let M be a nonempty set and $g, h : M \to M$. Then a point $z \in M$ is called a common fixed point of g and h if $gz = hz = z$. In other words, a solution of the equation $gz = hz = z$, if it exists, is called a common fixed point of g and h. For example, let g and h be self single valued maps of $\mathbf{R}^+ \cup \{0\}$ defined by $gx = 3\sqrt{x}$ and $hx = 3\sqrt{3}x^{\frac{1}{4}}$, then 0 and 9 are the common fixed points of g and h.

In 1998, Jungck and Rhoades introduced the notion of weakly compatible mappings in metric spaces [15].

Definition 2. [15] Let M be a nonempty set. Two self mappings g and h are called weakly compatible if $ghz = hgz$ for each z which $gz = hz$.

Example 1. Let $X = [3, \infty)$. Define $g, h : X \to X$ by

$$g(z) = \begin{cases} 3, & \text{if } z = 3 \\ z^2 + 9, & \text{if } z \neq 3 \end{cases}$$

$$h(z) = \begin{cases} 3, & \text{if } z = 3 \\ 6z, & \text{if } z \neq 3. \end{cases}$$

Then for $z = 3$, $ghz = hgz$ showing that g and h are weakly compatible maps on $[3, \infty)$.

In 2010, Chistyakov introduced and studied the concept of metric modular on an arbitrary set, more general than a metric space [11][12].

Definition 3. [11] A binary (M, w) is called a modular metric space if M is a nonempty set and w is a function on $(0, \infty) \times M^2$ such that the following axioms hold:

$(MM1)$ $w_\lambda(a, b) = 0$ if and only if $a = b$

$(MM2)$ $w_\lambda(a, b) = w_\lambda(b, a)$

$(MM3)$ $w_{\lambda+\mu}(a, b) \leq w_\lambda(a, c) + w_\mu(c, b)$

for all $a, b, c \in M$ and $\lambda, \mu > 0$.

Lemma 1. [11] By conditions $(MM1)$ and $(MM3)$, if $0 < \lambda < \mu$, then $w_\lambda(a, b) \leq w_{\lambda-\mu}(a, a) + w_\mu(a, b) = w_\mu(a, b)$ for each $a, b \in M$. This shows that $0 < \lambda \to w_\lambda(a, b) \in [0, \infty)$ is a nonincreasing function.

Example 2. Let (M, d) be a metric space. Define

$$w_\lambda(a, b) = \frac{d(a, b)}{\lambda}$$

for each $a, b \in M$ and $\lambda > 0$. In this instance, (M, w) is a modular metric space.

It is clear that w provides the properties $(MM1)$ and $(MM2)$. If the triangle inequality property of the d metric is used, we obtain

$$\begin{aligned} w_{\lambda+\mu}(a, b) = \tfrac{d(a,b)}{\lambda+\mu} &\leq \tfrac{d(a,c)}{\lambda+\mu} + \tfrac{d(c,b)}{\lambda+\mu} \\ &\leq \tfrac{d(a,c)}{\lambda} + \tfrac{d(c,b)}{\mu} \\ &= w_\lambda(a, c) + w_\mu(c, b) \end{aligned}$$

for all $a, b, c \in M$ and $\lambda, \mu > 0$. This indicates that w is a modular metric on M.

Definition 4. [11] Let (M, w) be a modular metric space. A sequence $\{a_k\}$ in M is called convergent to $a \in M$ if $\lim_{k\to\infty} w_\lambda(a_k, a) = 0$ for all $\lambda > 0$.

Definition 5. [11] Let (M, w) be a modular metric space. A sequence $\{a_k\}$ in M is called a Cauchy sequence if $\lim_{k\to\infty} w_\lambda(a_k, a_m) = 0$ for all $\lambda > 0$ and $k, m \in \mathbb{N}$.

Definition 6. [11] A modular metric space (M, w) in which every Cauchy sequence is convergent to a point in M is called complete.

Definition 7. [25] A pair of self mappings (g,h) of a modular metric space (M,w) is said to satisfy the common limit range property with respect to g, denoted by (CLR_g), if there exists a sequence $\{a_k\}\in M$ such that

$$\lim_{k\to\infty} ha_k = \lim_{k\to\infty} ga_k = ga$$

for some $a\in M$.

Example 3. Let $M=\mathbf{R}^+$ be a modular metric. Define a pair of self mappings g and h on M by $ha=2a+2$ and $ga=4a$ for each $a\in M$. Taking a sequence $\{a_k\}=\{1+\frac{1}{k}\}$ in M, we have

$$\lim_{k\to\infty} ha_k = \lim_{k\to\infty} ga_k = 4 = g(1)\in M,$$

which shows that h and g satisfy the (CLR_g) property.

We will use the following assumptions in our theorems:

Let ϕ be the class of all mappings $\psi:[0,\infty]\to[0,\infty]$ providing the folowing conditions :

$(\psi 1)$ ψ is continuous and nondecreasing on $[0,\infty]$,

$(\psi 2)$ $\psi(a)<a$ for each $a\in(0,\infty)$.

2 Fixed Point Results

Here, the existence and uniqueness of a common fixed point for weakly compatible mappings satisfying the (CLR_g) property in modular metric spaces is studied.

Theorem 2.1. Let $g,h:M\to M$ be weakly compatible mappings having the (CLR_g) property of a modular metric space (M,w). Assume that there exists $\psi\in\phi$ such that

(i) $w_\lambda(ha,gb)<\infty$

(ii) $w_\lambda(ha,hb)\le\psi\Big(\max\Big\{w_\lambda(ga,gb),w_\lambda(ha,ga),w_\lambda(hb,gb),w_\lambda(hb,ga),w_\lambda(ha,gb)\Big\}\Big)$

for each $a,b\in M$ and $\lambda>0$. Then, g and h have a unique common fixed point in M.

Proof . As g and h have the (CLR_g) property, then there exists a sequence $\{a_k\}$ in M such that

$$\lim_{k\to\infty} ha_k = \lim_{k\to\infty} ga_k = ga$$

for some $a\in M$. Then, let's write $a=a_k, b=a$ in (ii). We have

$$w_\lambda(ha_k,ha)\le\psi\Big(\max\Big\{w_\lambda(ga_k,ga),w_\lambda(ha_k,ga_k),w_\lambda(ha,ga),w_\lambda(ha,ga_k),w_\lambda(ha_k,ga)\Big\}\Big)$$

for all $k\geq 1$. Letting $k\to\infty$, we have

$$\begin{aligned}w_\lambda(ga,ha) &\le \psi\Big(\max\Big\{w_\lambda(ga,ga),w_\lambda(ga,ga),\\ &\quad w_\lambda(ha,ga),w_\lambda(ha,ga),w_\lambda(ga,ga)\Big\}\Big)\\ &= \psi\Big(\max\Big\{0,0,w_\lambda(ha,ga),w_\lambda(ha,ga),0\Big\}\Big)\\ &= \psi\Big(w_\lambda(ha,ga)\Big)\end{aligned}$$

for all $\lambda>0$. We claim that $ha=ga$. If not, then from (i) we have $w_\lambda(ha,gb)<\infty$. It follows from the condition of $(\psi 2)$ that

$$\psi\Big(w_\lambda(ha,ga)\Big)<w_\lambda(ha,ga)$$

which is a contradiction. Therefore, $ha=ga$.

Let $z=ha=ga$. Since the pair of self mappings (h,g) is weakly compatible, $hga=gha$ implies that $hz=hga=gha=gz$.

Now, we assert that $hz = z$. Suppose that $hz \neq z$. On using (i), we have $0 < w_\lambda(ha, gb) < \infty$ for some $\lambda > 0$. From the condition $(\psi 2)$, we have $\psi(w_\lambda(hz, z)) < w_\lambda(hz, z)$. If $a = z$ and $b = a$ is written in (ii), we obtain

$$\begin{aligned} w_\lambda(hz, z) &= w_\lambda(hz, ga) \\ &\leq \psi\Big(\max \Big\{ w_\lambda(gz, ga), w_\lambda(hz, gz), \\ &\quad w_\lambda(ha, ga), w_\lambda(ha, gz), w_\lambda(hz, ga) \Big\}\Big) \\ &= \psi\Big(\max \Big\{ w_\lambda(gz, ga), 0, 0, \\ &\quad w_\lambda(ha, gz), w_\lambda(hz, ga) \Big\}\Big) \\ &= \psi\Big(\max \Big\{ w_\lambda(hz, ha), 0, 0, \\ &\quad w_\lambda(ha, hz), w_\lambda(hz, ha) \Big\}\Big) \\ &= \psi\Big(w_\lambda(hz, ha) \Big) \\ &= \psi\Big(w_\lambda(hz, z) \Big) \end{aligned}$$

for all $\lambda > 0$, which is a contradiction because of the condition $(\psi 2)$. Thus $hz = z$, that is, $z = hz = gz$. As a result, z is a common fixed point of h and g.

In order to see that z is the only common fixed point of (h, g), let $w(\neq z)$ is another common fixed point, that is, $hw = gw = w$.

Since $w \neq z$, there exists $\lambda > 0$ such that $0 < w_\lambda(z, w) < \infty$. Then, we have $\psi(w_\lambda(z, w))$ by virtue of $(\psi 2)$. From (ii), we get

$$\begin{aligned} w_\lambda(z, w) &= w_\lambda(hz, hw) \\ &\leq \psi\Big(\max \Big\{ w_\lambda(gz, gw), w_\lambda(hz, gz), w_\lambda(hw, gw), \\ &\quad w_\lambda(hw, gz), w_\lambda(hz, gw) \Big\}\Big) \\ &= \psi\Big(\max \Big\{ w_\lambda(z, w), w_\lambda(z, z), w_\lambda(w, w), \\ &\quad w_\lambda(w, z), w_\lambda(z, w) \Big\}\Big) \\ &= \psi\Big(w_\lambda(z, w) \Big) \end{aligned}$$

for all $\lambda > 0$, which is a contradiction. Hence, z is a unique common fixed point of the mappings g and h. This concludes the proof. □

Theorem 2.2. Let $g, h : M \to M$ be weakly compatible mappings such that $h(M) \subset g(M)$ where M is a modular metric space. Assume that there exists a number $\gamma \in [0, \frac{1}{3})$ such that

(i) $w_\lambda(ha, gb) < \infty$

(ii) $w_\lambda(ha, hb) \leq \gamma[w_\lambda(ha, ga) + w_{2\lambda}(ha, gb) + w_{3\lambda}(hb, ga) + w_{4\lambda}(hb, gb) + w_{5\lambda}(gb, ga)]$

for each $a, b \in M$ and $\lambda > 0$. Then, g and h have a unique common fixed point provided that they satisfy the (CLR_g) property.

Proof . As h and g have the (CLR_g) property, then there exists a sequence $\{a_k\}$ in M such that

$$\lim_{k\to\infty} ha_k = \lim_{k\to\infty} ga_k = ga$$

for some $a \in M$. From (ii), we have

$$w_\lambda(ha_k, ha) \leq \gamma\Big[w_\lambda(ha_k, ga_k) + w_{2\lambda}(ha_k, ga) + w_{3\lambda}(ga, ga_k) + w_{4\lambda}(ha, ga) + w_{5\lambda}(ga, ga_k)\Big]$$

for all $k \geq 1$. Considering that λ is a nonincreasing function. Letting $k \to \infty$, we have

$$\begin{aligned} w_\lambda(ga, ha) &\leq \gamma\Big[w_\lambda(ga, ga) + w_{2\lambda}(ga, ga) + w_{3\lambda}(ha, ga) + w_{4\lambda}(ha, ga) + w_{5\lambda}(ga, ga)\Big] \\ &= \gamma\Big[w_{3\lambda}(ha, ga) + w_{4\lambda}(ha, ga)\Big] \\ &\leq 2\gamma w_\lambda(ha, ga). \end{aligned}$$

We obtain $ha = ga$. Let $z = ha = ga$. Taking into account that h and g are weakly compatible mappings, $hga = gha$ implies that $hz = hga = gha = gz$.

We claim that $hz = z$. From (ii), we have

$$\begin{aligned} w_\lambda(hz, z) &= w_\lambda(hz, ha) \\ &\leq \gamma\Big[w_\lambda(hz, gz) + w_{2\lambda}(hz, ga) + w_{3\lambda}(ha, gz) + w_{4\lambda}(ha, ga) + w_{5\lambda}(ga, gz)\Big] \\ &= \gamma\Big[w_{2\lambda}(hz, z) + w_{3\lambda}(z, hz) + w_{5\lambda}(z, hz)\Big] \end{aligned}$$

and, with Lemma (1) , since the function $\lambda \to w_\lambda(a, b)$ is nonincreasing, we have

$$w_\lambda(hz, z) \leq 3\gamma w_\lambda(hz, z).$$

This implies that $(1 - 3\gamma)w_\lambda(hz, z) \leq 0$ for all $\lambda > 0$, that is, $w_\lambda(hz, z) = 0$ and so $hz = z = gz$. Thus z is a common fixed point of h and g.

To prove the uniqueness of the common fixed point, let $w \in M$ is another common fixed point of h and g, that is, $hw = gw = w$. By (ii), we have

$$\begin{aligned} w_\lambda(gw, gz) &= w_\lambda(hw, hz) \\ &\leq \gamma\Big[w_\lambda(hw, gw) + w_{2\lambda}(hw, gz) + w_{3\lambda}(hz, gw) + w_{4\lambda}(hz, gz) + w_{5\lambda}(gz, gw)\Big] \\ &= \gamma\Big[w_{2\lambda}(hw, gz) + w_{3\lambda}(hz, gw) + w_{5\lambda}(gz, gw)\Big] \\ &= \gamma\Big[w_{2\lambda}(gw, gz) + w_{3\lambda}(gz, gw) + w_{5\lambda}(gz, gw)\Big] \end{aligned}$$

and, with Lemma (1), since the function $\lambda \to w_\lambda(x, y)$ is nonincreasing, we have

$$w_\lambda(gw, gz) \leq \gamma[w_\lambda(gw, gz) + w_\lambda(gw, gz) + w_\lambda(gz, gw].$$

This implies $gw = gz$. Thus, h and g have a unique common fixed point such that $hz = gz = z$. □

3 Application to Integral Type Contraction

In this part, we give an application related with Theorem (2.2) in modular metric spaces.

Theorem 3.1. Let $h, g : M \to M$ be weakly compatible mappings such that $h(M) \subset g(M)$ where M is a modular metric space. Assume that h and g satisfy the (CLR_g) property and there exists a number $\gamma \in [0, \frac{1}{3})$ such that each $a, b \in M$ and $\lambda > 0$ satisfying

$(C1)$ there exist $a, b \in M$ such that $w_\lambda(ha, gb) < \infty$

$(C2)$

$$\begin{aligned} &\int_0^{w_\lambda(ha,hb)} \chi(t)dt \leq \gamma\Big(\int_0^{w_\lambda(ha,ga)} \chi(t)dt + \int_0^{w_{2\lambda}(ha,gb)} \chi(t)dt + \int_0^{w_{3\lambda}(hb,ga)} \chi(t)dt \\ &+ \int_0^{w_{4\lambda}(hb,gb)} \chi(t)dt + \int_0^{w_{5\lambda}(gb,ga)} \chi(t)dt\Big) \end{aligned}$$

where $\chi : \mathbf{R}^+ \to \mathbf{R}^+$ is a Lebesque integrable function which is summable, nonnegative such that $\int_0^\epsilon \chi(t)dt > 0$ for all $\epsilon > 0$. Then, h and g have a unique common fixed point in M.

Proof . In proof of Theorem (2.2), as h and g have the (CLR_g) property, there exist a sequence $\{a_k\}$ and a point a in M such that $ha_k \to ga$ and $ga_k \to ga$. By $(C2)$, for $a = a_k$ and $b = a$

$$\begin{aligned} &\int_0^{w_\lambda(ha_k,hb)} \chi(t)dt \leq \gamma\Big(\int_0^{w_\lambda(ha_k,ga_k)} \chi(t)dt + \int_0^{w_{2\lambda}(ha_k,ga)} \chi(t)dt + \int_0^{w_{3\lambda}(ha,ga_k)} \chi(t)dt \\ &+ \int_0^{w_{4\lambda}(ha,ga)} \chi(t)dt + \int_0^{w_{5\lambda}(ga,ga_k)} \chi(t)dt\Big) \end{aligned}$$

and letting $k \to \infty$, we have

$$\int_0^{w_\lambda(ga,ha)} \chi(t)dt \leq \gamma\Big(\int_0^{w_{3\lambda}(ha,ga)} \chi(t)dt + \int_0^{w_{4\lambda}(ha,ga)} \chi(t)dt\Big).$$

With Lemma (1), we obtain

$$\int_0^{w_\lambda(ga,ha)} \chi(t)dt \leq 2\gamma\Big(\int_0^{w_\lambda(ha,ga)} \chi(t)dt$$

and thus $ha = ga$. Let $z = ha = ga$. Since h and g are weakly compatible mappings, $hz = hga = gha = gz$.

Suppose that $hz \neq z$. By $(C2)$, we have

$$\int_0^{w_\lambda(hz,z)} \chi(t)dt = \int_0^{w_\lambda(hz,ha)} \chi(t)dt \leq \gamma\Big(\int_0^{w_{2\lambda}(hz,z)} \chi(t)dt + \int_0^{w_3\lambda(z,hz)} \chi(t)dt + \int_0^{w_5\lambda(z,hz)} \chi(t)dt.$$

With Lemma (1), we obtain

$$\int_0^{w_\lambda(hz,z)} \chi(t)dt \leq 3\gamma \int_0^{w_\lambda(hz,z)} \chi(t)dt.$$

This implies that $hz = z = gz$. Thus, z is a common fixed point. The uniqueness of common fixed point is an easy consequence of inequality $(C2)$. □

Remark 1. If we take $\chi(t) = 1$ for all $t \geq 0$, we obtain Theorem (2.2).

4 Application to Volterra Type Integral Equations

The aim of this section is to present an existence theorem for a solution of the following system of integral equations (4.1) that belongs to $M = C([0, K], \mathbf{R})$ by using the obtained result in Theorem (2.2).

Consider the following system of nonlinear integral equations Volterra type :

$$\begin{aligned} \upsilon(a) &= f(a) + \int_0^a K\Big(a, s, \upsilon(s)\Big)ds \\ \varphi(a) &= f(a) + \int_0^a K\Big(a, s, \varphi(s)\Big)ds \end{aligned} \tag{4.1}$$

where $a, s \in M$. In this system, the functions $\upsilon(a)$ and $\varphi(a)$ are unknown, the function $K(a, s, \upsilon)$ is defined in

$$G = \Big\{(a, s, \upsilon) \in \mathbf{R}^3 : 0 \leq a, s \leq K, -\infty < \upsilon, \varphi < \infty\Big\}$$

and the function $f(a)$ is continuous on $[0, K]$. Let us consider $M = C([0, K], \mathbf{R})$ as the set of all continuous functions $\upsilon, \varphi : [0, K] \to \mathbf{R}$. Consider the complete metric space with the Bielecki's norm $\|\upsilon\| = sup_{a\in[0,K]}e^{-a}|\upsilon(a)|$ such that

$$d(\upsilon, \varphi) = sup_{a\in[0,K]}e^{-a}|\upsilon(a) - \varphi(a)|$$

for all $\upsilon, \varphi \in M$. We define

$$w_\lambda(\upsilon, \varphi) = \frac{d(\upsilon, \varphi)}{\lambda} = sup_{a\in[0,K]}\frac{e^{-a}|\upsilon(a) - \varphi(a)|}{\lambda} \tag{4.2}$$

for all $\upsilon, \varphi \in M$ and $\lambda > 0$. Obviously, (M, w) is a complete modular metric space.

Theorem 4.1. Assume that the following hypothesis hold:

$(V1)$ the functions K and g are continuous

$(V2)$ for all $a, s \in [0, K]$, if there exists a constant $\gamma \in [0, \frac{1}{2})$ such that

$$\begin{aligned} \textstyle\int_0^a |K(a, s, \upsilon(s)) - K(a, s, \upsilon(s))|ds \quad &\leq \quad \gamma\Big[|h\upsilon(a) - g\upsilon(a)| + \tfrac{|h\upsilon(a)-g\varphi(a)|}{2} + \tfrac{|g\upsilon(a)-h\varphi(a)|}{3} \\ &+ \quad \tfrac{|h\varphi(a)-g\varphi(a)|}{4} + \tfrac{|g\upsilon(a)-g\varphi(k)|}{5}\Big] \end{aligned}$$

then, the system of (4.1) has a unique solution in M.

Proof . Endow $M = C([0,K], \mathbf{R})$ with the modular metric defined by (4.2) and think the mappings $g, h : M \to M$ as follows:

$$hv(a) = f(a) + \int_0^a K\Big(a, s, v(s)\Big)ds, v \in M, a \in [0,K]$$
$$gv(a) = f(a) + \int_0^a K\Big(a, s, v(s)\Big)ds, v \in M, a \in [0,K].$$

For the inequality (ii) of the Theorem (2.2), we have

$$\begin{aligned} |hv(a) - h\varphi(a)| &= |f(a) + \textstyle\int_0^a |K(a,s,v(s))ds - f(a) - \int_0^a |K(a,s,\varphi(s))ds| \\ &= |\textstyle\int_0^a \Big(K(a,s,v(s)) - K(a,s,\varphi(s))\Big)ds| \\ &\le \textstyle\int_0^a |K(a,s,v(s)) - K(a,s,\varphi(s))|ds \\ &\le \gamma\Big[|hv(a) - gv(a)| + \tfrac{|hv(a)-g\varphi(a)|}{2} + \tfrac{|gv(a)-h\varphi(a)|}{3} \\ &+ \tfrac{|h\varphi(a)-g\varphi(a)|}{4} + \tfrac{|gv(a)-g\varphi(a)|}{5}\Big] \end{aligned}$$

If we multiply both sides of the inequality by $\frac{e^{-a}}{\lambda}$, we have

$$\begin{aligned} \frac{e^{-a}}{\lambda}\Big|hv(a) - h\varphi(a)\Big| \le \gamma\frac{e^{-a}}{\lambda}\Bigg[&|hv(a) - gv(a)| + \frac{|hv(a) - g\varphi(a)|}{2} + \frac{|gv(a) - h\varphi(a)|}{3} \\ &+ \frac{|h\varphi(a) - g\varphi(a)|}{4} + \frac{|gv(a) - g\varphi(a)|}{5}\Bigg] \end{aligned}$$

and thus, we obtain the following inequality:

$$w_\lambda(hv, h\varphi) \quad \le \quad \gamma\Bigg[w_\lambda(hv, gv) + w_{2\lambda}(hv, g\varphi) + w_{3\lambda}(gv, h\varphi) + w_{4\lambda}(h\varphi, g\varphi) + w_{5\lambda}(gv, g\varphi)\Bigg]$$

Hence, all the conditions of Theorem (2.2) are verify. The system of (4.1) has a unique solution in M. □

5 Conclusions

In the present paper, we prove some common fixed point theorems for weakly compatible mappings satisfying common limit in the range property in a modular metric spaces. As an application of our result, we study the existence and uniqueness of the solution a system of integral type contraction and Volterra type integral equations.

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