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New subclasses of bi-univalent functions associated with $q-$calculus operator

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Abstract

In the present paper, new subclasses of bi-univalent functions associated with $q-$calculus operator are introduced and coefficient estimates for functions in these classes are obtained. Several new (or known) consequences of the results are also pointed out.

Keywords: Analytic function, Univalent function, $q-$calculus operator, Bi-univalent function, Bi-starlike function, Bi-convex function, Coefficient bounds and Subordination
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1 Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$ in $\mathbb{U}$.
The convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as $(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$, where $f(z)$ is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

From Koebe one quarter theorem [16], it is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4)$$

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where

$$f^{-1}(w) = g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). The functions

$$\frac{z}{1-z}, \qquad -\log(1-z), \quad and \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

are in the class Σ (see details in [20]). However, the familiar Koebe function is not bi-univalent. Earlier, Brannan and Taha [5] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike functions of order α denoted by $\mathcal{S}^*_\Sigma(\alpha)$ and bi-convex functions of order α denoted by $\mathcal{K}_\Sigma(\alpha)$ corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively. Also, they determined non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (see also[25]). Many researchers have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (see[1, 6, 9, 20, 22, 26]also the refences cited there in).

An analytic function F is subordinate to an analytic function G, written $F(z) \prec G(z)$, provided there is an analytic function w defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $F(z) = G(\omega(z))$. Ma and Minda [11] unified various subclasses of starlike and convex functions, $f \in \mathcal{A}$ satisfying the subordination $\frac{z\, f'(z)}{f(z)} \prec \phi(z)$ and convex $1 + \frac{z\, f''(z)}{f'(z)} \prec \phi(z)$, respectively. For this purpose, it is assumed that ϕ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(\mathbb{U})$ is symmetric with respect to the real axis and has a series expansion of the form

$$\phi(z) = 1 + \mathfrak{C}_1 z + \mathfrak{C}_2 z^2 + \mathfrak{C}_3 z^3 + \cdots, \quad (\mathfrak{C}_1 > 0). \tag{1.3}$$

The study of operators plays an important role in the geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better.

We briefly recall here the notion of *q-operators* i.e.*q-difference operators* that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of *q-calculus* was initiated by Jackson [7] (also see [2, 3, 21]). Kanas and Răducanu [8] have used the fractional *q-calculus operators* to investigate certain classes of functions which are analytic in $\mathbb{U}$.

Consider $0 < q < 1$ and a non-negative integer n. The $q-$integer number or basic number n is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1},\ [0]_q = 0.$$

For a non-integer number t we will denote $[t]_q = \dfrac{1-q^t}{1-q}$. The $q-$ shifted factorial is defined as $[0]_q! = 1, \quad [n]_q! = [1]_q[2]_q \ldots [n]_q$. Note that $\lim\limits_{q\to 1^-}[n]_q = n$ and $\lim\limits_{q\to 1^-}[n]_q! = n!$.

The Jackson's *q-derivative operator* or *q-difference operator* for a function $f \in \mathcal{A}$ is defined by

$$\mathcal{D}_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{z(q-1)} & , z \neq 0 \\ f'(0) & , z = 0 \end{cases} \tag{1.4}$$

is given by

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} n_q a_n z^{n-1}$$

and

$$\mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z)).$$

Note that for $\mathcal{D}_q z^n = [n]_q z^{n-1}; n \in \mathbb{N} = \{1, 2, \ldots\}$ and $z \in \mathbb{U}$. For $t \in \mathbb{R}$ and $n \in \mathbb{N}$, the *q-generalized Pochhammer* symbol is defined by

$$[t]_n = [t]_q[t+1]_q[t+2]_q \ldots [t+n-1]_q.$$

Moreover, for $t > 0$ the $q-$ Gamma function is given by

$$\Gamma_q(t+1) = [t]_q\Gamma_q(t) \ \text{ and } \ \Gamma_q(1) = 1.$$

In [15], Obradovic et.al gave some criteria for univalence expressing by $\Re(f'(z)) > 0$, for the linear combinations

$$\mu\left(1+\frac{zf''(z)}{f'(z)}\right) + (1-\mu)\frac{1}{f'(z)} > 0, \qquad (\mu \geq 1, z \in \mathbb{U}).$$

In [19], Silverman investigated an expression involving the quotient of the analytic representations of convex and starlike functions. Precisely, for $0 < b \leq 1$he considered the class

$$\mathcal{G}_b = f \in A : \left|\frac{1+(zf''(z)/f'(z))}{zf'(z)/f(z)} - 1\right| < b, \quad z \in \mathbb{U}$$

and proved that $\mathcal{G}_b \subset \mathcal{S}^*(\frac{2}{1+\sqrt{1+8b}})$. That is the functions in the class $\mathcal{G}_b$ are starlike of order $\frac{2}{1+\sqrt{1+8b}}$. Based on the above definitions recently, in [10], Lashin introduced and studied the new subclasses of bi-univalent functions.

Motivated by the earlier works cited in [9, 12, 20] and Lashin [10], in this paper we introduce new subclasses of the function class Σ involving $q-$ calculus operator and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of Σ. Several related classes are also considered and connection to earlier known(or new) results are made.

Definition 1.1. A function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{L}_{\Sigma,q}(\mu,\phi)$ if it satisfies the following conditions :

$$\mu\left(1+\frac{z\mathcal{D}_q^2 f(z)}{\mathcal{D}_q f(z)}\right) + (1-\mu)\frac{1}{\mathcal{D}_q f(z)} \prec \phi(z) \tag{1.5}$$

and

$$\mu\left(1+\frac{w(\mathcal{D}_q^2 g(w)}{\mathcal{D}_q g(w)}\right) + (1-\mu)\frac{1}{\mathcal{D}_q g(w)} \prec \phi(w) \tag{1.6}$$

where $\mu \geq 1, z, w \in \mathbb{U}$ the function g is given by (1.2).

Definition 1.2. For a function $f(z) \in \Sigma$ given by (1.1),is said to be in the class $\mathcal{L}_{\Sigma,q}(1,\phi) \equiv K_{\Sigma,q}(\phi)$ if it satisfies the following conditions :

$$\left(1+\frac{z\mathcal{D}_q^2 f(z)}{\mathcal{D}_q f(z)}\right) \prec \phi(z) \ \text{ and } \ \left(1+\frac{w\mathcal{D}_q^2 g(w)}{\mathcal{D}_q g(w)}\right) \prec \phi(w)$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

Definition 1.3. A function $f(z) \in \Sigma$ given by (1.1) is said to be in $\mathcal{M}_{\Sigma,q}(\phi)$ if it satisfies the following conditions :

$$\left(\frac{1+\frac{z\mathcal{D}_q^2 f(z)}{\mathcal{D}_q f(z)}}{\frac{z\mathcal{D}_q f(z)}{f(z)}}\right) \prec \phi(z) \quad \text{and} \quad \left(\frac{1+\frac{w\mathcal{D}_q^2 g(w)}{(\mathcal{D}_q g(w)}}{\frac{w\mathcal{D}_q g(w)}{g(w)}}\right) \prec \phi(w) \tag{1.7}$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

Now, in the following section we determine the initial coefficients $|a_2|$ *and* $|a_3|$ for functions f in the subclasses $\mathcal{L}_{\Sigma,q}(\mu,\phi)$, and $\mathcal{K}_{\Sigma,q}(\phi)$.

2 Initial Coefficient estimates for $f \in \mathcal{L}_{\Sigma,q}(\mu,\phi)$, and $f \in \mathcal{K}_{\Sigma,q}(\phi)$,

Define the functions $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$ and suppose that

$$p(z) := p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := q_1 z + q_2 z^2 + \cdots .$$

It is well known that (see [13],p.172)

$$|p_1| \le 1; \qquad |p_2| \le 1 - |p_1|^2; \qquad |q_1| \le 1; \qquad |q_2| \le 1 - |q_1|^2. \tag{2.1}$$

It follows that,

$$\phi(p(z)) := 1 + \mathfrak{C}_1 p_1 z + (\mathfrak{C}_1 p_2 + \mathfrak{C}_2 p_1^2) z^2 + \cdots \tag{2.2}$$

and

$$\phi(q(w)) := 1 + \mathfrak{C}_1 q_1 w + (\mathfrak{C}_1 q_2 + \mathfrak{C}_2 q_1^2) w^2 + \cdots . \tag{2.3}$$

Theorem 2.1. If $f(z) \in \mathcal{L}_{\Sigma,q}(\mu,\phi)$ be given by (1.1) and $\mu \ge 1$, then

$$|a_2| \le \frac{\mathfrak{C}_1\sqrt{\mathfrak{C}_1}}{\sqrt{(2\mu-1)^2(1+q)^2\mathfrak{C}_1 + \left|A(\mu,q)\mathfrak{C}_1^2 - (2\mu-1)^2(1+q)^2\mathfrak{C}_2\right|}} \tag{2.4}$$

and

$$|a_3| \le \begin{cases} \dfrac{\mathfrak{C}_1}{A(\mu,q)} & \text{if} \quad \mathfrak{C}_1 \le \mathfrak{C}_2 \\ \dfrac{(2\mu-1)^2(1+q)^2\mathfrak{C}_1|\mathfrak{C}_2| + \mathfrak{C}_1\left|A(\mu,q)\mathfrak{C}_1^2 - (2\mu-1)^2(1+q)^2\mathfrak{C}_2\right|}{A(\mu,q)\left[(2\mu-1)^2(1+q)^2\mathfrak{C}_1 + \left|A(\mu,q)\mathfrak{C}_1^2 - (2\mu-1)^2(1+q)^2\mathfrak{C}_2\right|\right]} & \text{if} \quad \mathfrak{C}_1 > \mathfrak{C}_2 \end{cases}$$

where

$$A(\mu,q) = (1+q+q^2)(2\mu+\mu q-1) + (1-2\mu)(1+q)^2.$$

Proof . It follows from (1.5) and (1.6) that

$$\mu\left(1 + \frac{z\mathcal{D}_q^2 f(z)}{\mathcal{D}_q f(z)}\right) + (1-\mu)\frac{1}{\mathcal{D}_q f(z)} = \phi(u(z)) \tag{2.5}$$

and

$$\mu\left(1 + \frac{w(\mathcal{D}_q^2 g(w)}{\mathcal{D}_q g(w)}\right) + (1-\mu)\frac{1}{\mathcal{D}_q g(w)} = \phi(v(w)). \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned} 1 + (2\mu-1)(1+q)a_2 z + \left[(1+q+q^2)(2\mu+\mu q-1)a_3 + (1-2\mu)(1+q)^2 a_2^2\right] z^2 + \cdots \\ = 1 + \mathfrak{C}_1 p_1 z + (\mathfrak{C}_1 p_2 + \mathfrak{C}_2 p_1^2) z^2 + \cdots \end{aligned}$$

and

$$\begin{aligned} & 1 - (2\mu-1)(1+q)a_2 w \\ + & \left[\left[2(1+q+q^2)(2\mu+\mu q-1) + (1-2\mu)(1+q)^2\right] a_2^2 - (1+q+q^2)(2\mu+\mu q-1)a_3\right] w^2 \cdots \\ = & 1 + \mathfrak{C}_1 q_1 w + (\mathfrak{C}_1 q_2 + \mathfrak{C}_2 q_1^2) w^2 + \cdots . \end{aligned}$$

Now, equating the coefficients of z and z^2 , we get

$$(2\mu-1)(1+q)a_2 = \mathfrak{C}_1 p_1, \tag{2.7}$$

$$(1+q+q^2)(2\mu+\mu q-1)a_3+(1-2\mu)(1+q)^2a_2^2=\mathfrak{C}_1p_2+\mathfrak{C}_2p_1^2, \tag{2.8}$$

$$-(2\mu-1)(1+q)a_2=\mathfrak{C}_1q_1, \tag{2.9}$$

and

$$\begin{aligned}&\left[2(1+q+q^2)(2\mu+\mu q-1)+(1-2\mu)(1+q)^2\right]a_2^2\\&\qquad-(1+q+q^2)(2\mu+\mu q-1)a_3=\mathfrak{C}_1q_2+\mathfrak{C}_2q_1^2.\end{aligned} \tag{2.10}$$

From (2.7) and (2.9), we get

$$p_1=-q_1 \tag{2.11}$$

and

$$2(2\mu-1)^2(1+q)^2a_2^2=\mathfrak{C}_1^2(p_1^2+q_1^2). \tag{2.12}$$

Now from (2.8), (2.10) and (2.12), we obtain

$$\begin{aligned}&\Big(2(2\mu-1)^2(1+q)^2\mathfrak{C}_1+\left[2(1+q+q^2)(2\mu+\mu q-1)+2(1-2\mu)(1+q)^2\right]\mathfrak{C}_1^2\\&\qquad-2(2\mu-1)^2(1+q)^2\mathfrak{C}_2\Big)a_2^2=\mathfrak{C}_1^3(p_2+q_2).\end{aligned} \tag{2.13}$$

From(2.11), (2.13) and by using (2.1), for the coefficients p_2 and q_2, we have

$$\begin{aligned}&\Big|2(2\mu-1)^2(1+q)^2\mathfrak{C}_1+\left[2(1+q+q^2)(2\mu+\mu q-1)+2(1-2\mu)(1+q)^2\right]\mathfrak{C}_1^2\\&\qquad-2(2\mu-1)^2(1+q)^2\mathfrak{C}_2\Big||a_2|^2\le\mathfrak{C}_1^3|(1-|p_1|^2).\end{aligned} \tag{2.14}$$

From(2.7) and (2.14) we obtain

$$|a_2|\le\frac{\mathfrak{C}_1\sqrt{\mathfrak{C}_1}}{\sqrt{\left|\left[(1+q+q^2)(2\mu+\mu q-1)+(1-2\mu)(1+q)^2\right]\mathfrak{C}_1^2\right|-(2\mu-1)^2(1+q)^2(|\mathfrak{C}_2|-\mathfrak{C}_1)}}.$$

For convenience we let,

$$A(\mu,q)=(1+q+q^2)(2\mu+\mu q-1)+(1-2\mu)(1+q)^2. \tag{2.15}$$

Hence

$$|a_2|\le\frac{\mathfrak{C}_1\sqrt{\mathfrak{C}_1}}{\sqrt{(2\mu-1)^2(1+q)^2\mathfrak{C}_1+\left|A(\mu,q)\mathfrak{C}_1^2-(2\mu-1)^2(1+q)^2\mathfrak{C}_2\right|}}.$$

From (2.8) from (2.10) and using (2.11), we get

$$\begin{aligned}&2\left[(1+q+q^2)(2\mu+\mu q-1)+(1-2\mu)(1+q)^2\right](1+q+q^2)(2\mu+\mu q-1)a_3\\&=\left[2(1+q+q^2)(2\mu+\mu q-1)+(1-2\mu)(1+q)^2\right]\mathfrak{C}_1p_2\\&-(1-2\mu)(1+q)^2\mathfrak{C}_1q_2+2(1+q+q^2)(2\mu+\mu q-1)\mathfrak{C}_2p_1^2.\end{aligned}$$

Then by using (2.1), for the coefficients p_2 and q_2, we get

$$\left[(1+q+q^2)(2\mu+\mu q-1)+(1-2\mu)(1+q)^2\right]|a_3|\le\mathfrak{C}_1+\left[|\mathfrak{C}_2|-\mathfrak{C}_1\right]|p_1|^2. \tag{2.16}$$

From(2.7),

$$\begin{aligned}|p_1^2|&\le\frac{(2\mu-1)^2(1+q)^2}{\mathfrak{C}_1^2}|a_2|^2\\&=\frac{(2\mu-1)^2(1+q)^2\mathfrak{C}_1}{(2\mu-1)^2(1+q)^2\mathfrak{C}_1+\left|A(\mu,q)\mathfrak{C}_1^2-(2\mu-1)^2(1+q)^2\mathfrak{C}_2\right|}.\end{aligned}$$

Thus by substituting for $|p_1|^2$ in (2.16) and by simple computation we get,

$$|a_3| \leq \begin{cases} \dfrac{\mathfrak{C}_1}{A(\mu,q)} & \text{if} \quad \mathfrak{C}_1 \leq \mathfrak{C}_2 \\ \dfrac{(2\mu-1)^2(1+q)^2\mathfrak{C}_1|\mathfrak{C}_2| + \mathfrak{C}_1\big|A(\mu,q)\mathfrak{C}_1^2 - (2\mu-1)^2(1+q)^2\mathfrak{C}_2\big|}{A(\mu,q)\Big[(2\mu-1)^2(1+q)^2\mathfrak{C}_1 + \big|A(\mu,q)\mathfrak{C}_1^2 - (2\mu-1)^2(1+q)^2\mathfrak{C}_2\big|\Big]} & \text{if} \quad \mathfrak{C}_1 > \mathfrak{C}_2 \end{cases}$$

where $A(\mu,q)$ is as assumed in (2.15). $\square$

By taking $\mu = 1$ we state the following :

Theorem 2.2. Let $f(z)$ be given by (1.1) and $f \in \mathcal{K}_{\Sigma,q}(\phi)$, then

$$|a_2| \leq \frac{\mathfrak{C}_1\sqrt{\mathfrak{C}_1}}{\sqrt{(1+q)^2\mathfrak{C}_1 + \big|A(q)\mathfrak{C}_1^2 - (1+q)^2\mathfrak{C}_2\big|}} \tag{2.17}$$

and

$$|a_3| \leq \begin{cases} \dfrac{\mathfrak{C}_1}{A(q)} & \text{if} \quad \mathfrak{C}_1 \leq \mathfrak{C}_2 \\ \dfrac{(1+q)^2\mathfrak{C}_1|\mathfrak{C}_2| + \mathfrak{C}_1\big|A(q)\mathfrak{C}_1^2 - (1+q)^2\mathfrak{C}_2\big|}{A(q)\Big[(1+q)^2\mathfrak{C}_1 + \big|A(q)\mathfrak{C}_1^2 - (1+q)^2\mathfrak{C}_2\big|\Big]} & \text{if} \quad \mathfrak{C}_1 > \mathfrak{C}_2 \end{cases} \tag{2.18}$$

where $A(1,q) = A(q) = (1+q+q^2)(1+q) + (1+q)^2$.

3 Coefficient estimates for the function class $\mathcal{M}_{\Sigma,q}(\phi)$

Theorem 3.1. Let $f(z)$ is given by (1.1) and $f \in \mathcal{M}_{\Sigma,q}(\phi)$. Then

$$|a_2| \leq \frac{\mathfrak{C}_1\sqrt{\mathfrak{C}_1}}{|q(q^2-1)\mathfrak{C}_1^2 - \mathfrak{C}_2| + \mathfrak{C}_1} \tag{3.1}$$

and

$$|a_3| \leq \begin{cases} \frac{\mathfrak{C}_1}{(1+q)(1+q^2)} & \text{if} \quad |\mathfrak{C}_1| \leq \frac{1}{(1+q)(1+q^2)}, \\ \frac{\mathfrak{C}_1}{(1+q)(1+q^2)} + \big(1 - \frac{1}{(1+q)(1+q^2)\mathfrak{C}_1}\big)\frac{\mathfrak{C}_1^3}{|q(q^2-1)\mathfrak{C}_1^2 - \mathfrak{C}_2| + \mathfrak{C}_1} & \text{if} \quad |\mathfrak{C}_1| > \frac{1}{(1+q)(1+q^2)}. \end{cases}$$

Proof . We can write the argument inequalities in (1.7) equivalently as follows:

$$\frac{1 + \frac{z\mathcal{D}_q^2 f(z)}{\mathcal{D}_q f(z)}}{\frac{z\mathcal{D}_q f(z)}{f(z)}} = \phi(u(z)) \quad \textit{and} \quad \frac{1 + \frac{w\mathcal{D}_q^2 g(w)}{g(w)}}{\frac{w\mathcal{D}_q g(w)}{g(w)}} = \phi(v(w)), \tag{3.2}$$

and proceeding as in the proof of Theorem 2.1, we can arrive the following relations from (3.2)

$$a_2 = \mathfrak{C}_1 p_1, \tag{3.3}$$

$$(1+q)(1+q^2)a_3 - (1+q)^2 a_2^2 = \mathfrak{C}_1 p_2 + \mathfrak{C}_2 p_1^2 \tag{3.4}$$

and

$$-a_2 = \mathfrak{C}_1 q_1, \tag{3.5}$$

$$(1+q^2+2q^3)a_2^2 - (1+q)(1+q^2)a_3 = \mathfrak{C}_1 q_2 + \mathfrak{C}_2 q_1^2. \tag{3.6}$$

From (3.3) and (3.5), we get

$$p_1 = -q_1 \tag{3.7}$$

and

$$2a_2^2 = \mathfrak{C}_1^2(p_1^2 + q_1^2). \tag{3.8}$$

Now from (3.4), (3.6) and (3.8), we obtain

$$2q(q^2-1)\mathfrak{C}_1^2 - 2\mathfrak{C}_2 a_2^2 = -\mathfrak{C}_1^3(p_2+q_2). \tag{3.9}$$

From (3.7), (3.9) and by using (2.1), we get

$$\begin{aligned} 2|q(q^2-1)\mathfrak{C}_1^2 - \mathfrak{C}_2||a_2|^2 &\leq 2\mathfrak{C}_1^3(1-|p_1|^2) \\ |q(q^2-1)\mathfrak{C}_1^2 - \mathfrak{C}_2||a_2|^2 &\leq \mathfrak{C}_1^3 - \frac{\mathfrak{C}_1^3|a_2|^2}{\mathfrak{C}_1^2}) \\ \{|q(q^2-1)\mathfrak{C}_1^2 - \mathfrak{C}_2| + \mathfrak{C}_1\}|a_2|^2 &\leq \mathfrak{C}_1^3. \end{aligned} \tag{3.10}$$

From (3.5)and (3.10)

$$|a_2| \leq \frac{\mathfrak{C}_1\sqrt{\mathfrak{C}_1}}{|q(q^2-1)\mathfrak{C}_1^2 - \mathfrak{C}_2| + \mathfrak{C}_1}.$$

Next, in order to find the bound on $|a_3|$, from (3.4) (3.6), and using (3.7), we get

$$2(1+q)(1+q^2)a_3 = \mathfrak{C}_1(p_2-q_2) + \mathfrak{C}_2(p_1^2-q_1^2) + [(1+q)^2 + (2q^3+q^2+1)]a_2^2.$$

Thus by using(3.5) ,(3.7) and (3.10) in above equation, we get

$$\begin{aligned} 2(1+q)(1+q^2)|a_3| &\leq 2\mathfrak{C}_1(1-|p_1|^2) + 2(1+q)(1+q^2)|a_2|^2 \\ (1+q)(1+q^2)|a_3| &\leq \mathfrak{C}_1(1-\frac{|a_2|^2}{\mathfrak{C}_1^2}) + (1+q)(1+q^2)|a_2|^2 \\ &= \mathfrak{C}_1 + \{\,(1+q)(1+q^2) - \frac{1}{\mathfrak{C}_1}\}\frac{\mathfrak{C}_1^3}{|q(q^2-1)\mathfrak{C}_1^2 - \mathfrak{C}_2| + \mathfrak{C}_1} \\ |a_3| &\leq \begin{cases} \frac{\mathfrak{C}_1}{(1+q)(1+q^2)} & \text{if} \quad |\mathfrak{C}_1| \leq \frac{1}{(1+q)(1+q^2)}, \\ \frac{\mathfrak{C}_1}{(1+q)(1+q^2)} + \big(1 - \frac{1}{(1+q)(1+q^2)\mathfrak{C}_1}\big)\frac{\mathfrak{C}_1^3}{|q(q^2-1)\mathfrak{C}_1^2 - \mathfrak{C}_2| + \mathfrak{C}_1} & \text{if} \quad |\mathfrak{C}_1| > \frac{1}{(1+q)(1+q^2)}. \end{cases} \end{aligned}$$

□

Taking $q \to 1$ in Theorem 2.1, we obtain Theorem 1 given by Lashin [10] .

Remark 3.2. Suitable choices of $\phi(z)$ as given below

1. For $(0 < \alpha \leq 1)$ and $-1 \leq B < A \leq 1$, taking the function ϕ as

$$\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^\alpha = 1 + \alpha(A-B)z - \frac{\alpha}{2}[2B(A-B) + (1-\alpha)(A-B)^2]z^2 + \cdots \tag{3.11}$$

which gives $\mathfrak{C}_1 = \alpha(A-B)$ and $\mathfrak{C}_2 = -\frac{\alpha}{2}[2B(A-B) + (1-\alpha)(A-B)^2]$.

2. If we take $\alpha = 1$ and $-1 \leq B < A \leq 1$, then we have

$$\phi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z + B(A-B)z^2 + \cdots \tag{3.12}$$

thus we have $\mathfrak{C}_1 = A - B$ and $\mathfrak{C}_2 = B(A-B)$.

3. By fixing $A = 1$ and $B = -1$ we have

$$\phi(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots \tag{3.13}$$

thus we have $\mathfrak{C}_1 = 2$ and $\mathfrak{C}_2 = 2$

4. Further for some $c \in (0,1]$, taking

$$\phi(z) = \sqrt{1+cz} = 1 + \frac{c}{2}z - \frac{c^2}{8}z^2 + \ldots \tag{3.14}$$

then the class is said to be associated with the right -loop of the Cassinian Ovals [4]. In particular if $c = 1$ then the class is associated with right-half of the lemniscate of Bernoulli [23] is given by

$$\phi(z) = \sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \ldots. \tag{3.15}$$

5. Taking
$$\phi(z) = z + \sqrt{1+z^2} = 1 + z + \frac{1}{2}z^2 - \frac{1}{8}z^4 + \dots. \tag{3.16}$$
then the class is said to be associated with the right crescent [17].
6. Again by taking
$$\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2 \tag{3.17}$$
then the class is said to be associated with the cardioid [18].

and thus by fixing the values of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ as given in 3.11 - 3.17, from Theorems 2.1-3.1 we can state the estimates $|a_2|$ and $|a_3|$ for f in the classes $\mathcal{L}_{\Sigma,q}(\mu,\phi)$, $\mathcal{K}_{\Sigma,q}(\phi)$, and $\mathcal{M}_{\Sigma,q}(\phi)$.

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