Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 67–78
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23952.2641

A new general integral transform for solving Caputo fractional-order differential equations

Ali Khalouta*

Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, Faculty of Sciences, Ferhat Abbas Sétif University 1, 19000 Sétif, Algeria

(Communicated by Shahram Saeidi)

Abstract

There are many integral transforms that are widely used to solve many problems arising in applied mathematics and engineering. The main goal of this paper is to use a new general integral transform to solve fractional differential equations with the fractional derivative in the sense of Caputo. In addition, several illustrative examples are given to demonstrate the accurateness and effectiveness of this approach. The obtained results confirm the applicability and high accuracy of the proposed approach to solving fractional differential equations.

Keywords: fractional differential equations, Caputo fractional derivative, Mittag-Leffler function, integral transform
2010 MSC: Primary 34A08, 35A22; Secondary 33E12, 35C10.

1 Introduction

Fractional differential equations are generalizations of differential equations from integer order to non-integer order. Recently, fractional differential equations have attracted the attention of many researchers due to a wide range of applications in many fields of pure and applied mathematics such as: physics, fluid mechanics, viscoelasticity, electrochemistry, electrodynamics, nonlinear biological systems and other fields of science and engineering, see, for example [5, 6, 7, 21]. Consequently, considerable attentions have been given to the solutions of fractional differential equations of physical interest.

There are various types of tools and techniques for solving the problems involving differential equations, among them integral transform method such as: Laplace transform method [15], Sumudu transform method [23], Elzaki transform method [4], Natural transform method [13], Aboodh transform method [1], Pourreza transform method [2], Kamal transform method [9], Mohand transform method [16], Shehu transform method [17], Complex integral transform method[18] etc,. The advantage of this method is that it transforms the differential problem to an algebraic problem that can be easily solved.

The purpose of this paper is to use a new general integral transform to obtain an analytical solution of fractional differential equations with Caputo fractional derivative.

*Corresponding author
Email address: nadjibkh@yahoo.fr; ali.khalouta@univ-setif.dz (Ali Khalouta)

Received: July 2021 *Accepted:* December 2021

This paper is organized as follows. In Section 2, we give some basic definitions and important properties of the theory of fractional calculus. In Section 3, we present the definition and fundamental properties of the new general integral transform. In Section 4, we present our main results related to the solutions of Caputo fractional-order differential equations using the new general integral transform method. Finally, in Section 5 some conclusions are presented.

2 Preliminaries and basic definitions

In this section, we present the important basic definitions and properties of theory of fractional calculus, which will be used later in this paper. For more details about the theory of fractional calculus can be found in [14], [19].

Definition 2.1. [14] The Euler gamma function, which is generalization of factorial function from set of integers to the set of complex numbers, defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, z \in \mathbb{C}, \text{with } Re(z) > 0. \tag{2.1}$$

Definition 2.2. [14] The Riemann-Liouville fractional integral of function $u : (0, +\infty) \to \mathbb{R}$, for $\alpha \in \mathbb{R}^+$ is defined as

$$I^\alpha u(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{2.2}$$

Definition 2.3. [14] The Caputo fractional derivative of function $u : (0, +\infty) \to \mathbb{R}$ is defined as

$$D^\alpha u(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(\tau) d\tau, & \text{for } \alpha \neq n \in \mathbb{R} - \mathbb{N}, \\ u^{(n)}(t), & \text{for } \alpha = n \in \mathbb{N}. \end{cases} \tag{2.3}$$

Definition 2.4. [14] Mittag-Leffler function is the generalization of exponential function denoted by $E_\alpha(z)$ (for one parameter), $E_{\alpha,\beta}(z)$ (for two parameters) defined as

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \alpha \in \mathbb{R}^+, z \in \mathbb{C}. \tag{2.4}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \alpha, \beta \in \mathbb{R}^+, z \in \mathbb{C}. \tag{2.5}$$

3 A new general integral transform

In this section, we present the definition and some fundamental properties of the new general integral transform.

Definition 3.1. [8] Let $u(t)$ be a integrable function defined for $t \geq 0$, $p(s) \neq 0$ and $q(s)$ are positive real functions, we define the general integral transform $\mathcal{T}$ of $u(t)$ by the formula

$$T[u(t)] = \mathcal{T}(s) = p(s) \int_0^{+\infty} u(t) e^{-q(s)t} dt, \tag{3.1}$$

provided the integral exists for some $q(s)$.

Some basic properties of the new general integral transform are given as follows

Property 1: The new general integral transform is also a linear operator

$$T[au_1(t) + bu_2(t)] = aT[u_1(t)] + bT[u_2(t)], a, b \in \mathbb{R}.$$

Property 2: If $u(t)$ is n^{th} differentiable and $p(s)$ and $q(s)$ are positive real functions, then

$$T\left[u^{(n)}(t)\right] = q^n(s)\mathcal{T}(s) - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)u^{(k)}(0).$$

Property 3: (Convolution) Let $u_1(t)$ and $u_2(t)$ have new general integral transform $\mathcal{T}_1(s)$ and $\mathcal{T}_2(s)$. Then the new general integral transform of the convolution of u_1 and u_2 is

$$T\left[(u_1 * u_2)(t)\right] = T\left[\int_0^{+\infty} u_1(t)u_2(t-\tau)d\tau\right] = \frac{1}{p(s)}\mathcal{T}_1(s)\mathcal{T}_2(s).$$

Property 4: Some special new general integral transform

$$\begin{aligned} T(1) &= \frac{p(s)}{q(s)}, \\ T(t) &= \frac{p(s)}{q^2(s)}, \\ T\left(\frac{t^n}{n!}\right) &= \frac{p(s)}{q^{n+1}(s)}, n = 0, 1, 2, ... \end{aligned}$$

Property 5: The new general integral transform of t^α is given by

$$T\left[t^\alpha\right] = \frac{p(s)}{q^{\alpha+1}(s)}\Gamma\left(\alpha+1\right), \alpha > 0.$$

The advantages of the new general integral transform are that it covers all classes of integral transforms. Hence, all the transforms in the class of Laplace transform, introduced during the last few decades, are a special case of the new general integral transform. This transform can be applied to solve differential equations with constant and variable coefficients and it can be easily applied to solve fractional-order differential equations. Furthermore, from the definition of the new general integral transform, several new integral transforms can be defined by choosing new forms for $p(s)$ and $q(s)$.

4 Main Result

In this section, we present our main results related to the solutions of Caputo fractional-order differential equations using the new general integral transform method and give some illustrative examples to demonstrate the accuracy and efficiency of the proposed method. To achieve our goal, we have to prove the following two theorems which are crucial to the results we obtained.

Theorem 4.1. If $\mathcal{T}(s)$ is the new general integral transform of the function $u(t)$, then the new general integral transform of Riemann-Liouville fractional integral of order $\alpha > 0$, is

$$T\left[I^\alpha u(t)\right] = \frac{1}{q^\alpha(s)}\mathcal{T}(s). \tag{4.1}$$

Proof . The Riemann-Liouville fractional integral for the function $u(t)$ defined by (2.2), can be expressed as the convolution

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * u(t). \tag{4.2}$$

By applying the new general integral transform to both sides of the equation (4.2) and using properties (3) and

(5), we get

$$\begin{aligned} T\left[I^{\alpha}u(t)\right] &= T\left[\frac{1}{\Gamma(\alpha)}t^{\alpha-1} * u(t)\right] \\ &= \frac{1}{p(s)}T\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]T\left[u(t)\right] \\ &= \frac{1}{p(s)}\frac{p(s)}{q^{\alpha}(s)}\mathcal{T}(s) \\ &= \frac{1}{q^{\alpha}(s)}\mathcal{T}(s). \end{aligned}$$

The proof is complete. □

Theorem 4.2. If $n \in \mathbb{Z}^{+}$ where $n-1 < \alpha \leq n$ and $\mathcal{T}(s)$ be the new general integral transform of the function $u(t)$, then, the new general integral transform of the Caputo fractional derivative of order $\alpha > 0$, is

$$T\left[D^{\alpha}u(t)\right] = q^{\alpha}(s)\mathcal{T}(s) - p(s)\sum_{k=0}^{n-1} q^{\alpha-1-k}(s)u^{(k)}(0). \tag{4.3}$$

Proof . We put

$$v(t) = u^{(n)}(t).$$

Then, equation (2.3), can be expressed as follows

$$\begin{aligned} D^{\alpha}u(t) &= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}u^{(n)}(\tau)d\tau \\ &= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}v(\tau)d\tau \\ &= I^{n-\alpha}v(t). \end{aligned} \tag{4.4}$$

By applying the new general integral transform on both sides of equation (4.4) and using the Theorem *4.1*, we get

$$T\left[D^{\alpha}u(t)\right] = T\left[I^{n-\alpha}v(t)\right] = \frac{1}{q^{n-\alpha}(s)}\mathcal{V}(s), \tag{4.5}$$

where $\mathcal{V}(s)$ is the new general integral transform of the function $v(t)$.

According to properties (1) and (2), we have

$$T\left[v(t)\right] = T\left[u^{(n)}(t)\right],$$

and

$$\mathcal{V}(s) = q^{n}(s)\mathcal{T}(s) - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)u^{(k)}(0).$$

Therefore, the equation (4.5) becomes

$$\begin{aligned} T\left[D^{\alpha}u(t)\right] &= \frac{1}{q^{n-\alpha}(s)}\left(q^{n}(s)\mathcal{T}(s) - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)u^{(k)}(0)\right) \\ &= q^{\alpha}(s)\mathcal{T}(s) - p(s)\sum_{k=0}^{n-1} q^{\alpha-1-k}(s)u^{(k)}(0). \end{aligned}$$

The proof is complete. □

Corollary 4.3. • If $p(s) = 1$ and $q(s) = s$, then the Laplace transform of the Caputo fractional derivative [10] is obtained

$$\mathbb{L}\left[D^{\alpha}u(t)\right] = s^{\alpha}\mathbb{L}\left[u(t)\right] - \sum_{k=0}^{n-1} s^{\alpha-(k+1)}u^{(k)}(0).$$

• If $p(s) = s$ and $q(s) = \frac{1}{s}$, then the Elzaki transform of the Caputo fractional derivative [12] is obtained

$$\begin{aligned}\mathbb{E}\left[D^{\alpha}u(t)\right] &= \frac{1}{s^{\alpha}}\mathbb{E}\left[u(t)\right] - s\sum_{k=0}^{n-1}\frac{1}{s^{\alpha-1-k}}u^{(k)}(0)\\ &= \frac{1}{s^{\alpha}}\mathbb{E}\left[u(t)\right] - \sum_{k=0}^{n-1} s^{2-\alpha+k}u^{(k)}(0).\end{aligned}$$

• If $p(s) = \frac{1}{s}$ and $q(s) = s$, then the Aboodh transform of the Caputo fractional derivative [22] is obtained

$$\begin{aligned}\mathbb{A}\left[D^{\alpha}u(t)\right] &= s^{\alpha}\mathbb{A}\left[u(t)\right] - \frac{1}{s}\sum_{k=0}^{n-1} s^{\alpha-1-k}u^{(k)}(0)\\ &= s^{\alpha}\mathbb{A}\left[u(t)\right] - \sum_{k=0}^{n-1} s^{\alpha-2-k}u^{(k)}(0).\end{aligned}$$

• If $p(s) = q(s) = \frac{1}{s}$, then the Sumudu transform of the Caputo fractional derivative [11] is obtained

$$\begin{aligned}\mathbb{S}\left[D^{\alpha}u(t)\right] &= \frac{1}{s^{\alpha}}\mathbb{S}\left[u(t)\right] - \frac{1}{s}\sum_{k=0}^{n-1}\frac{1}{s^{\alpha-1-k}}u^{(k)}(0)\\ &= s^{-\alpha}\left[\mathbb{S}\left[u(t)\right] - \sum_{k=0}^{n-1} s^{k}u^{(k)}(0)\right].\end{aligned}$$

• If $p(s) = \frac{1}{v}$ and $q(s) = \frac{s}{v}$, then the natural transform of the Caputo fractional derivative [20] is obtained

$$\begin{aligned}\mathbb{N}^{+}\left[D^{\alpha}u(t)\right] &= \left(\frac{s}{v}\right)^{\alpha}\mathbb{N}^{+}\left[u(t)\right] - \frac{1}{v}\sum_{k=0}^{n-1}\left(\frac{s}{v}\right)^{\alpha-1-k}u^{(k)}(0)\\ &= \left(\frac{s}{v}\right)^{\alpha}\mathbb{N}^{+}\left[u(t)\right] - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{v^{\alpha-k}}u^{(k)}(0).\end{aligned}$$

• If $p(s) = 1$ and $q(s) = \frac{s}{v}$, then, the Shehu transform of the Caputo fractional derivative [3] is obtained

$$\mathbb{H}\left[D^{\alpha}u(t)\right] = \left(\frac{s}{v}\right)^{\alpha}\mathbb{H}\left[u(t)\right] - \sum_{k=0}^{n-1}\left(\frac{s}{v}\right)^{\alpha-1-k}u^{(k)}(0).$$

Our main results are given by the following theorems.

Theorem 4.4. Let $1 < \alpha \leq 2$ and $a, b \in \mathbb{R}$. Then the Caputo fractional-order differential equation

$$D^{\alpha}u(t) + au'(t) + bu(t) = 0, \tag{4.6}$$

with the initial conditions

$$u(0) = c_0, u'(0) = c_1, \tag{4.7}$$

has a solution given by

$$\begin{aligned} u(t) &= c_0 \sum_{k=0}^{+\infty} \frac{(-b)^k}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)(-a)^l t^{(\alpha-1)l+\alpha k}}{\Gamma((\alpha-1)l+\alpha k+1)\, l!} \\ &+ c_1 \sum_{k=0}^{+\infty} \frac{(-b)^k}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)(-a)^l t^{(\alpha-1)l+\alpha k+1}}{\Gamma((\alpha-1)l+\alpha k+2)\, l!} \\ &+ c_0 a \sum_{k=0}^{+\infty} \frac{(-b)^k}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)(-a)^l t^{(\alpha-1)l+\alpha k+\alpha-1}}{\Gamma((\alpha-1)l+\alpha k+\alpha)\, l!}. \end{aligned}$$

Proof . Taking the new general integral transform of equation (4.6) and using the Theorem 4.2, we have

$$T\left[D^\alpha u(t) + a u'(t) + b u(t)\right] = 0,$$

$$q^\alpha(s)\mathcal{T}(s) - p(s)q^{\alpha-1}(s)u(0) - p(s)q^{\alpha-2}(s)u'(0) + a q(s)\mathcal{T}(s) - a p(s) u(0) + b\mathcal{T}(s) = 0,$$

$$\left(q^\alpha(s) + a q(s) + b\right)\mathcal{T}(s) = p(s)q^{\alpha-1}(s)c_0 + p(s)q^{\alpha-2}(s)c_1 + a p(s) c_0,$$

$$\mathcal{T}(s) = \frac{c_0 p(s) q^{\alpha-1}(s) + c_1 p(s) q^{\alpha-2}(s) + c_0 a p(s)}{q^\alpha(s) + a q(s) + b}. \tag{4.8}$$

Since

$$\begin{aligned} \frac{1}{q^\alpha(s) + a q(s) + b} &= \frac{q^{-1}(s)}{q^{\alpha-1}(s) + a + b q^{-1}(s)} \\ &= \frac{q^{-1}(s)}{\left(q^{\alpha-1}(s) + a\right)\left(1 + \dfrac{b q^{-1}(s)}{q^{\alpha-1}(s) + a}\right)} \\ &= \frac{q^{-1}(s)}{\left(q^{\alpha-1}(s) + a\right)} \left(1 + \frac{b q^{-1}(s)}{q^{\alpha-1}(s) + a}\right)^{-1} \\ &= \frac{q^{-1}(s)}{\left(q^{\alpha-1}(s) + a\right)} \sum_{k=0}^{\infty} (-1)^k \left(\frac{b q^{-1}(s)}{q^{\alpha-1}(s) + a}\right)^k \\ &= \sum_{k=0}^{+\infty} (-b)^k \frac{q^{-k-1}(s)}{\left(q^{\alpha-1}(s) + a\right)^{k+1}} \\ &= \sum_{k=0}^{+\infty} (-b)^k \frac{q^{-k-1}(s)}{\left(q^{\alpha-1}(s)\left(1 + a q^{1-\alpha}(s)\right)\right)^{k+1}} \\ &= \sum_{k=0}^{+\infty} (-b)^k q^{-\alpha k-\alpha}(s) \left(\left(1 + a q^{1-\alpha}(s)\right)^{k+1}\right)^{-1} \\ &= \sum_{k=0}^{+\infty} (-b)^k q^{-\alpha k-\alpha}(s) \sum_{l=0}^{+\infty} \binom{k+l}{l} \left(-a q^{1-\alpha}(s)\right)^l \\ &= \sum_{k=0}^{+\infty} (-b)^k \sum_{l=0}^{\infty} \binom{k+l}{l} (-a)^l q^{l-\alpha l-\alpha k-\alpha}(s). \end{aligned}$$

Therefore, by using equation (4.8), we have

$$\begin{aligned} \mathcal{T}(s) &= \left(c_0 p(s) q^{\alpha-1}(s) + c_1 p(s) q^{\alpha-2}(s) + c_0 a p(s)\right) \times \left(\sum_{k=0}^{+\infty} (-b)^k \sum_{l=0}^{+\infty} \binom{k+l}{l} (-a)^l q^{l-\alpha l-\alpha k-\alpha}(s)\right) \\ &= c_0 p(s) \sum_{k=0}^{+\infty} (-b)^k \sum_{l=0}^{+\infty} \binom{k+l}{l} (-a)^l q^{l-\alpha l-\alpha k-1}(s) + c_1 p(s) \sum_{k=0}^{+\infty} (-b)^k \sum_{l=0}^{+\infty} \binom{k+l}{l} (-a)^l q^{l-\alpha l-\alpha k-2}(s) \\ &+ c_0 a p(s) \sum_{k=0}^{+\infty} (-b)^k \sum_{l=0}^{+\infty} \binom{k+l}{l} (-a)^l q^{l-\alpha l-\alpha k-\alpha}(s). \end{aligned} \tag{4.9}$$

Now, applying the inverse new general integral transform to equation (4.9), we get

$$\begin{aligned} u(t) &= c_0 \sum_{k=0}^{+\infty} \frac{(-b)^k}{k!} \sum_{l=0}^{+\infty} \frac{(k+l)!(-a)^l}{\Gamma\left((\alpha-1)\,l+\alpha k+1\right)} \frac{t^{(\alpha-1)l+\alpha k}}{l!} \\ &+ c_1 \sum_{k=0}^{+\infty} \frac{(-b)^k}{k!} \sum_{l=0}^{+\infty} \frac{(k+l)!(-a)^l}{\Gamma\left((\alpha-1)\,l+\alpha k+2\right)} \frac{t^{(\alpha-1)l+\alpha k+1}}{l!} \\ &+ c_0 a \sum_{k=0}^{+\infty} \frac{(-b)^k}{k!} \sum_{l=0}^{+\infty} \frac{(k+l)!(-a)^l}{\Gamma\left((\alpha-1)\,l+\alpha k-1\right)} \frac{t^{(\alpha-1)l+\alpha k+\alpha-1}}{l!} \end{aligned}$$

Therefore, the solution of equations (4.6) and (4.7), is

$$\begin{aligned} u(t) &= c_0 \sum_{k=0}^{+\infty} \frac{(-b)^k}{k!} \sum_{l=0}^{\infty} \frac{\Gamma\left(k+l+1\right)(-a)^l t^{(\alpha-1)l+\alpha k}}{\Gamma\left((\alpha-1)\,l+\alpha k+1\right) l!} \\ &+ c_1 \sum_{k=0}^{+\infty} \frac{(-b)^k}{k!} \sum_{l=0}^{\infty} \frac{\Gamma\left(k+l+1\right)(-a)^l t^{(\alpha-1)l+\alpha k+1}}{\Gamma\left((\alpha-1)\,l+\alpha k+2\right) l!} \\ &+ c_0 a \sum_{k=0}^{+\infty} \frac{(-b)^k}{k!} \sum_{l=0}^{\infty} \frac{\Gamma\left(k+l+1\right)(-a)^l t^{(\alpha-1)l+\alpha k+\alpha-1}}{\Gamma\left((\alpha-1)\,l+\alpha k+\alpha\right) l!}. \end{aligned}$$

The proof is complete. □

Example: The Caputo fractional-order differential equation

$$D^{\frac{5}{4}} u(t) + u'(t) + u(t) = 0,$$

with the initial conditions

$$u(0) = c_0, u'(0) = c_1,$$

has a solution given by

$$\begin{aligned} u(t) &= c_0 \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma\left(k+l+1\right)(-1)^l t^{\frac{l}{4}+\frac{5k}{4}}}{\Gamma\left(\frac{l}{4}+\frac{5k}{4}+1\right) l!} \\ &+ c_1 \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma\left(k+l+1\right)(-1)^l t^{\frac{l}{4}+\frac{5k}{4}+1}}{\Gamma\left(\frac{l}{4}+\frac{5k}{4}+2\right) l!} \\ &+ c_0 \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma\left(k+l+1\right)(-1)^l t^{\frac{l}{4}+\frac{5k}{4}+\frac{1}{4}}}{\Gamma\left(\frac{l}{4}+\frac{5k}{4}+\frac{5}{4}\right) l!}. \end{aligned}$$

Lemma 4.5. If $a = 0$ in equation (4.6), then the equation

$$D^{\alpha} u(t) + bu(t) = 0, 1 < \alpha \leq 2,$$

with the initial conditions

$$u(0) = c_0, u'(0) = c_1,$$

has a solution given by

$$u(t) = c_0 \sum_{k=0}^{+\infty} \frac{(-bt^{\alpha})^k}{\Gamma\left(\alpha k+1\right)} + c_1 t \sum_{k=0}^{+\infty} \frac{(-bt^{\alpha})^k}{\Gamma\left(\alpha k+2\right)} = c_0 E_{\alpha}\left(-bt^{\alpha}\right) + c_1 t E_{\alpha,2}\left(-bt^{\alpha}\right).$$

where $E_{\alpha}\left(-bt^{\alpha}\right)$ and $E_{\alpha,2}\left(-bt^{\alpha}\right)$ are the Mittag-Leffler functions defined by equations (2.4) and (2.5) respectively.

Example: The Caputo fractional-order differential equation

$$D^{\frac{3}{2}}u(t)+2u(t)=0,$$

with the initial conditions

$$u(0)=c_0, u'(0)=c_1,$$

as a solution given by

$$u(t)=c_0\sum_{k=0}^{+\infty}\frac{\left(-2t^{\frac{3}{2}}\right)^k}{\Gamma\left(\frac{3}{2}k+1\right)}+c_1t\sum_{k=0}^{+\infty}\frac{\left(-2t^{\frac{3}{2}}\right)^k}{\Gamma\left(\frac{3}{2}k+2\right)}=c_0E_{\frac{3}{2}}\left(-2t^{\frac{3}{2}}\right)+c_1tE_{\frac{3}{2},2}\left(-2t^{\frac{3}{2}}\right).$$

Theorem 4.6. Let $0<\alpha\leq 1$ and $b\in\mathbb{R}$. Then the Caputo fractional-order differential equation

$$D^{\alpha}u(t)-bu(t)=0, \tag{4.10}$$

with the initial condition

$$u(0)=c_0, \tag{4.11}$$

has a solution given by

$$u(t)=c_0\sum_{k=0}^{+\infty}\frac{(bt^{\alpha})^k}{\Gamma(\alpha k+1)}=c_0E_{\alpha}\left(bt^{\alpha}\right),$$

where $E_{\alpha}\left(bt^{\alpha}\right)$ is the Mittag-Leffler function defined by equation (2.4).

Proof . Taking the new general integral transform of equation (4.10) and using the Theorem 4.2, we have

$$T\left[D^{\alpha}u(t)-bu(t)\right]=0,$$
$$q^{\alpha}(s)\mathcal{T}(s)-p(s)q^{\alpha-1}(s)u(0)-b\mathcal{T}(s)=0,$$
$$\left(q^{\alpha}(s)-b\right)\mathcal{T}(s)=p(s)q^{\alpha-1}(s)c_0,$$

$$\begin{aligned}\mathcal{T}(s) &= \frac{c_0p(s)q^{\alpha-1}(s)}{q^{\alpha}(s)-b}\\ &= \frac{c_0p(s)q^{-1}(s)}{1-bq^{-\alpha}(s)}\\ &= c_0p(s)q^{-1}(s)\left(1-bq^{-\alpha}(s)\right)^{-1}\\ &= c_0p(s)q^{-1}(s)\sum_{k=0}^{+\infty}\left(bq^{-\alpha}(s)\right)^k\\ &= c_0p(s)\sum_{k=0}^{+\infty}b^kq^{-\alpha k-1}(s).\end{aligned} \tag{4.12}$$

Now, applying the inverse new general integral transform to equation (4.12), we get

$$u(t)=c_0\sum_{k=0}^{+\infty}\frac{b^kt^{k\alpha}}{\Gamma(\alpha k+1)}=c_0\sum_{k=0}^{+\infty}\frac{(bt^{\alpha})^k}{\Gamma(\alpha k+1)}=c_0E_{\alpha}\left(bt^{\alpha}\right).$$

The proof is complete. □

Example: The Caputo fractional-order differential equation

$$D^{\frac{4}{5}}u(t)-4u(t)=0,$$

with the initial condition

$$u(0)=c_0,$$

as a solution given by

$$u(t)=c_0\sum_{k=0}^{+\infty}\frac{\left(4t^{\frac{4}{5}}\right)^k}{\Gamma\left(\frac{4}{5}k+1\right)}=c_0E_{\frac{4}{5}}\left(4t^{\frac{4}{5}}\right).$$

Theorem 4.7. Let $1 < \alpha \leq 2$ and $a, b \in \mathbb{R}$. Then the Caputo fractional-order differential equation

$$u''(t) + aD^{\alpha}u(t) + bu(t) = 0, \tag{4.13}$$

with the initial conditions

$$u(0) = c_0, u'(0) = c_1, \tag{4.14}$$

has a solution given by

$$\begin{aligned} u(t) &= c_0 \sum_{k=0}^{+\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2k+1)\, l!} \\ &+ c_1 \sum_{k=0}^{+\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2k+2)\, l!} \\ &+ c_0 a \sum_{k=0}^{+\infty} \frac{(-b)^k t^{2k-\alpha+2}}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2k-\alpha+3)\, l!} \\ &+ c_1 a \sum_{k=0}^{+\infty} \frac{(-b)^k t^{2k-\alpha+3}}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2k-\alpha+4)\, l!}. \end{aligned}$$

Proof . Taking the new general integral transform of equation (4.13) and using the Theorem 4.2 , we have

$$T\left[u''(t) + aD^{\alpha}u(t) + bu(t)\right] = 0,$$

$$q^2(s)\mathcal{T}(s) - p(s)q(s)u(0) - p(s)u'(0) + a(q^{\alpha}(s)\mathcal{T}(s) - p(s)q^{\alpha-1}(s)u(0) - p(s)q^{\alpha-2}(s)u'(0)) + b\mathcal{T}(s) = 0,$$

$$\left(q^2(s) + aq^{\alpha}(s) + b\right)\mathcal{T}(s) = p(s)q(s)c_0 + p(s)c_1 + ap(s)q^{\alpha-1}(s)c_0 + ap(s)q^{\alpha-2}(s)c_1 = 0,$$

$$\mathcal{T}(s) = \frac{c_0 p(s)q(s) + c_1 p(s) + c_0 ap(s)q^{\alpha-1}(s) + c_1 ap(s)q^{\alpha-2}(s)}{q^2(s) + aq^{\alpha}(s) + b}. \tag{4.15}$$

Since

$$\begin{aligned} \frac{1}{q^2(s) + aq^{\alpha}(s) + b} &= \frac{q^{-\alpha}(s)}{q^{2-\alpha}(s) + a + bq^{-\alpha}(s)} \\ &= \frac{q^{-\alpha}(s)}{(q^{2-\alpha}(s) + a)\left(1 + \frac{bq^{-\alpha}(s)}{q^{2-\alpha}(s) + a}\right)} \\ &= \frac{q^{-\alpha}(s)}{(q^{2-\alpha}(s) + a)}\left(1 + \frac{bq^{-\alpha}(s)}{q^{2-\alpha}(s) + a}\right)^{-1} \\ &= \frac{q^{-\alpha}(s)}{(q^{2-\alpha}(s) + a)} \sum_{k=0}^{\infty} (-1)^k \left(\frac{bq^{-\alpha}(s)}{q^{2-\alpha}(s) + a}\right)^k \\ &= \sum_{k=0}^{+\infty} (-b)^k \frac{q^{-\alpha k-\alpha}(s)}{(q^{2-\alpha}(s) + a)^{k+1}} \\ &= \sum_{k=0}^{+\infty} (-b)^k \frac{q^{-\alpha k-\alpha}(s)}{(q^{2-\alpha}(s)(1 + aq^{\alpha-2}(s)))^{k+1}} \\ &= \sum_{k=0}^{+\infty} (-b)^k q^{-2k-2}(s)\left(\left(1 + aq^{\alpha-2}(s)\right)^{k+1}\right)^{-1} \\ &= \sum_{k=0}^{+\infty} (-b)^k q^{-2k-2}(s) \sum_{l=0}^{+\infty} \binom{k+l}{l} \left(-aq^{\alpha-2}(s)\right)^l \\ &= \sum_{k=0}^{+\infty} (-b)^k \sum_{l=0}^{\infty} \binom{k+l}{l} (-a)^l q^{(\alpha-2)l-2k-2}(s). \end{aligned}$$

Therefore, by using equation (4.15), we have

$$
\begin{aligned}
\mathcal{T}(s) &= \big(c_0 p(s)q(s) + c_1 p(s) + c_0 a p(s) q^{\alpha-1}(s) + c_1 a p(s) q^{\alpha-2}(s)\big) \\
&\quad \times \left(\sum_{k=0}^{+\infty}(-b)^k \sum_{l=0}^{\infty}\binom{k+l}{l}(-a)^l q^{(\alpha-2)l-2k-2}(s)\right) \\
&= c_0 p(s)\sum_{k=0}^{+\infty}(-b)^k \sum_{l=0}^{\infty}\binom{k+l}{l}(-a)^l q^{(\alpha-2)l-2k-1}(s) \\
&\quad + c_1 p(s)\sum_{k=0}^{+\infty}(-b)^k \sum_{l=0}^{\infty}\binom{k+l}{l}(-a)^l q^{(\alpha-2)l-2k-2}(s) \\
&\quad + c_0 a p(s)\left(\sum_{k=0}^{+\infty}(-b)^k \sum_{l=0}^{\infty}\binom{k+l}{l}(-a)^l q^{(\alpha-2)l-2k+\alpha-3}(s))\right) \\
&\quad + c_1 a p(s)\left(\sum_{k=0}^{+\infty}(-b)^k \sum_{l=0}^{\infty}\binom{k+l}{l}(-a)^l q^{(\alpha-2)l-2k+\alpha-4}(s))\right).
\end{aligned} \tag{4.16}
$$

Now, applying the inverse new general integral transform to equation (4.16), we get

$$
\begin{aligned}
u(t) &= c_0\sum_{k=0}^{+\infty}\frac{(-b)^k}{k!}\sum_{l=0}^{+\infty}\frac{(k+l)!(-a)^l}{\Gamma\left((2-\alpha)l+2k+1\right)}\frac{t^{(2-\alpha)l+2k}}{l!} \\
&\quad + c_1\sum_{k=0}^{+\infty}\frac{(-b)^k}{k!}\sum_{l=0}^{+\infty}\frac{(k+l)!(-a)^l}{\Gamma\left((2-\alpha)l+2k+2\right)}\frac{t^{(2-\alpha)l+2k+1}}{l!} \\
&\quad + c_0 a\sum_{k=0}^{+\infty}\frac{(-b)^k}{k!}\sum_{l=0}^{+\infty}\frac{(k+l)!(-a)^l}{\Gamma\left((2-\alpha)l+2k-\alpha+3\right)}\frac{t^{(2-\alpha)l+2k-\alpha+2}}{l!} \\
&\quad + c_1 a\sum_{k=0}^{+\infty}\frac{(-b)^k}{k!}\sum_{l=0}^{+\infty}\frac{(k+l)!(-a)^l}{\Gamma\left((2-\alpha)l+2k-\alpha+4\right)}\frac{t^{(2-\alpha)l+2k-\alpha+3}}{l!}.
\end{aligned}
$$

Therefore, the solution of equations (4.13) and (4.14), is

$$
\begin{aligned}
u(t) &= c_0\sum_{k=0}^{+\infty}\frac{(-b)^k t^{2k}}{k!}\sum_{l=0}^{+\infty}\frac{\Gamma\left(k+l+1\right)\left(-at^{2-\alpha}\right)^l}{\Gamma\left((2-\alpha)l+2k+1\right)l!} \\
&\quad + c_1\sum_{k=0}^{+\infty}\frac{(-b)^k t^{2k+1}}{k!}\sum_{l=0}^{+\infty}\frac{\Gamma\left(k+l+1\right)\left(-at^{2-\alpha}\right)^l}{\Gamma\left((2-\alpha)l+2k+2\right)l!} \\
&\quad + c_0 a\sum_{k=0}^{+\infty}\frac{(-b)^k t^{2k-\alpha+2}}{k!}\sum_{l=0}^{+\infty}\frac{\Gamma\left(k+l+1\right)\left(-at^{2-\alpha}\right)^l}{\Gamma\left((2-\alpha)l+2k-\alpha+3\right)l!} \\
&\quad + c_1 a\sum_{k=0}^{+\infty}\frac{(-b)^k t^{2k-\alpha+3}}{k!}\sum_{l=0}^{+\infty}\frac{\Gamma\left(k+l+1\right)\left(-at^{2-\alpha}\right)^l}{\Gamma\left((2-\alpha)l+2k-\alpha+4\right)l!}.
\end{aligned}
$$

The proof is complete. □

Example: The Caputo fractional-order differential equation

$$u''(t) + \sqrt{2}D^{\frac{5}{3}}u(t) + 5u(t) = 0,$$

with the initial conditions

$$u(0) = c_0, u'(0) = c_1,$$

has a solution given by

$$
\begin{aligned}
u(t) &= c_0 \sum_{k=0}^{+\infty} \frac{(-5)^k t^{2k}}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)\left(-\sqrt{2}t^{\frac{1}{3}}\right)^l}{\Gamma\left(\frac{1}{3}l+2k+1\right) l!} \\
&+ c_1 \sum_{k=0}^{+\infty} \frac{(-5)^k t^{2k+1}}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)\left(-\sqrt{2}t^{\frac{1}{3}}\right)^l}{\Gamma\left(\frac{1}{3}l+2k+2\right) l!} \\
&+ c_0\sqrt{2} \sum_{k=0}^{+\infty} \frac{(-5)^k t^{2k+\frac{1}{3}}}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)\left(-\sqrt{2}t^{\frac{1}{3}}\right)^l}{\Gamma\left(\frac{1}{3}l+2k+\frac{4}{3}\right) l!} \\
&+ c_1\sqrt{2} \sum_{k=0}^{+\infty} \frac{(-5)^k t^{2k+\frac{4}{3}}}{k!} \sum_{l=0}^{+\infty} \frac{\Gamma(k+l+1)\left(-\sqrt{2}t^{\frac{1}{3}}\right)^l}{\Gamma\left(\frac{1}{3}l+2k+\frac{7}{3}\right) l!}.
\end{aligned}
$$

5 Conclusion

In this paper, a new genaral integral transform was applied to solve Caputo fractional-order differential equations. Many theorems related to this approach were proved. Various illustrative examples were presented to show the accuracy and effectiveness of the proposed approach. The obtained results demonstrated that the proposed approach is very efficient, useful, and easy to use for solving fractional differential equations.

Acknowledgements

The author wish to thank Professor Madjid Eshaghi Gordji (IJNAA Editor-in-Chief) as well as the anonymous reviewers for their critical suggestions, contributions and comments which greatly improved this paper.

References

[1] K.S. Aboodh, *The new integral transform "Aboodh transform"*, Glob. J. Pure Appl. Math. **9** (2013), 35–43.

[2] S.A.P. Ahmadi, H. Hosseinzadeh and A.Y Cherati, *A new integral transform for solving higher order linear ordinary differential equations*, Nonlinear Dyn. Syst. Theory **19** (2019), no. 2, 243–52.

[3] R. Belgacem, D. Baleanu and A. Bokhari, *Shehu transform and applications to Caputo-fractional differential equations*, Int. J. Anal. Appl. **17** (2019), no. 6, 917–927.

[4] T.M. Elzaki, *The new integral transform Elzaki Transform*, Glob. J. Pure Appl. Math. **7** (2011), no. 1, 57–64.

[5] K. Hosseini, M. Ilie, M. Mirzazadeh, A. Yusuf, T. A. Sulaiman, D. Baleanu and S. Salahshour, *An effective computational method to deal with a time-fractional nonlinear water wave equation in the Caputo sense*, Math. Comput. Simul. **187** (2021), 248–260.

[6] K. Hosseini, M. Ilie, M. Mirzazadeh and D. Baleanu, *An analytic study on the approximate solution of a nonlinear time-fractional Cauchy reaction-diffusion equation with the Mittag–Leffler law*, Math. Methods Appl. Sci. **44** (2021), no. 8, 6247–6258.

[7] K. Hosseini, Kh. Sadri, M. Mirzazadeh, A. Ahmadian, Y.M Chu and S. Salahshour, *Reliable methods to look for analytical and numerical solutions of a nonlinear differential equation arising in heat transfer with the conformable derivative*, Math. Methods Appl. Sci. 2021, https://doi.org/10.1002/mma.7582.

[8] H. Jafari, *A new general integral transform for solving integral equations*, Journal of Advanced Research, 32 (2021) 133–138.

[9] H. Kamal and A Sedeeg: *The new integral transform Kamal transform*, Adv. Theor. Appl. Math. **11** (2016), no. 4, 451–458.

[10] S. Kazem, *Exact solution of some linear fractional differential equations by Laplace transform*, Int. J. Nonlinear Sci. **16** (2013), no. 1, 3–11.

[11] Q.D. Katatbeh and F.B.M. Belgacem, *Applications of the Sumudu transform to fractional differential equations*, Nonlinear Stud. **18** (2011), no. 1, 99–112.

[12] M. Khalid, M. Sultana, F. Zaidi and U. Arshad, *Application of Elzaki transform method on some fractional differential equations*, Math. Theory Model. **5** (2015), no. 1, 89–96.

[13] Z.H. Khan and W.A. Khan, *N-transform properties and applications*, NUST J. Engin. Sci. **1** (2008), no. 1, 127–33.

[14] A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and application of fractional differential equations*, Elsevier, North-Holland, 2006.

[15] S. Liang, R. Wu and L. Chen, *Laplace transform of fractional order differential equations*, Electron. J. Differ. Equ. **2015** (2015), 1–15.

[16] M.M.A. Mahgoub, *The new integral transform mohand transform*, Adv. Theor. Appl. Math. **12** (2017), no. 2, 113–20.

[17] Sh. Maitama and W. Zhao, *New integral transform: Shehu transform a genaralization of Sumudu and Laplace transform for solving differential equations*, Int. J. Anal. Appl. 17 (2019), no. 2, 167–190.

[18] E.A. Mansoura, S. Mehdia and E.A. Kuffi, *The new integral transform and its applications*, Int. J. Nonlinear Anal. Appl. **12** (2021), no. 2, 849–856.

[19] I. Podlubny, I. Petras, B.M. Vinagre, P. O'leary and L. Dorcak, *Analogue realizations of fractional-order controllers. fractional order calculus and its applications*, Nonlinear Dyn. **29** (2002), no. 4, 281–296.

[20] M.S. Rawashdeh and H. Al-Jammal, *Theories and applications of the inverse fractional natural transform method*, Adv. Differ. Equ. **2018** (2018), 222.

[21] K. Sadri, K. Hosseini, D. Baleanu, A. Ahmadian and S. Salahshour, *Bivariate Chebyshev polynomials of the fifth kind for variable-order time-fractional partial integro-differential equations with weakly singular kernel*, Adv. Differ. Equ. **2021** (2021), 348.

[22] T.G. Thange and A.R. Gade, *On Aboodh transform for fractional differential operato*, Malaya J. Mat., 8(1) (2020) 225–229.

[23] G.K Watugala, *Sumudu transform: a new integral transform to solve differential equations and control engineering problems*, Internat. J. Math. Ed. Sci. Technol. **24** (1993), no 1, 35–43.