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Sakaguchi type function defined by $(\mathfrak{p},\mathfrak{q})$-derivative operator using Gegenbauer polynomials

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Abstract

An introduction of a new subclass of bi-univalent functions involving Sakaguchi type functions defined by $(\mathfrak{p},\mathfrak{q})$-Derivative operators using Gegenbauer polynomials have been obtained. Further, the bounds for initial coefficients $|a_2|$, $|a_3|$ and Fekete Szegö inequality have been estimated.

Keywords: Analytic function, Bi-Univalent function, $(\mathfrak{p},\mathfrak{q})$- Derivative operator, Sakaguchi type function, Gegenbauer polynomials
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1 Introduction and preliminaries

A function of one or more complex variables which is complex-valued is said to be analytic if it is differentiable at every point of the domain. Every normalized analytic function can be expressed as a series of the form

$$\mathfrak{f}(z) = z + \sum_{t=2}^{\infty} a_t z^t \tag{1.1}$$

in the complex variable z, that is convergent in $\mathfrak{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Let A consists of every such function. A subclass $\mathcal{S}$ of A is defined by $\mathcal{S} = \{\mathfrak{f}(z) \in A : \mathfrak{f}(z_1) = \mathfrak{f}(z_2) \Rightarrow z_1 = z_2\}$ (i.e.,) $\mathcal{S}$ consists of all univalent functions.

A function $\mathfrak{f}(z) \in A$ is called bi-univalent in $\mathfrak{U}$, if $\mathfrak{f}(z) \in \mathcal{S}$ and its inverse function has an analytic continuation to $|w| < 1$. Let $\sigma = \{\mathfrak{f} \in \mathcal{S} : \mathfrak{f} \text{ is bi-iunivalent}\}$.

Though Lewin [7] introduced the class of bi-univalent functions, the passion on the bounds for the coefficients of these classes was upraised by Netanyahu, Clunie, Brannan and many others [1, 2, 8, 13, 14, 18, 15, 16, 17, 19, 20].

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This has been a field of fascination for young researchers till date.

If, for $\mathfrak{f}_1(z)$ and $\mathfrak{f}_2(z)$ analytic in $\mathfrak{U}$, there exists a Schwarz function $\mathfrak{w}(z)$ with $\mathfrak{w}(0)=0$ and $|\mathfrak{w}(z)|<1$ in $\mathfrak{U}$ such that $\mathfrak{f}_1(z)=\mathfrak{f}_2(\mathfrak{w}(z))$, then we say that $\mathfrak{f}_1(z)\prec\mathfrak{f}_2(z)$.

A subclass consisting of functions satisfying the analytic criterion $Re\left(\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)-\mathfrak{f}(-z)}\right)>\alpha$ was introduced by Sakaguchi [11] and these functions were named after him as Sakaguchi type functions [9, 10]. Sakaguchi type functions are Starlike with respect to symmetric points. Frasin [5] generalized Sakaguchi type class which had functions of the form (1.1) given by $Re\left(\frac{(\mathsf{s}_1-\mathsf{s}_2)z\mathfrak{f}'(z)}{\mathfrak{f}(\mathsf{s}_1 z)-\mathfrak{f}(\mathsf{s}_2 z)}\right)>\alpha$, $0\le\alpha<1, \mathsf{s}_1,\mathsf{s}_2\in\mathbb{C}$ with $\mathsf{s}_1\neq\mathsf{s}_2, |\mathsf{s}_2|\le 1, \forall z\in\mathfrak{U}$.

Definition 1.1. For $\mathfrak{p},\mathfrak{q}\in(0,1]$ and $\mathfrak{q}<\mathfrak{p}$, the $(\mathfrak{p},\mathfrak{q})$-derivative operator $\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))$ [3] is defined as

$$\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))=\frac{\mathfrak{f}(\mathfrak{p}z)-\mathfrak{f}(\mathfrak{q}z)}{(\mathfrak{p}-\mathfrak{q})(z)}, z\neq 0 \tag{1.2}$$

and $\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(0))=\mathfrak{f}'(0)$ provided $\mathfrak{f}'(0)$ exists. It can be easily deduced that

$$\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))=1+\sum_{t=2}^{\infty}[t]_{\mathfrak{p}\mathfrak{q}}a_t z^{t-1},$$

where $[t]_{\mathfrak{p}\mathfrak{q}}=\frac{\mathfrak{p}^t-\mathfrak{q}^t}{\mathfrak{p}-\mathfrak{q}}$, the $(\mathfrak{p},\mathfrak{q})$-bracket of t. It is also called a twin-basic number. It is to be noted that $\mathfrak{D}_{p,q}(z^t)=[t]_{\mathfrak{p}\mathfrak{q}}z^{t-1}$. Also for $\mathfrak{p}=1$, the $(\mathfrak{p},\mathfrak{q})$-derivative operator $\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}$ reduces to the $\mathfrak{q}$-derivative operator $\mathfrak{D}_{\mathfrak{q}}$.

The inverse series of (1.2) is given by

$$\begin{aligned}\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w)) &= \frac{\mathfrak{g}(\mathfrak{p}w)-\mathfrak{g}(\mathfrak{q}w)}{(\mathfrak{p}-\mathfrak{q})w}\\ &= 1-[2]_{\mathfrak{p}\mathfrak{q}}a_2 w+[3]_{\mathfrak{p}\mathfrak{q}}(2a_2^2-a_3)w^2\\ &\quad -[4]_{\mathfrak{p}\mathfrak{q}}(5a_2^3-5a_2a_3+a_4)w^3+\cdots.\end{aligned}$$

For non-zero real constant α, a generating function of Gegenbauer polynomials is defined by

$$\mathfrak{H}_\alpha(y,z)=\frac{1}{(1-2yz+z^2)^\alpha}, \tag{1.3}$$

where $y\in[-1,1]$ and $z\in\mathfrak{U}$. The function $\mathfrak{H}_\alpha$ which is analytic in $\mathfrak{U}$, for fixed y, is expanded in a Taylor series form such as

$$\mathfrak{H}_\alpha(y,z)=\sum_{t=0}^{\infty}C_t^\alpha(y)z^t, \tag{1.4}$$

where $C_t^\alpha(y)$ is Gegenbauer polynomial of degree t. We can see that when $\alpha=0$, $\mathfrak{H}_\alpha$ does not exist. Therefore, the Gegenbauer polynomial is generated by following function

$$\mathfrak{H}_0(y,z)=1-\log(1-2yz+z^2)=\sum_{t=0}^{\infty}C_t^0(y)z^t,$$

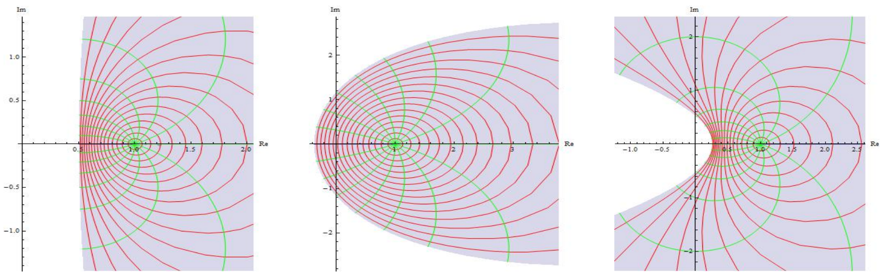
for $\alpha=0$. The function gets normalized when $\alpha>-1/2$ [4, 12].
The images of the unit disk under $\mathfrak{H}_\alpha(y,z)$ are shown in figure 1.
The Gegenbauer polynomials are defined by the following recurrence relation

$$C_t^\alpha(y)=\frac{1}{t}[2y(t+\alpha-1)C_{t-1}^\alpha(y)-(t+2\alpha-2)C_{t-2}^\alpha(y)], (t\ge 2) \tag{1.5}$$

with initial coefficients $C_0^\alpha(y)=1$ and $C_1^\alpha(y)=2\alpha y$.
From the above , we get

$$C_2^\alpha(y)=2\alpha(1+\alpha)y^2-\alpha. \tag{1.6}$$

The special cases of Gegenbauer polynomials:

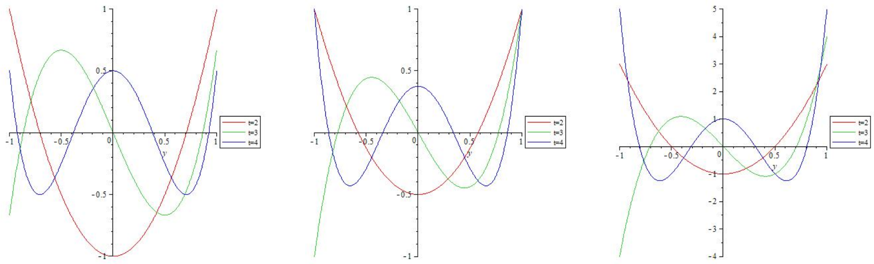


$y=-1$ and $\alpha=0.5$ $\quad$ $y=-1$ and $\alpha=0$ $\quad$ $y=1$ and $\alpha=1$

Figure 1: Image of $\mathfrak{U}$ under $\mathfrak{H}_\alpha(y,z)$.

1. For $\alpha=1$, we get the Chebyshev Polynomials.
2. For $\alpha=1/2$, we get the Legendre Polynomials.

The Graphs of the Gegenbauer polynomials $C_t^\alpha(y)$ are shown in figure 2.



$\alpha=0$ $\quad$ $\alpha=0.5$ $\quad$ $\alpha=1$

Figure 2: Graph of $C_t^\alpha(y)$.

2 Main results

Definition 2.1. A function $\mathfrak{f}\in\sigma$ is said to be in the class $\mathcal{S}_\sigma^{\mathfrak{pq}}(y,\alpha,\mathsf{s}_1,\mathsf{s}_2)$, if the following subordination relations hold

$$\frac{(\mathsf{s}_1-\mathsf{s}_2)z\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1z)-\mathfrak{f}(\mathsf{s}_2z)}\prec\mathfrak{H}_\alpha(y,z), \tag{2.1}$$

and

$$\frac{(\mathsf{s}_1-\mathsf{s}_2)w\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1w)-\mathfrak{g}(\mathsf{s}_2w)}\prec\mathfrak{H}_\alpha(y,w), \tag{2.2}$$

where $\mathfrak{g}(w)=\mathfrak{f}^{-1}(w),\mathsf{s}_1,\mathsf{s}_2\in\mathbb{C}$ with $\mathsf{s}_1\neq\mathsf{s}_2,|\mathsf{s}_2|\leq1$.

Theorem 2.2. Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_\sigma^{\mathfrak{pq}}(y,\alpha,\mathsf{s}_1,\mathsf{s}_2)$. Then

$$|a_2|\leq\frac{2|\alpha y|\sqrt{2|\alpha y|}}{\sqrt{\left|4\alpha^2y^2L-(2\alpha(1+\alpha)y^2-\alpha)M^2\right|}} \tag{2.3}$$

and

$$|a_3|\leq\left|\frac{2\alpha y}{N}\right|+\frac{4\alpha^2y^2}{M^2} \tag{2.4}$$

where
$L=[3]_{\mathfrak{pq}}-[2]_{\mathfrak{pq}}(\mathsf{s}_1+\mathsf{s}_2)+\mathsf{s}_1\mathsf{s}_2$,
$M=[2]_{\mathfrak{pq}}-\mathsf{s}_1-\mathsf{s}_2$,
$N=[3]_{\mathfrak{pq}}-\mathsf{s}_1^2-\mathsf{s}_2^2-\mathsf{s}_1\mathsf{s}_2$.

Proof . Let $\mathfrak{f} \in \mathcal{S}_{\sigma}^{\mathfrak{pq}}(y, \alpha, \mathsf{s}_1, \mathsf{s}_2)$. Then, there exist analytic functions $\phi(z)$, $\psi(w) : \mathfrak{U} \to \mathfrak{U}$ given by the (2.1) and (2.2) such that

$$\frac{(\mathsf{s}_1 - \mathsf{s}_2)z\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1 z) - \mathfrak{f}(\mathsf{s}_2 z)} = \mathfrak{H}_{\alpha}(y, \phi(z)), \tag{2.5}$$

and

$$\frac{(\mathsf{s}_1 - \mathsf{s}_2)w\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1 w) - \mathfrak{g}(\mathsf{s}_2 w)} = \mathfrak{H}_{\alpha}(y, \psi(w)). \tag{2.6}$$

Define the functions $\phi(z)$ and $\psi(w)$ as

$$\phi(z) = c_1 z + c_2 z^2 + c_3 z^3 + ..., \tag{2.7}$$

and

$$\psi(w) = d_1 w + d_2 w^2 + d_3 w^3 + ... \tag{2.8}$$

which are analytic in $\mathfrak{U}$ with $\phi(0)$=0,$\psi(0) = 0$ and $|\phi(z)| < 1$, $|\psi(w)| < 1$, $(z, w \in \mathfrak{U})$.
It is to be noted that if

$$|\phi(z)| = |c_1 z + c_2 z^2 + c_3 z^3 + \cdots| < 1 \quad (z \in \mathfrak{U})$$

and

$$|\psi(w)| = |d_1 w + d_2 w^2 + d_3 w^3 + \cdots| < 1 \quad (w \in \mathfrak{U})$$

then

$$|c_i| \leq 1, \quad |d_i| \leq 1 \quad (i = 1, 2, 3, ...). \tag{2.9}$$

Since

$$\begin{aligned}\frac{(\mathsf{s}_1 - \mathsf{s}_2)z\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1 z) - \mathfrak{f}(\mathsf{s}_2 z)} =& 1 + \left([2]_{\mathfrak{pq}} - \mathsf{s}_1 - \mathsf{s}_2\right) a_2 z + \left\{\left([3]_{\mathfrak{pq}} - {\mathsf{s}_1}^2 - {\mathsf{s}_2}^2 - \mathsf{s}_1\mathsf{s}_2\right) a_3\right. \\ &\left. - \left([2]_{\mathfrak{pq}}\mathsf{s}_1 + [2]_{\mathfrak{pq}}\mathsf{s}_2 - {\mathsf{s}_1}^2 - {\mathsf{s}_2}^2 - 2\mathsf{s}_1\mathsf{s}_2\right) a_2^2\right\} \times z^2 + \cdots\end{aligned} \tag{2.10}$$

$$\begin{aligned}\frac{(\mathsf{s}_1 - \mathsf{s}_2)w\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1 w) - \mathfrak{g}(\mathsf{s}_2 w)} =& 1 - \left([2]_{\mathfrak{pq}} - \mathsf{s}_1 - \mathsf{s}_2\right) a_2 w - \left\{\left([3]_{\mathfrak{pq}} - {\mathsf{s}_1}^2 - {\mathsf{s}_2}^2 - \mathsf{s}_1\mathsf{s}_2\right) a_3\right. \\ &\left. - \left(2[3]_{\mathfrak{pq}} - {\mathsf{s}_1}^2 - {\mathsf{s}_2}^2 - [2]_{\mathfrak{pq}}\mathsf{s}_1 - [2]_{\mathfrak{pq}}\mathsf{s}_2\right) a_2^2\right\} \times w^2 + \cdots.\end{aligned} \tag{2.11}$$

$$\frac{(\mathsf{s}_1 - \mathsf{s}_2)z\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1 z) - \mathfrak{f}(\mathsf{s}_2 z)} = [C_1^{\alpha}(y)c_1]z + [C_1^{\alpha}(y)c_2 + C_2^{\alpha}(y)c_1^2]z^2 + \cdots \tag{2.12}$$

$$\frac{(\mathsf{s}_1 - \mathsf{s}_2)w\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1 w) - \mathfrak{g}(\mathsf{s}_2 w)} = [C_1^{\alpha}(y)d_1]w + [C_1^{\alpha}(y)d_2 + C_2^{\alpha}(y)d_1^2]w^2 + \cdots. \tag{2.13}$$

We get following equations

$$[[2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2]\, a_2 = C_1^{\alpha}(y)c_1 \tag{2.14}$$

$$\begin{aligned}\left[[3]_{pq} - {\mathsf{s}_1}^2 - {\mathsf{s}_2}^2 - \mathsf{s}_1\mathsf{s}_2\right] a_3 - &\left[[2]_{pq}\mathsf{s}_1 + [2]_{pq}\mathsf{s}_2 - {\mathsf{s}_1}^2 - {\mathsf{s}_2}^2 - 2\mathsf{s}_1\mathsf{s}_2\right] a_2^2 \\ &= C_1^{\alpha}(y)c_2 + C_2^{\alpha}(y)c_1^2\end{aligned} \tag{2.15}$$

$$-[[2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2]a_2 = C_1^{\alpha}(y)d_1 \tag{2.16}$$

$$\begin{aligned}\left[2[3]_{pq} - {\mathsf{s}_1}^2 - {\mathsf{s}_2}^2 - [2]_{pq}\mathsf{s}_1 - [2]_{pq}\mathsf{s}_2\right] a_2^2 - \left[[3]_{pq} - {\mathsf{s}_1}^2 - {\mathsf{s}_2}^2 - \mathsf{s}_1\mathsf{s}_2\right] a_3 \\ = C_1^{\alpha}(y)d_2 + C_2^{\alpha}(y)d_1^2.\end{aligned} \tag{2.17}$$

Adding (2.14) and (2.16), we get the following equation

$$c_1 = -d_1. \tag{2.18}$$

Further squaring and adding (2.14) and (2.16), we have

$$2[[2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2]^2 a_2^2 = [C_1^{\alpha}(y)]^2[c_1^2 + d_1^2]. \tag{2.19}$$

Then the addition of (2.15) and (2.17) gives

$$2[[3]_{pq} - [2]_{pq}(\mathsf{s}_1 + \mathsf{s}_2) + \mathsf{s}_1\mathsf{s}_2]a_2^2 = C_1^{\alpha}(y)(c_2 + d_2) + C_2^{\alpha}(y)(c_1^2 + d_1^2). \tag{2.20}$$

From above equations, we obtain

$$\left[2[[3]_{pq} - [2]_{pq}(\mathsf{s}_1 + \mathsf{s}_2) + \mathsf{s}_1\mathsf{s}_2][C_1^{\alpha}(y)]^2 - 2([2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2)^2 C_2^{\alpha}(y)\right] a_2^2 = [C_1^{\alpha}(y)]^3(c_2 + d_2). \tag{2.21}$$

A small computation leads to

$$|a_2| \leq \frac{2|\alpha y|\sqrt{2|\alpha y|}}{\sqrt{\left|4\alpha^2 y^2 L-(2\alpha(1+\alpha)y^2-\alpha)M^2\right|}}.$$

Next, in order to obtain the bound for $|a_3|$, subtracting (2.17) from (2.15) we have

$$2[[3]_{pq} - \mathsf{s}_1^2 - \mathsf{s}_2^2 - \mathsf{s}_1\mathsf{s}_2][a_3 - a_2^2] = C_1^\alpha(y)(c_2 - d_2) + C_2^\alpha(y)(c_1^2 - d_1^2). \tag{2.22}$$

Using the equations (2.18) and (2.19) in (2.22), we get

$$a_3 = \frac{C_1^\alpha(y)(c_2-d_2)}{2N} + \frac{(C_1^\alpha(y))^2(c_1^2+d_1^2)}{2M^2}. \tag{2.23}$$

Applying the value of $C_1^\alpha(y)$ and taking modulus, we have the desired bound for $|a_3|$

$$|a_3| \leq \left|\frac{2\alpha y}{N}\right| + \frac{4\alpha^2 y^2}{M^2}.$$

□

Corollary 2.3. Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_\sigma^{\mathfrak{p}\mathfrak{q}}(y,1,\mathsf{s}_1,\mathsf{s}_2)$. Then

$$|a_2| \leq \frac{2|y|\sqrt{2|y|}}{\sqrt{\left|4y^2 L-(4y^2-1)M^2\right|}}$$

and

$$|a_3| \leq \left|\frac{2y}{N}\right| + \frac{4y^2}{M^2}$$

where L, M, N are as defined in Theorem 1.2.

Corollary 2.4. Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_\sigma^{\mathfrak{p}\mathfrak{q}}(y,1/2,\mathsf{s}_1,\mathsf{s}_2)$. Then

$$|a_2| \leq \frac{|y|\sqrt{2|y|}}{\sqrt{\left|2y^2 L-(3y^2-1)M^2\right|}}$$

and

$$|a_3| \leq \left|\frac{y}{N}\right| + \frac{y^2}{M^2},$$

where L, M, N are as defined in Theorem 1.2.

Corollary 2.5. Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_\sigma(y,\alpha,\mathsf{s}_1,\mathsf{s}_2)$. Then

$$|a_2| \leq \frac{2|\alpha y|\sqrt{2|\alpha y|}}{\sqrt{\left|4\alpha^2 y^2 L_1-(2\alpha(1+\alpha)y^2-\alpha)M_1^2\right|}}$$

and

$$|a_3| \leq \left|\frac{2\alpha y}{N_1}\right| + \frac{4\alpha^2 y^2}{M_1^2},$$

where
$L_1 = 3 - 2(\mathsf{s}_1 + \mathsf{s}_2) + \mathsf{s}_1\mathsf{s}_2$,
$M_1 = 2 - \mathsf{s}_1 - \mathsf{s}_2$,
$N_1 = 3 - \mathsf{s}_1^2 - \mathsf{s}_2^2 - \mathsf{s}_1\mathsf{s}_2$.

Corollary 2.6. Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_\sigma(y,\alpha,1,-1)$. Then

$$|a_2| \leq \frac{|\alpha y|\sqrt{2|\alpha y|}}{\sqrt{\left|2\alpha^2 y^2-(2\alpha(1+\alpha)y^2-\alpha)\right|}}$$

and

$$|a_3| \leq |\alpha y| + \alpha^2 y^2.$$

Corollary 2.7. Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_\sigma(y,\alpha,1,0)$. Then

$$|a_2| \leq \frac{2|\alpha y|\sqrt{2|\alpha y|}}{\sqrt{\left|4\alpha^2y^2-(2\alpha(1+\alpha)y^2-\alpha)\right|}}$$

and

$$|a_3| \leq |\alpha y| + 4\alpha^2y^2.$$

2.1 Fekete-Szegö Problem for the Functions in the Class $\mathcal{S}_\sigma^{\mathfrak{pq}}(y,\alpha,\mathfrak{s}_1,\mathfrak{s}_2)$

In this section, for functions belonging to the class $\mathcal{S}_\sigma^{\mathfrak{pq}}(y,\alpha,\mathfrak{s}_1,\mathfrak{s}_2)$, we have estimated the bounds for the linear functional.

Theorem 2.8. Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_\sigma^{\mathfrak{pq}}(y,1,\mathfrak{s}_1,\mathfrak{s}_2)$. Then

$$|a_3-\gamma a_2^2| \leq \begin{cases} |\frac{2\alpha y}{N}| \quad , & if \quad 0 \leq |\gamma-1| \leq \left|\frac{D}{N}\right|, \\ \frac{|4\alpha^3y^3(1-\gamma)|}{|2\alpha^2y^2L-(2\alpha(1+\alpha)y^2-\alpha)M^2|}, & if \quad |\gamma-1| \geq \left|\frac{D}{N}\right|. \end{cases}$$

where L,M,N are as defined in Theorem 1.2 and $D = L - \frac{(2\alpha(1+\alpha)y^2-\alpha)M^2}{4\alpha^2y^2}$.

Proof . From (2.22), for $\gamma \in \mathbb{R}$, we have

$$a_3-\gamma a_2^2 = (1-\gamma)a_2^2 + \frac{(c_2-d_2)C_1^\alpha(y)}{2N} \tag{2.24}$$

By using (2.21) in (2.24), we have

$$\begin{aligned} a_3-\gamma a_2^2 &= (1-\gamma)\left[\frac{(c_2+d_2)(C_1^\alpha(y))^3}{2(C_1^\alpha(y))^2L-2C_2^\alpha(y)M^2}\right] + \frac{(c_2-d_2)C_1^\alpha(y)}{2N} \\ &= C_1^\alpha(y)[(\xi(\gamma,y)+\tfrac{1}{2N})c_2 + (\xi(\gamma,y)-\tfrac{1}{2N})d_2] \end{aligned}$$

where

$$\xi(\gamma,y) = \frac{(1-\gamma)[C_1^\alpha(y)]^2}{2[C_1^\alpha(y)]^2L-2M^2C_2^\alpha(y)}.$$

Taking modulus, we have

$$|a_3-\gamma a_2^2| \leq \begin{cases} |\frac{2\alpha y}{N}|, & if \quad 0 \leq |\xi(\gamma,y)| \leq \frac{1}{2|N|}, \\ 4|\alpha y\xi(\gamma,y)|, & if \quad |\xi(\gamma,y)| \geq \frac{1}{2|N|}. \end{cases}$$

□

Corollary 2.9. Let $\mathfrak{f} \in \sigma$ given by (1.1) belongs to the class $\mathcal{S}_\sigma^{\mathfrak{pq}}(y,1,\mathfrak{s}_1,\mathfrak{s}_2)$. Then

$$|a_3-\gamma a_2^2| \leq \begin{cases} |\frac{2y}{N}| \quad , & if \quad 0 \leq |\gamma-1| \leq \left|\frac{D_1}{N}\right|, \\ \frac{|4y^3(1-\gamma)|}{|2y^2L-(4y^2-1)M^2|} & , if \quad |\gamma-1| \geq \left|\frac{D_1}{N}\right|. \end{cases} \tag{2.25}$$

where L,M,N are as defined in Theorem 1.2 and $D_1 = L - \frac{(4y^2-1)M^2}{4y^2}$.

Corollary 2.10. Let $\mathfrak{f} \in \sigma$ given by (1.1) belongs to the class $\mathcal{S}_\sigma^{\mathfrak{pq}}(y,1/2,\mathfrak{s}_1,\mathfrak{s}_2)$. Then

$$|a_3-\gamma a_2^2| \leq \begin{cases} |\frac{y}{N}| & \text{if} \quad 0 \leq |\gamma-1| \leq \left|\frac{D_2}{N}\right| \\ \frac{|y^3(1-\gamma)|}{|y^2L-(3y^2-1)M^2|} & \text{if} \quad |\gamma-1| \geq \left|\frac{D_2}{N}\right|. \end{cases} \tag{2.26}$$

where L,M,N are as defined in Theorem 1.2 and $D_2 = L - \frac{(3y^2-1)M^2}{2y^2}$.

Corollary 2.11. Let $\mathfrak{f} \in \sigma$ given by (1.1) be in the class $\mathcal{S}_\sigma(y,\alpha,\mathsf{s}_1,\mathsf{s}_2)$. Then

$$|a_3-\gamma a_2^2| \leq \begin{cases} |\frac{2\alpha y}{N_1}| & ,if \quad 0 \leq |\gamma-1| \leq \left|\frac{D_3}{N_1}\right|, \\ \frac{|4\alpha^3 y^3(1-\gamma)|}{|2\alpha^2y^2L_1-(2\alpha(1+\alpha)y^2-\alpha)M_1{}^2|} & ,if \quad |\gamma-1| \geq \left|\frac{D_3}{N_1}\right|. \end{cases} \tag{2.27}$$

where
$L_1 = 3-2(\mathsf{s}_1+\mathsf{s}_2)+\mathsf{s}_1\mathsf{s}_2$,
$M_1 = 2-\mathsf{s}_1-\mathsf{s}_2$,
$N_1 = 3-\mathsf{s}_1^2-\mathsf{s}_2^2-\mathsf{s}_1\mathsf{s}_2$,
$D_3 = L_1 - \frac{(2\alpha(1+\alpha)y^2-\alpha)M_1{}^2}{4\alpha^2y^2}$.

Corollary 2.12. Let $\mathfrak{f} \in \sigma$ given by (1.1) be in the class $\mathcal{S}_\sigma(y,\alpha,1,-1)$. Then

$$|a_3-\gamma a_2^2| \leq \begin{cases} |\alpha y| & \text{if} \quad 0 \leq |\gamma-1| \leq \left|\frac{D_4}{2}\right|, \\ \frac{|\alpha^3 y^3(1-\gamma)|}{|\alpha^2y^2-(2\alpha(1+\alpha)y^2-\alpha)|} & \text{if} \quad |\gamma-1| \geq \left|\frac{D_4}{2}\right|. \end{cases} \tag{2.28}$$

where
$D_4 = 2 - \frac{(2\alpha(1+\alpha)y^2-\alpha)}{\alpha^2y^2}$.

Corollary 2.13. Let $\mathfrak{f} \in \sigma$ given by (1.1) be in the class $\mathcal{S}_\sigma(y,\alpha,1,0)$. Then

$$|a_3-\gamma a_2^2| \leq \begin{cases} |\alpha y| & , \quad if \quad 0 \leq |\gamma-1| \leq \left|\frac{D_5}{2}\right|, \\ \frac{|4\alpha^3 y^3(1-\gamma)|}{|2\alpha^2y^2-(2\alpha(1+\alpha)y^2-\alpha)|}, if & |\gamma-1| \geq \left|\frac{D_5}{2}\right|. \end{cases} \tag{2.29}$$

where $D_5 = 1 - \frac{(2\alpha(1+\alpha)y^2-\alpha)}{4\alpha^2y^2}$.

3 Conclusion

We have calculated the bounds for $|a_2|$ and $|a_3|$ and Fekete-Szegö inequality for the Sakaguchi-Type function defined by $(\mathfrak{p},\mathfrak{q})$-Derivative operator using Gegenbauer polynomials defined by us in this paper.

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