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Wavelet frames associated with linear canonical transform on spectrum

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Abstract

The linear canonical transform(LCT) provides a unified treatment of the generalized Fourier transforms in the sense that it is an embodiment of several well-known integral transforms including the Fourier transform, fractional Fourier transform, Fresnel transform. Using this fascinating property of LCT, we in this paper construct associated wavelet frames. To be precise we introduce wavelet frames whose construction depends on the nonuniform multiresolution analysis associated with linear canonical transform(LCT-NUMRA) whose translation set is not necessarily a group. The translation set is taken for elements in $\Lambda = \{0, r/N\} + 2\,\mathbb{Z}, \mathbb{N} \geq 1$ (an integer) and r is an odd integer with $1 \leq r \leq 2N-1$ such that r and N are relatively prime and $\mathbb{Z}$ is the set of all integers. Furthermore we establish necessary and sufficient condition for such nonuniform wavelet frames associated with linear canonical transform.

Keywords: Frame, nonuniform LCT wavelets, wavelet frame, Linear canonical transform
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1 Introduction

The frame was first introduced by Duffin and Schaeffer [12] in the study of non-harmonic Fourier series in 1952, reintroduced in 1986 by Daubechies et al. [10], and popularized from then on. Frames and their duals have very important and interesting properties which make them very useful in the characterization of function spaces, signal and image processing, sampling theory and many other fields. A frame is a family of elements in a separable Hilbert space which allows stable not necessarily unique decomposition of arbitrary elements into expansions of frame elements. A sequence $\{f_k\}_{k=1}^{\infty}$ of elements of a Hilbert space $\mathbb{H}$ is called a *frame* for $\mathbb{H}$ if there exist constants $A, B > 0$ such that for all $f \in \mathbb{H}$,

$$A\big\|f\big\|_2^2 \leq \sum_{k=1}^{\infty} \big|\langle f, f_k\rangle\big|^2 \leq B\big\|f\big\|_2^2,$$

where A and B are the lower and upper frame bounds, respectively. If $A = B$, then the frame is called a *tight frame*. If $A = B = 1$, then the frame is called a *normalized tight frame*. The particular frames of interest to us will be the frames in the space $L^2(\mathbb{R})$ which are generated by the combined action of dilations and translations of finite number of functions. In order to describe these frames, for $a, b \in \mathbb{R}$ with $a > 1$, and $b > 0$, we define wavelet systems as

$$\mathcal{F}(\psi, a, b) = \Big\{\psi_{j,k} =: a^{j/2}\psi\big(a^j x - kb\big) : j, k \in \mathbb{Z}\Big\}.$$

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Wavelet systems $\mathcal{F}(\psi, a, b)$ that form frames for $L^2(\mathbb{R})$ have a wide variety of applications [15, 17, 19, 22]. Therefore, one of the fundamental problems in the study of wavelet frames is to find conditions on ψ, a and b such that the system $\mathcal{F}(\psi, a, b)$ forms a frame. In 1990, Daubechies obtained the first result on the necessary conditions for wavelet frames, and then in 1993, Chui and Shi obtained an improved result. Cassaza and Christensen provided a stronger version of Daubechies's sufficient condition for wavelet frames in $L^2(\mathbb{R})$. In recent years, these conditions have been further improved and investigated by many authors. All these concepts are developed on regular lattices, that is the translation set is always a group. Recently, Gabardo and Nashed [13, 14] developed the theory of nonuniform wavelets and wavelet sets in $L^2(\mathbb{R})$ for which the translation set is no longer a discrete subgroup of $\mathbb{R}$, but a union of two lattices, which is associated with a famous open conjecture of Fuglede on spectral pairs. Subsequently, nonuniform wavelet frames associated with spectral pairs were constructed by Sharma and Manchanda [24] using the machinery of Fourier transforms. In fact, they obtained a necessary and sufficient conditions for the nonuniform wavelet system to be a frame for $L^2(\mathbb{R})$. Recent results in this direction can also be found in [11, 12, 21, 23] and the references therein.

The concept of novel multiresolution analysis in nonuniform settings was established by Shah and Lone [20]. They call it Nonuniform Multiresolution analysis associated with linear canonical transform (LCT-NUMRA). More results in this direction can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 16, 18, 25] and the references therein. In this paper, we continue the study based on this nonstandard setting and introduce nonuniform wavelet frames associated with linear canonical transform. We establish a necessary and sufficient condition for the existence such frames.

The paper is organised in the following manner. In Section 2, we obtain necessary condition for the existence of such frames. In Section 3, we obtain a sufficient condition for nonuniform wavelet frames associated with linear canonical transform.

2 Nonuniform LCT Wavelet Frames in $L^2(\mathbb{R})$

For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N-1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z} = \left\{\frac{rk}{N} + 2n : n \in \mathbb{Z}, k = 0, 1\right\}. \tag{2.1}$$

It is easy to verify that Λ is not necessarily a group nor a uniform discrete set, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Moreover, the set Λ is the spectrum for the spectral set $\Upsilon = \left[0, \frac{1}{2}\right) \cup \left[\frac{N}{2}, \frac{N+1}{2}\right)$ and the pair (Λ, Υ) is called a *spectral pair* [6,7].

For a given $\psi \in L^2(\mathbb{R})$, define the nonuniform LCT wavelet system

$$\mathcal{L}_M(\psi, j, \lambda) = \left\{\psi_{j,\lambda}^M =: (2N)^{j/2}\psi\big((2N)^j x - \lambda\big)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} : j \in \mathbb{Z}, \lambda \in \Lambda\right\}. \tag{2.2}$$

We call the wavelet system $\mathcal{L}_M(\psi^M, j, \lambda)$ a nonuniform LCT wavelet frame for $L^2(\mathbb{R})$, if there exist positive numbers $0 < P \leq Q < \infty$ such that

$$P\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda} \left|\langle f, \psi_{j,\lambda}^M\rangle\right|^2 \leq Q\|f\|_2^2, \tag{2.3}$$

holds for every $f \in L^2(\mathbb{R})$.We call P, Q the lower frame bound and the upper frame bound for the frame, respectively. If $P = Q$, then the frame is called a tight frame. If $P = Q = 1$, then the frame is called a normalized tight frame. First we prove a lemma which will be used in the proofs of the main results.

Lemma 2.1. If $f, \psi^M \in L^2(\mathbb{R})$ then

$$\begin{aligned}\sum_{\lambda\in\Lambda} \left|\langle f, \psi_{j,\lambda}^M\rangle\right|^2 \;=\; & \frac{1}{4N}\sum_{p=0}^{2N-1}\int_{\mathbb{R}}\left\{\overline{\hat{f}\left(\frac{\xi}{B} + (2N)^j\frac{p}{2}\right)}\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{p}{2}\right)\right\}\mathcal{G}_1\left(\frac{\xi}{B}\right)d\xi \\ & + \frac{1}{4N}\sum_{p=0}^{2N-1}\int_{\mathbb{R}}\left\{\overline{\hat{f}\left(\frac{\xi}{B} + (2N)^j\frac{p}{2}\right)}\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{p}{2}\right)\left(e^{-\pi i\frac{r}{N}p}\right)\right\}\mathcal{G}_2\left(\frac{\xi}{B}\right)d\xi\end{aligned} \tag{2.4}$$

where

$$\mathcal{G}_1\left(\frac{\xi}{B}\right) = \sum_{l\in\mathbb{Z}}\int_{\mathbb{R}}\left\{\hat{f}\left(\frac{\xi}{B} + (2N)^j\frac{l}{2}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{l}{2}\right)}\right\} \tag{2.5}$$

and

$$\mathcal{G}_2\left(\frac{\xi}{B}\right)=\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}}\left\{\hat{f}\left(\frac{\xi}{B}+(2N)^j\frac{l}{2}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}\right)}\left(e^{\pi i\frac{r}{N}l}\right)\right\} \tag{2.6}$$

Proof . For $f\in L^2(\mathbb{R})$

$$\begin{aligned}\langle f,\psi^M_{j,\lambda}\rangle &= \frac{1}{(2N)^{j/2}}\int_{\mathbb{R}}\hat{f}\left(\frac{\xi}{B}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}\right)}e^{2\pi i\frac{\xi}{(2NB)^j}\lambda}d\xi\\
&= \frac{1}{(2N)^{j/2}}\sum_{l\in\mathbb{Z}}\int_{\frac{B(2N)^j}{2}l}^{\frac{B(2N)^j}{2}(l+1)}\hat{f}\left(\frac{\xi}{B}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}\right)}e^{2\pi i\frac{\xi}{(2NB)^j}\lambda}d\xi\\
&= \frac{1}{(2N)^{j/2}}\sum_{l\in\mathbb{Z}}\int_0^{\frac{B(2N)^j}{2}}\hat{f}\left(\frac{\xi}{B}+(2N)^j\frac{l}{2}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}\right)}\\
&\times e^{\pi i\lambda l}e^{2\pi i\frac{\xi}{(2NB)^j}\lambda}d\xi\\
&= \frac{1}{(2N)^{j/2}}\int_0^{\frac{B(2N)^j}{2}}\sum_{l\in\mathbb{Z}}\hat{f}\left(\frac{\xi}{B}+(2N)^j\frac{l}{2}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}\right)}e^{\pi i\lambda l}\\
&\times e^{2\pi i\frac{\xi}{(2NB)^j}\lambda}d\xi\end{aligned}$$

Now,

$$\sum_{\lambda\in\Lambda}\left|\langle f,\psi^M_{j,\lambda}\rangle\right|^2=\sum_{\lambda\in 2\mathbb{Z}}\left|\langle f,\psi^M_{j,\lambda}\rangle\right|^2+\sum_{\lambda\in(r/N+2\mathbb{Z})}\left|\langle f,\psi^M_{j,\lambda}\rangle\right|^2 \tag{2.7}$$

Since $\{\frac{\sqrt{2}}{(2N)^{j/2}}e^{2\pi i(2N)^{-j}2m\xi/B}\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $L^2(0,\frac{B(2N)^j}{2})$ Parseval's formula gives:

$$\begin{aligned}\sum_{\lambda\in 2\mathbb{Z}}\left|\langle f,\psi^M_{j,\lambda}\rangle\right|^2 &= \frac{1}{2}\sum_{m\in\mathbb{Z}}\left|\int_0^{\frac{B(2N)^j}{2}}\sum_{l\in\mathbb{Z}}\hat{f}\left(\frac{\xi}{B}+(2N)^j\frac{l}{2}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}\right)}\frac{\sqrt{2}}{(2N)^{j/2}}e^{2\pi i(2N)^{-j}2m\xi/B}d\xi\right|^2\\
&= \frac{1}{2}\int_0^{\frac{B(2N)^j}{2}}\left|\sum_{l\in\mathbb{Z}}\hat{f}\left(\frac{\xi}{B}+(2N)^j\frac{l}{2}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}\right)}\right|^2d\xi\\
&= \frac{1}{2}\int_0^{\frac{B(2N)^j}{2}}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2d\xi\end{aligned}$$

Using Parseval's formula again, we obtain

$$\begin{aligned}\sum_{\lambda\in(r/N+2\mathbb{Z})}\left|\langle f,\psi^M_{j,\lambda}\rangle\right|^2 &= \frac{1}{2}\sum_{m\in\mathbb{Z}}\left|\int_0^{\frac{B(2N)^j}{2}}\sum_{l\in\mathbb{Z}}\hat{f}\left(\frac{\xi}{B}+(2N)^j\frac{l}{2}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}\right)}\right.\\
&\left.\times\, e^{\pi irl/N}\frac{\sqrt{2}e^{2\pi i(2N)^{-j}(r/N+2m)\xi/B}}{(2N)^{j/2}}d\xi\right|^2\\
&= \frac{1}{2}\sum_{m\in\mathbb{Z}}\left|\int_0^{\frac{B(2N)^j}{2}}\sum_{l\in\mathbb{Z}}\hat{f}\left(\frac{\xi}{B}+(2N)^j\frac{l}{2}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}\right)}e^{\pi irl/N}d\xi\right|^2\\
&= \frac{1}{2}\int_0^{\frac{B(2N)^j}{2}}\left|\mathcal{G}_2\left(\frac{\xi}{B}\right)\right|^2d\xi\end{aligned}$$

Therefore (2.7) becomes

$$\sum_{\lambda\in\Lambda}\left|\left\langle f,\psi^{M}_{j,\lambda}\right\rangle\right|^2 = \frac{1}{2}\int_0^{\frac{B(2N)^j}{2}}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2 d\xi+\int_0^{\frac{B(2N)^j}{2}}\left|\mathcal{G}_2(\frac{\xi}{B})\right|^2 d\xi \tag{2.8}$$

$$= \frac{1}{2}[I+II] \tag{2.9}$$

where, $I=\int_0^{\frac{B(2N)^j}{2}}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2 d\xi$ and $II=\int_0^{\frac{B(2N)^j}{2}}\left|\mathcal{G}_2(\frac{\xi}{B})\right|^2 d\xi$
For $1\le p\le 2N-1$, we have

$$\mathcal{G}_1\left(\frac{\xi}{B}+\frac{1}{2}(2N)^j p\right) = \sum_{l\in\mathbb{Z}}\hat{f}\left(\frac{\xi}{B}+\frac{1}{2}(2N)^j(l+p)\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}(l+p)\right)}$$

$$= \mathcal{G}_1(\frac{\xi}{B}) \tag{2.10}$$

and

$$\mathcal{G}_2\left(\frac{\xi}{B}+\frac{1}{2}(2N)^j p\right) = \sum_{l\in\mathbb{Z}}\hat{f}\left(\frac{\xi}{B}+\frac{1}{2}(2N)^j(l+p)\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}(l+p)\right)}e^{\pi irl/N}$$

$$= e^{-\pi irp/N}\mathcal{G}_2(\frac{\xi}{B}) \tag{2.11}$$

Since

$$I=\int_0^{\frac{B(2N)^j}{2}}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2 d\xi=\int_0^{\frac{B(2N)^{j+1}}{2}}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2 d\xi-\int_{\frac{B(2N)^j}{2}}^{\frac{B(2N)^{j+1}}{2}}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2 d\xi$$

Using (2.10), we get

$$\int_{\frac{B(2N)^j}{2}}^{\frac{B(2N)^{j+1}}{2}}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2 d\xi = \int_{\frac{B(2N)^j}{2}}^{B(2N)^j}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2 d\xi+\int_{B(2N)^j}^{\frac{3B(2N)^j}{2}}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2 d\xi+\cdots$$

$$+\int_{\frac{B(2N-1)(2N)^j}{2}}^{\frac{B(2N)^{j+1}}{2}}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2 d\xi=(2N-1)I \tag{2.12}$$

Therefore

$$I=\frac{1}{2N}\int_0^{\frac{B(2N)^{j+1}}{2}}\left|\mathcal{G}_1(\frac{\xi}{B})\right|^2 d\xi \tag{2.13}$$

Similarly

$$II=\frac{1}{2N}\int_0^{\frac{B(2N)^{j+1}}{2}}\left|\mathcal{G}_2(\frac{\xi}{B})\right|^2 d\xi \tag{2.14}$$

The function $\mathcal{G}_1(\frac{\xi}{B})$ can be expressed as

$$\mathcal{G}_1\left(\frac{\xi}{B}\right) = \sum_{l\in\mathbb{Z}} \hat{f}\left(\frac{\xi}{B} + \frac{(2N)^j}{2} l\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{l}{2}\right)}$$
$$= \sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{\xi}{B} + \frac{(2N)^j 2Nm}{2} l\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{2Nm}{2}\right)}$$
$$+ \sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{\xi}{B} + \frac{(2N)^j (2Nm+1)}{2} l\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{(2Nm+1)}{2}\right)}$$
$$+$$
$$\vdots$$
$$+ \sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{\xi}{B} + \frac{(2N)^j (2Nm+2N-1)}{2} l\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{(2Nm+2N-1)}{2}\right)}$$
$$= \sum_{p=0}^{2N-1} \sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{\xi}{B} + \frac{(2N)^{j+1} m}{2} + \frac{(2N)^j p}{2}\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{(2Nm+p)}{2}\right)}$$

By similar procedure we can write $\mathcal{G}_2(\frac{\xi}{B})$ as

$$\mathcal{G}_2\left(\frac{\xi}{B}\right) = \sum_{p=0}^{2N-1} \sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{\xi}{B} + \frac{(2N)^{j+1} m}{2} + \frac{(2N)^j p}{2}\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{(2Nm+p)}{2}\right)} e^{\pi irp/N}$$

Functions $\mathcal{G}_1(\frac{\xi}{B})$ and $\mathcal{G}_2(\frac{\xi}{B})$ are $\frac{B}{2}(2N)^{j+1}$-periodic,which gives

$$\begin{aligned} I &= \frac{1}{2N} \sum_{p=0}^{2N-1} \int_0^{\frac{B(2N)^{j+1}}{2}} \sum_{m\in\mathbb{Z}} \overline{\hat{f}\left(\frac{\xi}{B} + \frac{(2N)^{j+1} m}{2} + \frac{(2N)^j p}{2}\right)} \hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{(2Nm+p)}{2}\right) \mathcal{G}_1(\frac{\xi}{B}) d\xi \\ &= \frac{1}{2N} \sum_{p=0}^{2N-1} \sum_{m\in\mathbb{Z}} \int_{\frac{B(2N)^{j+1}}{2} m}^{\frac{B(2N)^{j+1}(m+1)}{2}} \overline{\hat{f}\left(\frac{\xi}{B} + \frac{(2N)^j p}{2}\right)} \hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{p}{2}\right) \mathcal{G}_1(\frac{\xi}{B}) d\xi \\ &= \frac{1}{2N} \sum_{p=0}^{2N-1} \int_{\mathbb{R}} \overline{\hat{f}\left(\frac{\xi}{B} + \frac{(2N)^j p}{2}\right)} \hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{p}{2}\right) \mathcal{G}_1(\frac{\xi}{B}) d\xi \end{aligned} \tag{2.15}$$

and similarly

$$II = \frac{1}{2N} \sum_{p=0}^{2N-1} \int_{\mathbb{R}} \overline{\hat{f}\left(\frac{\xi}{B} + \frac{(2N)^j p}{2}\right)} \hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{p}{2}\right) \mathcal{G}_2(\frac{\xi}{B}) e^{-\pi irp/N} d\xi \tag{2.16}$$

Combining all these results, we get

$$\sum_{\lambda\in\Lambda} \left|\langle f, \psi^M_{j,\lambda}\rangle\right|^2 = \frac{1}{4N} \sum_{p=0}^{2N-1} \int_{\mathbb{R}} \overline{\hat{f}\left(\frac{\xi}{B} + \frac{(2N)^j p}{2}\right)} \hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{p}{2}\right) \left(\mathcal{G}_1(\frac{\xi}{B}) + e^{\pi irp/N} \mathcal{G}_2(\frac{\xi}{B})\right) d\xi$$

Hence completes the proof.□

Analogous to the result of Chui and Shi we prove a necessary condition for the system $\left\{\psi^M_{j,\lambda}(t) : j \in \mathbb{Z}, \lambda \in \Lambda\right\}$. to be a wavelet frame for $L^2(\mathbb{R})$.

Theorem 2.2. If $\left\{\psi^M_{j,\lambda}(t) : j \in \mathbb{Z}, \lambda \in \Lambda\right\}$ is a wavelet frame for $L^2(\mathbb{R})$ with bounds P and Q, then

$$P \le \mathcal{S}_{\psi^M}(\xi/B) \le Q \quad a.e\, \xi \in \mathbb{R} \tag{2.17}$$

where

$$\mathcal{S}_{\psi^M}(\xi/B) = \sum_{j\in\mathbb{Z}} \left|\hat{\psi}(\frac{\xi}{(2NB)^J})\right|^2 \tag{2.18}$$

Proof . By definition, we have

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^{M}\rangle\right|^2 \le Q||f||^2$$

for $f\in L^2(\mathbb{R})$ and for any $K>0, K\in\mathbb{Z}$ we write

$$\sum_{j=-K}^{K}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^{M}\rangle\right|^2 \le Q||f||^2$$

Using Lemma 2.1, we get

$$\begin{aligned}&\frac{1}{4N}\sum_{j=-K}^{K}\sum_{p=0}^{2N-1}\Bigg[\int_{\mathbb{R}}\overline{\hat f\left(\frac{\xi}{B}+\frac{(2N)^jp}{2}\right)}\hat\psi\left(\frac{\xi}{(2NB)^j}+\frac{p}{2}\right)\\&\times\sum_{l\in\mathbb{Z}}\hat f\left(\frac{\xi}{B}+\frac{(2N)^jl}{2}\right)\overline{\hat\psi\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}\right)}d\xi\\&+\int_{\mathbb{R}}\overline{\hat f\left(\frac{\xi}{B}+\frac{(2N)^jp}{2}\right)}\hat\psi\left(\frac{\xi}{(2NB)^j}+\frac{p}{2}\right)e^{-\pi irp/N}\\&\times\sum_{l\in\mathbb{Z}}\hat f\left(\frac{\xi}{B}+\frac{(2N)^jl}{2}\right)\overline{\hat\psi\left(\frac{\xi}{(2NB)^j}+\frac{l}{2}\right)e^{-\pi irl/N}}d\xi\Bigg]\le Q||\hat f||^2\end{aligned}\tag{2.19}$$

Let $\xi_0\in\mathbb{R}$, given $\epsilon>0$, let us consider

$$\hat f(\frac{\xi}{B})=\begin{cases}\dfrac{1}{\sqrt{2\epsilon}},\xi\in(\xi_0-\epsilon,\xi_0+\epsilon),\\ \quad 0,\qquad \text{otherwise}\end{cases}\tag{2.20}$$

For very small $\epsilon, j\ge -K$, the intervals $(\xi_0-\epsilon-(2N)^jl/2,\xi_0+\epsilon-(2N)^jl/2), l\in\mathbb{Z}$ are mutually disjoint,we have

$$\hat f(\frac{\xi}{B}+(2N)^jl/2)\hat f(\frac{\xi}{B}+(2N)^jm/2)=0, l\ne m; l,m\in\mathbb{Z}\tag{2.21}$$

Then (2.19)becomes

$$\frac{1}{2N}\sum_{j=-K}^{K}\sum_{p=0}^{2N-1}\int_{\xi_0-\epsilon-(2N)^jp/2}^{\xi_0+\epsilon-(2N)^jp/2}\left|\hat f\left(\frac{\xi}{B}+\frac{(2N)^jp}{2}\right)\right|^2\left|\hat\psi\left(\frac{\xi}{(2NB)^j}+\frac{p}{2}\right)\right|^2\le Q||\hat f||^2\tag{2.22}$$

or

$$\frac{1}{2N}\sum_{j=-K}^{K}\sum_{p=0}^{2N-1}\frac{1}{2\epsilon}\int_{\xi_0-\epsilon-(2N)^jp/2}^{\xi_0+\epsilon-(2N)^jp/2}\left|\hat\psi\left(\frac{\xi}{(2NB)^j}+\frac{p}{2}\right)\right|^2d\xi\le Q\tag{2.23}$$

By taking $\epsilon\to 0$ and$K\to\infty$ consecutively, we have

$$\frac{1}{2N}\sum_{j\in\mathbb{Z}}\sum_{p=0}^{2N-1}\left|\hat\psi\left(\frac{\xi}{(2NB)^j}\right)\right|^2\le Q\Rightarrow\sum_{j\in\mathbb{Z}}\left|\hat\psi\left(\frac{\xi}{(2NB)^j}\right)\right|^2\le Q\tag{2.24}$$

which is the right inequality of (2.17). To prove left inequality of (2.17), consider

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^{M}\rangle\right|^2=I_1+I_2\tag{2.24}$$

where

$$I_1=\sum_{j>-K}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^{M}\rangle\right|^2$$

and

$$I_2 = \sum_{j\leq -K}\sum_{\lambda\in\Lambda}\left|\langle f, \psi^M_{j,\lambda}\rangle\right|^2$$

By the frame condition $I_1 \geq P - I_2$. As we have shown that $I_1 \to \mathcal{S}_{\psi^M}(\frac{\xi}{B})$, now we have to prove that $I_2 \to 0$as $K \to \infty$

Using (2.5), (2.6) and (2.9), we obtain

$$\begin{aligned}
I_2 &= \sum_{j\leq -K}\sum_{\lambda\in\Lambda}\left|\langle f, \psi^M_{j,\lambda}\rangle\right|^2\\
&= \frac{1}{2}\sum_{j\leq -K}\int_0^{B(2N)^j/2}\left|\sum_{l\in\mathbb{Z}}\hat{f}(\frac{\xi}{B}+(2N)^j l/2)\overline{\hat{\psi}(\frac{\xi}{(2NB)^j}+l/2)}\right|^2 d\xi\\
&+ \frac{1}{2}\sum_{j\leq -K}\int_0^{B(2N)^j/2}\left|\sum_{l\in\mathbb{Z}}\hat{f}(\frac{\xi}{B}+(2N)^j l/2)\overline{\hat{\psi}(\frac{\xi}{(2NB)^j}+l/2)}e^{\pi i r l/N}\right|^2 d\xi\\
&\leq \frac{1}{2}\sum_{j\leq -K}\int_0^{B(2N)^j/2}\left|\sum_{l\in\mathbb{Z}}|\hat{f}(\frac{\xi}{B}+(2N)^j l/2)\overline{\hat{\psi}(\frac{\xi}{(2NB)^j}+l/2)|}\right|^2 d\xi\\
&+ \frac{1}{2}\sum_{j\leq -K}\int_0^{B(2N)^j/2}\left|\sum_{l\in\mathbb{Z}}|\hat{f}(\frac{\xi}{B}+(2N)^j l/2)\overline{\hat{\psi}(\frac{\xi}{(2NB)^j}+l/2)e^{\pi i r l/N}|}\right|^2 d\xi\\
&= \sum_{j\leq -K}\int_0^{B(2N)^j/2}\left|\sum_{l\in\mathbb{Z}}|\hat{f}(\frac{\xi}{B}+(2N)^j l/2)\overline{\hat{\psi}(\frac{\xi}{(2NB)^j}+l/2)|}\right|^2 d\xi\\
&= \sum_{j\leq -K}\int_0^{B(2N)^j/2}\left\{\sum_{l\in\mathbb{Z}}|\hat{f}(\frac{\xi}{B}+(2N)^j l/2)\overline{\hat{\psi}(\frac{\xi}{(2NB)^j}+l/2)|}\right\}\\
&\times\left\{\sum_{m\in\mathbb{Z}}|\hat{f}(\frac{\xi}{B}+(2N)^j m/2)\hat{\psi}(\frac{\xi}{(2NB)^j}+m/2)|\right\}d\xi
\end{aligned}$$

Since the function$\sum_{l\in\mathbb{Z}}|\hat{f}(\frac{\xi}{B}+(2N)^j l/2)\overline{\hat{\psi}(\frac{\xi}{(2NB)^j}+l/2)}|$ is$\frac{B(2N)^j}{2}$−periodic, we can continue with

$$\begin{aligned}
I_2 &\leq \sum_{j\leq -K}\sum_{l\in\mathbb{Z}}\int_{\frac{B(2N)^j}{2}l}^{\frac{B(2N)^j}{2}(l+1)}|\hat{f}(\frac{\xi}{B})\overline{\hat{\psi}(\frac{\xi}{(2NB)^j})}|\\
&\times\sum_{m\in\mathbb{Z}}|\hat{f}(\frac{\xi}{B}+(2N)^j m/2)\hat{\psi}(\frac{\xi}{(2NB)^j}+m/2)|d\xi\\
&= \sum_{j\leq -K}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}|\hat{f}(\frac{\xi}{B})\overline{\hat{\psi}(\frac{\xi}{(2NB)^j})}||\overline{\hat{f}(\frac{\xi}{B}+(2N)^j m/2)}\hat{\psi}(\frac{\xi}{(2NB)^j}+m/2)|d\xi
\end{aligned}$$

By Cauchy-Schwarz inequality, we get

$$\begin{aligned}
I_2 &\leq \sum_{j\leq -K}\sum_{m\in\mathbb{Z}}\left\{\int_{\mathbb{R}}|\hat{f}(\frac{\xi}{B})\overline{\hat{\psi}(\frac{\xi}{(2NB)^j})}|^2 d\xi\right\}^{1/2}\\
&\times\left\{|\overline{\hat{f}(\frac{\xi}{B}+(2N)^j m/2)}\hat{\psi}(\frac{\xi}{(2NB)^j}+m/2)|^2 d\xi\right\}^{1/2}
\end{aligned}$$

Let $\xi_0 \in \mathbb{R}$ and $\hat{f}$ be defined as above. If $\left\{\frac{\xi}{B}+(2N)^j m/2\right\} \in (\xi_0-\epsilon, \xi_0+\epsilon)$, then $|(2N)^J m/2)| < \epsilon$ for fixed $j \leq -K$.By the hypothesis of $\hat{f}$, the number of summation index m is bounded by $\frac{2\epsilon}{(2NB)^j}$. Thus

$$I_2 \leq \frac{2\epsilon}{(2NB)^j}\sum_{j\leq -K}\int_R|\hat{f}(\frac{\xi}{B})\overline{\hat{\psi}(\frac{\xi}{(2NB)^j})}|^2 d\xi = \sum_{j\leq -K}\int_{\frac{B(\xi_0-\epsilon)}{(2N)^j}}^{\frac{B(\xi_0-\epsilon)}{(2N)^j}}|\hat{\psi}(\frac{\xi}{B})|^2 d\xi \tag{2.25}$$

Also for any $\xi_0 \neq 0$ given $\eta > 0$,a positive integer K may be chosen so that

$$\int_{\frac{2B(2N)^K\xi_0}{1+2N}}^{\infty} |\hat{\psi}(\frac{\xi}{B})|^2 d\xi < \eta \tag{2.26}$$

It is clear that for $0 < \epsilon < \frac{2N-1}{2N+1}\xi_0$, the intervals$\left(\frac{B(\xi_0-\epsilon)}{(2N)^j}, \frac{B(\xi_0-\epsilon)}{(2N)^j}\right)$, $j \in \mathbb{Z}$ are mutually disjoint. It follows that

$$I_2 \leq \sum_{j \leq -K} \int_{\frac{B(\xi_0-\epsilon)}{(2N)^j}}^{\frac{B(\xi_0-\epsilon)}{(2N)^j}} |\hat{\psi}(\frac{\xi}{B})|^2 d\xi \leq \int_{\frac{2B(2N)^K\xi_0}{1+2N}}^{\infty} |\hat{\psi}(\frac{\xi}{B})|^2 d\xi < \eta$$

hence completes proof. □

3 A Sufficient Condition

In this section, we establish a sufficient condition for the system$\left\{\psi_{j,\lambda}^M(t) : j \in \mathbb{Z}, \lambda \in \Lambda\right\}$ to be a wavelet frame for$L^2(\mathbb{R})$. For this we prove the lemma.

Lemma 3.1. Let f be in $L^2(\mathbb{R})$ such that $\hat{f} \in \mathcal{C}_c(\mathbb{R})$.If $\psi^M \in L^2(\mathbb{R})$ and $\text{esssup}_{\xi\in\mathbb{R}}$ $\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(\frac{\xi}{(2NB)^j})\right|^2 \leq \infty$, then

$$\sum_{j\in\mathbb{R}} \sum_{\lambda\in\Lambda} \left|\langle f, \psi_{j,\lambda}^M \rangle\right|^2 = \int_{\mathbb{R}} \left|\hat{f}(\frac{\xi}{B})\right|^2 \sum_{j\in\mathbb{Z}} \left|\hat{\psi}(\frac{\xi}{(2NB)^j})\right|^2 d\xi + R_{\psi^M}(f), \tag{3.1}$$

where $R_{\psi^M}(f) = R_0 + R_1 + \cdots + R_{2N-1}$ and for $0 \leq p \leq 2N-1, R_p$ is given by

$$R_p = \frac{1}{4N} \sum_{j\in\mathbb{Z}} \sum_{\ell\neq p} \int_{\mathbb{R}} \left\{ \overline{\hat{f}\left(\frac{\xi}{B} + (2N)^j \frac{p}{2}\right)} \hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{p}{2}\right) \right\} \left\{ \hat{f}\left(\frac{\xi}{B} + (2N)^j \frac{\ell}{2}\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{\ell}{2}\right)} \left(1 + e^{\pi i \frac{r}{N}(\ell-p)}\right) \right\} d\xi. \tag{3.2}$$

Proof . By (2.4), (2.5) and (2.6), we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\langle f, \psi_{j,\lambda}^M \rangle\right|^2 &= \frac{1}{4N} \sum_{j\in\mathbb{Z}} 2 \int_{\mathbb{R}} |\hat{f}(\frac{\xi}{B}|^2 \hat{\psi}(\frac{\xi}{2NB})|^2 d\xi \\
&+ \frac{1}{4N} \sum_{j\in\mathbb{Z}} \sum_{\ell\neq 0} \int_{\mathbb{R}} \overline{\hat{f}(\frac{\xi}{B})} \hat{\psi}(\frac{\xi}{(2NB^j)}) \hat{f}(\frac{\xi}{B} + 2N^j/2) \\
&\times \overline{\hat{\psi}(\frac{\xi}{(2NB)^j} + l/2)} (1 + e^{\pi irl/N}) d\xi \\
&+ \frac{1}{4N} \sum_{j\in\mathbb{Z}} 2 \int_{\mathbb{R}} |\hat{f}(\frac{\xi}{B} + (2N)^j/2)|^2 \hat{\psi}\left(\frac{\xi}{(2NB)^j} + 1/2\right)|^2 d\xi \\
&+ \frac{1}{4N} \sum_{j\in\mathbb{Z}} \sum_{l\neq 1} \int_{\mathbb{R}} \left\{ \overline{\hat{f}(\frac{\xi}{B} + (2N)^j/2)} \hat{\psi}(\frac{\xi}{(2NB)^j} + 1/2) \right\} \\
&\times \left\{ \hat{f}(\frac{\xi}{B} + (2N)^j l/2) \overline{\hat{\psi}(\frac{\xi}{(2NB)^j} + l/2)} (1 + e^{\pi i(l-1)r/N}) \right\} d\xi \\
&\vdots \\
&+ \frac{1}{4N} \sum_{j\in\mathbb{Z}} 2 \int |\hat{f}(\frac{\xi}{B} + (2N-1)(2N)^j/2)|^2 |\hat{\psi}(\frac{\xi}{(2NB)^j} + (2N-1)/2)|^2 d\xi \\
&+ \frac{1}{4N} \sum_{j\in\mathbb{Z}} \sum_{l\neq 2N-1} \int \left\{ \overline{\hat{f}(\frac{\xi}{B} + (2N-1)(2N)^j/2)} \hat{\psi}(\frac{\xi}{(2NB)^j} + (2N-1)/2) \right\} \\
&\times \left\{ \hat{f}(\frac{\xi}{B} + (2N)^j l/2) \overline{\psi(\frac{\xi}{(2NB)^j} + l/2)} (1 + e^{\pi i(l-2N+1)r/N}) \right\} d\xi
\end{aligned}$$

For $0 \le p \le 2N-1$, we have

$$\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}|\hat{f}(\frac{\xi}{B}+(2N)^j p)|^2|\psi(\frac{\xi}{(2NB)^j}+p)|^2 d\xi=\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}|\hat{f}(\frac{\xi}{B})|^2|\psi(\frac{\xi}{(2NB)^j})|^2 d\xi. \tag{3.3}$$

Therefore, we obtain

$$\sum_{j\in\mathbb{R}}\sum_{\lambda\in\Lambda}|\langle f,\psi^M_{j,\lambda}\rangle|^2=\int_{\mathbb{R}}|\hat{f}(\frac{\xi}{B})|^2\sum_{j\in\mathbb{Z}}|\psi(\frac{\xi}{(2NB)^j})|^2 d\xi+R_0+R_1+\cdots+R_{2N-1},$$

which is desired result. □

We now establish our first sufficient condition for the nonuniform wavelet system $\mathcal{L}_{\mathcal{M}}(\psi,j,\lambda)$ to be a frame for $L^2(\mathbb{R})$. For this, we set

$$\Xi^M_\psi(m)=\operatorname{ess\,sup}\left\{\sum_{j\in\mathbb{Z}}\left|t^M_\psi\left(m,\frac{\xi}{(2NB)^j}\right)\right|:\xi\in[1,2NB]\right\}, \tag{3.4}$$

where

$$t^M_\psi\left(m,\frac{\xi}{B}\right)=\sum_{k\in\mathbb{N}_0}\hat{\psi}\Big((2N)^k\frac{\xi}{B}\Big)\overline{\hat{\psi}\left((2N)^k\Big(\frac{\xi}{B}+\frac{m}{2}\Big)\right)}. \tag{3.5}$$

and

$$\underline{\mathcal{S}}_{\psi^M}=\operatorname{ess\,inf}\left\{\mathcal{S}\left(\frac{\xi}{B}\right):\xi\in[1,2NB]\right\} \tag{3.6}$$

$$\overline{\mathcal{S}}_{\psi^M}=\operatorname{ess\,sup}\left\{\mathcal{S}\left(\frac{\xi}{B}\right):\xi\in[1,2NB]\right\} \tag{3.7}$$

where

$$\mathcal{S}\left(\frac{\xi}{B}\right)=\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\frac{\xi}{(2NB)^j}\right)\right|^2 \tag{3.8}$$

We also use the following set:

$$\Omega=\Big\{(2N)k+\ell:k\in\mathbb{N}_0,1\le\ell\le 2N-1\Big\}.$$

Analogous to the uniform case, we give the first sufficient condition as follows.

Theorem 3.2. Suppose $\psi^M\in L^2(\mathbb{R})$ such that

$$A^M_\psi=\underline{\mathcal{S}}_{\psi^M}-\sum_{\alpha\neq\beta\in\Omega}\left[\Xi^M_\psi\left(\frac{\alpha-\beta}{2}\right)\cdot\Xi^M_\psi\left(\frac{\beta-\alpha}{2}\right)\right]^{1/2}>0,$$

$$B^M_\psi=\overline{\mathcal{S}}_{\psi^M}+\sum_{\alpha\neq\beta\in\Omega}\left[\Xi^M_\psi\left(\frac{\alpha-\beta}{2}\right)\cdot\Xi^M_\psi\left(\frac{\beta-\alpha}{2}\right)\right]^{1/2}<\infty.$$

Then $\left\{\psi^M_{j,\lambda}:j\in\mathbb{Z},\lambda\in\Lambda\right\}$ is a frame for $L^2(\mathbb{R})$ with bounds A^M_ψ and B^M_ψ.

Proof . Since the last series in (3.2) is absolutely convergent for every $f\in\mathcal{D}$ a dense subset of $\mathbb{H}$, we can estimate $R_{\psi^M}(f)$ by rearranging the series, changing the orders of summation and integration by Levi Lemma so that we deduce that

$$\begin{aligned}
\left|R_\psi^M(f)\right| &\le \frac{1}{4N}\sum_{p=0}^{2N-1}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\overline{\hat{f}(\frac{\xi}{B})}\hat{\psi}\left(\frac{\xi}{(2NB)^j}\right)\right|\left\{\sum_{\ell\neq p}\left|\hat{f}\left(\frac{\xi}{B}+(2N)^j\left(\frac{\ell-p}{2}\right)\right)\right.\right.\\
&= \frac{1}{4N}\sum_{\alpha=0}^{2N-1}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\overline{\hat{f}(\frac{\xi}{B})}\right|\left\{\sum_{k\in\mathbb{N}_0}\sum_{\alpha\neq\beta\in\Omega}\left|\hat{\psi}\left(\frac{\xi}{(2NB)^j}\right)\right.\right.\\
&\qquad\times\hat{f}\left(\frac{\xi}{B}+(2N)^{j+k}\left(\frac{\alpha-\beta}{2}\right)\right)\left.\left.\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+(2N)^k\left(\frac{\alpha-\beta}{2}\right)\right)}\right|\right\}d\xi\\
&= \frac{1}{4N}\sum_{\alpha=0}^{2N-1}\int_{\mathbb{R}}\left|\overline{\hat{f}(\frac{\xi}{B})}\right|\left\{\sum_{k\in\mathbb{N}_0}\sum_{\alpha\neq\beta\in\Omega}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\frac{\xi}{(2NB)^{j-k}}\right)\right.\right.\\
&\qquad\times\hat{f}\left(\frac{\xi}{B}+(2N)^{j}\left(\frac{\alpha-\beta}{2}\right)\right)\left.\left.\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^{j-k}}+(2N)^k\left(\frac{\alpha-\beta}{2}\right)\right)}\right|\right\}d\xi\\
&= \frac{1}{4N}\sum_{\alpha=0}^{2N-1}\int_{\mathbb{R}}\left|\overline{\hat{f}(\frac{\xi}{B})}\right|\left\{\sum_{j\in\mathbb{Z}}\sum_{\alpha\neq\beta\in\Omega}\left|\hat{f}\left(\frac{\xi}{B}+(2N)^j\left(\frac{\alpha-\beta}{2}\right)\right)\right.\right.\\
&\qquad\times\sum_{k\in\mathbb{N}_0}\hat{\psi}\left(\frac{\xi}{(2NB)^{j-k}}\right)\left.\left.\overline{\hat{\psi}\left((2N)^k\left(\frac{\xi}{(2NB)^j}+\frac{\alpha-\beta}{2}\right)\right)}\right|\right\}d\xi\\
&= \frac{1}{4N}\sum_{\alpha=0}^{2N-1}\int_{\mathbb{R}}\left|\overline{\hat{f}(\frac{\xi}{B})}\right|\left\{\sum_{j\in\mathbb{Z}}\sum_{\alpha\neq\beta\in\Omega}\left|\hat{f}\left(\frac{\xi}{B}+(2N)^j\left(\frac{\alpha-\beta}{2}\right)\right)\right.\right.\\
&\qquad\times\left.\left.\left|t_\psi^M\left(\frac{\alpha-\beta}{2},\frac{\xi}{(2NB)^j}\right)\right|\right|\right\}d\xi\\
&= \frac{1}{4N}\sum_{\alpha=0}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{\alpha\neq\beta\in\Omega}\int_{\mathbb{R}}\left\{\left|\hat{f}(\frac{\xi}{B})\right|\left|t_\psi^M\left(\frac{\alpha-\beta}{2},\frac{\xi}{(2NB)^j}\right)\right|^{1/2}\right\}\\
&\qquad\times\left\{\left|\hat{f}\left(\frac{\xi}{B}+(2N)^j\left(\frac{\alpha-\beta}{2}\right)\right)\right|\left|t_\psi^M\left(\frac{\alpha-\beta}{2},\frac{\xi}{(2NB)^j}\right)\right|^{1/2}\right\}d\xi\\
&\le \frac{1}{4N}\sum_{\alpha=0}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{\alpha\neq\beta\in\Omega}\left\{\int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\left|t_\psi^M\left(\frac{\alpha-\beta}{2},\frac{\xi}{(2NB)^j}\right)\right|d\xi\right\}^{1/2}\\
&\qquad\times\left\{\int_{\mathbb{R}}\left|\hat{f}\left(\frac{\xi}{B}+(2N)^j\left(\frac{\alpha-\beta}{2}\right)\right)\right|^2\left|t_\psi^M\left(\frac{\alpha-\beta}{2},\frac{\xi}{(2NB)^j}\right)\right|d\xi\right\}^{1/2}\\
&\le \frac{1}{4N}\sum_{\alpha=0}^{2N-1}\sum_{\alpha\neq\beta\in\Omega}\left\{\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\left|t_\psi^M\left(\frac{\alpha-\beta}{2},\frac{\xi}{(2NB)^j}\right)\right|d\xi\right\}^{1/2}\\
&\qquad\times\left\{\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\hat{f}\left(\frac{\xi}{B}+(2N)^j\left(\frac{\alpha-\beta}{2}\right)\right)\right|^2\left|t_\psi^M\left(\frac{\alpha-\beta}{2},\frac{\xi}{(2NB)^j}\right)\right|d\xi\right\}^{1/2}
\end{aligned}$$

$$= \frac{1}{4N}\sum_{\alpha=0}^{2N-1}\sum_{\alpha\neq\beta\in\Omega}\left\{\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\left|t_{\psi}^{M}\left(\frac{\alpha-\beta}{2},\frac{\xi}{(2NB)^j}\right)\right|d\xi\right\}^{1/2}$$
$$\times\left\{\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\hat{f}(\frac{\omega}{B})\right|^2\left|t_{\psi}^{M}\left(-\frac{\alpha-\beta}{2},\frac{\omega}{(2NB)^j}\right)\right|d\xi\right\}^{1/2}$$
$$\leq \frac{1}{4N}\sum_{\alpha=0}^{2N-1}\sum_{\alpha\neq\beta\in\Omega}\left\{\int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\Xi_{\psi}^{M}\left(\frac{\alpha-\beta}{2}\right)d\xi\right\}^{1/2}$$
$$\times\left\{\int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\Xi_{\psi}^{M}\left(\frac{-(\alpha-\beta)}{2}\right)d\xi\right\}^{1/2}$$
$$= \frac{1}{4N}\sum_{\alpha=0}^{2N-1}\int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2 d\xi\sum_{\alpha\neq\beta\in\Omega}\left[\Xi_{\psi}^{M}\left(\frac{\alpha-\beta}{2}\right)\cdot\Xi_{\psi}^{M}\left(\frac{-(\alpha-\beta)}{2}\right)\right]^{1/2}.$$

Consequently, it follows from the expression (3.1) in Lemma 3.1 that

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^{M}\rangle\right|^2 \geq \int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\frac{\xi}{(2NB)^j}\right)\right|^2 - \sum_{\alpha\neq\beta\in\Omega}\left[\Xi_{\psi}^{M}\left(\frac{\alpha-\beta}{2}\right)\cdot\Xi_{\psi}^{M}\left(\frac{-(\alpha-\beta)}{2}\right)\right]^{1/2}\right\}d\xi, \tag{3.9}$$

and

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^{M}\rangle\right|^2 \leq \int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\frac{\xi}{(2NB)^j}\right)\right|^2 + \sum_{\alpha\neq\beta\in\Omega}\left[\Xi_{\psi}^{M}\left(\frac{\alpha-\beta}{2}\right)\cdot\Xi_{\psi}^{M}\left(\frac{-(\alpha-\beta)}{2}\right)\right]^{1/2}\right\}d\xi. \tag{3.10}$$

Taking infimum in (3.9) and supremum in (3.10), respectively, we obtain that

$$A_{\psi}^{M}\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^{M}\rangle\right|^2 \leq B_{\psi}^{M}\|f\|_2^2,$$

hold for all $f\in\mathcal{D}$. This completes the proof of Theorem. □

Before, we state our next sufficient condition, we introduce some notations. Similar to the a-adic number, we call an element $\alpha\in\mathbb{R}$, a $2N$-adic number if it has the form $\alpha=(2N)^j(\lambda-\sigma)/2$, $j\in\mathbb{Z}$, $\lambda\neq\sigma\in\Lambda$. With this concept, we consider the set

$$\Gamma=\left\{\alpha\in\mathbb{R}:\text{there exists }(j,\lambda)\in\mathbb{Z}\times\Lambda\text{ such that }\alpha=\frac{(2N)^j(\lambda-\sigma)}{2};\lambda\neq\sigma\right\},$$

and for all $\alpha\in\Gamma$, we define

$$I(\alpha) = \left\{(j,\lambda)\in\mathbb{Z}\times\Lambda:\alpha=\frac{(2N)^j(\lambda-\sigma)}{2}\right\}, \tag{3.11}$$

$$\Xi_{\alpha}^{+}(\frac{\xi}{B}) = \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)}\hat{\psi}\left(\frac{\xi}{(2NB)^j}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{\lambda-\sigma}{2}\right)}, \tag{3.12}$$

$$\Xi_{\alpha}^{-}(\frac{\xi}{B}) = \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)}\hat{\psi}\left(\frac{\xi}{(2NB)^j}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}-\frac{\lambda-\sigma}{2}\right)}. \tag{3.13}$$

With the notations above we state the following result.

Theorem 3.3. Suppose $\psi^M \in L^2(\mathbb{R})$ such that

$$C_\psi^M = ess \inf_{\xi\in[1,2NB]} \left\{ \Xi_0^+(\frac{\xi}{B}) - \sum_{\alpha\in\Gamma\backslash\{0\}} \left| \Xi_\alpha^+(\frac{\xi}{B}) \right| \right\} > 0,$$

$$D_\psi^M = ess \sup_{\xi\in[1,2NB]} \sum_{\alpha\in\Gamma\backslash\{0\}} \left| \Xi_\alpha^+(\frac{\xi}{B}) \right| < +\infty.$$

Then $\left\{\psi_{j,\lambda}^M : j \in \mathbb{Z},\, \lambda \in \Lambda\right\}$ is a wavelet frame for $L^2(\mathbb{R})$ with bounds C_ψ^M and D_ψ^M.

Proof . We first note that $\Xi_0^+(\frac{\xi}{B}) = \sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(\xi/(2NB)^j\right)\right|^2$ by the definition of $\Xi_\alpha^+(\frac{\xi}{B})$. We apply Lemma 3.1 to re-estimate $R_\psi^M(f)$ for $f \in \mathcal{D}$ as

$$\begin{aligned}
\left|R_\psi^M(f)\right| &= \Bigg| \frac{1}{4N} \sum_{p=0}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\ell\neq p} \int_{\mathbb{R}} \Bigg\{ \overline{\hat{f}\left(\xi + (2NB)^j \frac{p}{2}\right)} \hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{p}{2}\right) \hat{f}\left(\frac{\xi}{B} + (2N)^j \frac{\ell}{2}\right) \\
&\qquad \times \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{\ell}{2}\right)} \left(1 + e^{\pi i \frac{r}{N}(\ell - p)}\right) \Bigg\} d\xi \Bigg| \\
&\le \frac{1}{4N} \sum_{p=0}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\ell\neq p} \int_{\mathbb{R}} \left| \overline{\hat{f}\left(\frac{\xi}{B} + (2N)^j \frac{p}{2}\right)} \hat{f}\left(\frac{\xi}{B} + (2N)^j \frac{\ell}{2}\right) \right| \\
&\qquad \times \left| \hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{p}{2}\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{\ell}{2}\right)} \right| d\xi \\
&= \sum_{\alpha\in\Gamma\backslash\{0\}} \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \int_{\mathbb{R}} \left| \overline{\hat{f}\left(\frac{\xi}{B} + (2N)^j \frac{\lambda}{2}\right)} \hat{f}\left(\frac{\xi}{B} + (2N)^j \frac{\sigma}{2}\right) \right| \\
&\qquad \times \left| \hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{\lambda}{2}\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{\sigma}{2}\right)} \right| d\xi \\
&\le \sum_{\alpha\in\Gamma\backslash\{0\}} \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \int_{\mathbb{R}} \left| \overline{\hat{f}(\frac{\xi}{B})} \hat{f}\left(\frac{\xi}{B} + (2N)^j \left(\frac{\lambda - \sigma}{2}\right)\right) \right| \\
&\qquad \times \left| \hat{\psi}\left(\frac{\xi}{(2NB)^j}\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{\lambda - \sigma}{2}\right)} \right| d\xi \\
&= \sum_{\alpha\in\Gamma\backslash\{0\}} \int_{\mathbb{R}} \left| \overline{\hat{f}(\frac{\xi}{B})} \hat{f}\left(\frac{\xi}{B} + \frac{\alpha}{2}\right) \right| \left\{ \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \left| \hat{\psi}\left(\frac{\xi}{(2NB)^j}\right) \overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j} + \frac{\lambda - \sigma}{2}\right)} \right| \right\} d\xi \\
&= \sum_{\alpha\in\Gamma\backslash\{0\}} \int_{\mathbb{R}} \left| \overline{\hat{f}(\frac{\xi}{B})} \hat{f}\left(\frac{\xi}{B} + \frac{\alpha}{2}\right) \right| \left| \Xi_\alpha^+(\frac{\xi}{B}) \right| d\xi \qquad \text{(By Eq. (3.12))} \\
&\le \sum_{\alpha\in\Gamma\backslash\{0\}} \left\{ \int_{\mathbb{R}} \left| \hat{f}(\frac{\xi}{B}) \right|^2 \left| \Xi_\alpha^+(\frac{\xi}{B}) \right| d\xi \right\}^{1/2} \left\{ \int_{\mathbb{R}} \left| \hat{f}\left(\frac{\xi}{B} + \frac{\alpha}{2}\right) \right|^2 \left| \Xi_\alpha^+(\frac{\xi}{B}) \right| d\xi \right\}^{1/2} \\
&\le \left\{ \sum_{\alpha\in\Gamma\backslash\{0\}} \int_{\mathbb{R}} \left| \hat{f}(\frac{\xi}{B}) \right|^2 \left| \Xi_\alpha^+(\frac{\xi}{B}) \right| d\xi \right\}^{1/2} \left\{ \sum_{\alpha\in\Gamma\backslash\{0\}} \int_{\mathbb{R}} \left| \hat{f}\left(\frac{\xi}{B} + \frac{\alpha}{2}\right) \right|^2 \left| \Xi_\alpha^+(\frac{\xi}{B}) \right| d\xi \right\}^{1/2}. \qquad (3.14)
\end{aligned}$$

Let $\frac{\omega}{B}=\frac{\xi}{B}+\alpha/2$. We deduce from $\alpha=(2N)^j(\lambda-\sigma)$ for $(j,\lambda)\neq(j,\sigma)\in I(\alpha)$ that

$$\begin{aligned}\Xi_\alpha^+(\frac{\xi}{B}) &= \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \hat{\psi}\left(\frac{\xi}{(2NB)^j}\right)\overline{\hat{\psi}\left(\frac{\xi}{(2NB)^j}+\frac{\lambda-\sigma}{2}\right)} \\ &= \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \hat{\psi}\left((2NB)^{-j}\left(\omega-\frac{\alpha}{2}\right)\right)\overline{\hat{\psi}\left((2NB)^{-j}\left(\omega-\frac{\alpha}{2}\right)+\frac{\lambda-\sigma}{2}\right)} \\ &= \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \hat{\psi}\left(\frac{\omega}{(2NB)^j}-(2N)^{-j}\frac{\alpha}{2}\right)\overline{\hat{\psi}\left(\frac{\omega}{(2NB)^j}-(2N)^{-j}\frac{\alpha}{2}+\frac{\lambda-\sigma}{2}\right)} \\ &= \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \hat{\psi}\left(\frac{\omega}{(2NB)^j}-\frac{\lambda-\sigma}{2}\right)\overline{\hat{\psi}\left(\frac{\omega}{(2NB)^j}\right)} \\ &= \overline{\Xi_\alpha^-(\frac{\omega}{B})} \qquad \text{(By Eq. (3.13).}\end{aligned}$$

Therefore

$$\sum_{\alpha\in\Gamma\setminus\{0\}}\left|\Xi_\alpha^+(\frac{\omega}{B})\right| = \sum_{\alpha\in\Gamma\setminus\{0\}}\left|\Xi_\alpha^-(\frac{\omega}{B})\right|. \tag{3.15}$$

Replacing $\xi+\alpha/2$ by ω in the last integration of (3.14), we derive from (3.14) and (3.15) that

$$\begin{aligned}\left|R_\psi^M(f)\right| &\leq \left\{\sum_{\alpha\in\Gamma\setminus\{0\}}\int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\left|\Xi_\alpha^+(\frac{\xi}{B})\right|d\xi\right\}^{1/2}\left\{\sum_{\alpha\in\Gamma\setminus\{0\}}\int_{\mathbb{R}}\left|\hat{f}(\frac{\omega}{B})\right|^2\left|\overline{\Xi_\alpha^-(\frac{\omega}{B})}\right|d\omega\right\}^{1/2} \\ &= \int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\left\{\sum_{\alpha\in\Gamma\setminus\{0\}}\left|\Xi_\alpha^+(\frac{\xi}{B})\right|\right\}d\xi. \end{aligned}\tag{3.16}$$

Therefore, by (3.16) and (3.1), we have

$$\int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\left\{\Xi_0^+(\frac{\xi}{B})-\sum_{\alpha\in\Gamma\setminus\{0\}}\left|\Xi_\alpha^+(\frac{\xi}{B})\right|\right\}d\xi \leq \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^M\rangle\right|^2, \tag{3.17}$$

and

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^M\rangle\right|^2 \leq \int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\left\{\Xi_0^+(\frac{\xi}{B})+\sum_{\alpha\in\Gamma\setminus\{0\}}\left|\Xi_\alpha^+(\frac{\xi}{B})\right|\right\}d\xi,$$

or, equivalently

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^M\rangle\right|^2 \leq \int_{\mathbb{R}}\left|\hat{f}(\frac{\xi}{B})\right|^2\left\{\sum_{\alpha\in\Gamma}\left|\Xi_\alpha^+(\frac{\xi}{B})\right|\right\}d\xi. \tag{3.18}$$

Taking infimum in (3.17) and supremum in (3.18), respectively, we obtain again that

$$C_{\psi^M}\left\|f\right\|_2^2 \leq \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}^M\rangle\right|^2 \leq D_{\psi^M}\left\|f\right\|_2^2.$$

The proof of Theorem 3.3 is complete.□

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