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New refinements for integral form of Jensen's and Holder's inequalities and related results

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Abstract

In this paper we establish two new refinements for integral forms of Jensen's and Holder's inequalites. Several applications are given on special means.

Keywords: Jensen's inequality, Holder's inequality, Integral inequality
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1 Introduction

Let μ be a positive measure on X such that $\mu(X) = 1$. If h is a real-valued function in $L^1(\mu), a < f(x) < b$ for all $x \in X$ and φ is convex on (a, b), then

$$\varphi(\int_X h d\mu) \leq \int_X (\varphi \circ f) d\varphi \tag{1.1}$$

The inequality (1.1) is known as Jensen's inequality. Another verstion of Jensen's inequality is the following form

$$\varphi(\frac{\int_a^b p(t)h(t)dt}{\int_a^b p(t)dt}) \leq \frac{1}{\int_a^b p(t)dt} \int_a^b p(t)\varphi(h(t))dt \tag{1.2}$$

where p is a non-negative function on $[a, b]$ such that $\int_a^b p(t)dt > 0$, see [1, 9, 14].

Let $\varphi : [a, b] \to \mathbb{R}$ be a convex function, then the inequality

$$\varphi(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b \varphi(x)dx \leq \frac{\varphi(a) + \varphi(b)}{2} \tag{1.3}$$

is known as Hermite-Hadamard inequality (H-H inequality). It is well known that Jensen's, Holder's and H-H inequalities play an important role in non-linear analysis. In recent years there have been many extentions, generalizations and refinements of these inequalities, see [1, 2, 4, 5, 6, 7, 8, 9, 14] and the references therein.

In this paper we establish two refinements of Jensen's, Holder's and H-H inequalities via a partition of $[a, b]$, identity

$$\sum_{k=0}^{m} \binom{m}{k} (\frac{x-a}{b-a})^k (\frac{b-x}{b-a})^{m-k} = 1$$

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and Beta integral

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad (x,y>0)$$

Then we apply these inequalities on special means.

2 Main results

Theorem 2.1. Let h be a real-valued function on $[a,b]$ and $m\le h(x)\le M$ for all $x\in[a,b]$. If φ be a convex function on $[m,M]$ and $h\in L^1[a,b]$, then the following inequalities hold

(i) $$\varphi(\frac{1}{b-a}\int_a^b h(x)dx)\le\frac{1}{n}\sum_{i=1}^{n}\varphi(\frac{n}{b-a}\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}h(x)dx)\le\frac{1}{b-a}\int_a^b(\varphi\circ h)(x)dx$$

(ii)

$$\begin{aligned}\varphi(\frac{1}{b-a}\int_a^b h(x)dx)&\le\frac{1}{m+1}\sum_{k=0}^{m}\varphi(\frac{\Gamma(m+2)}{\Gamma(k+1)\Gamma(m-k+1)}\int_0^1 t^k(1-t)^{m-k}h(a+t(b-a))dt)\\&=\frac{1}{m+1}\sum_{k=0}^{m}\varphi(\frac{\Gamma(m+2)}{\Gamma(k+1)\Gamma(m-k+1)(b-a)^{m+1}}\int_a^b(x-a)^k(b-x)^{m-k}h(x)dx)\\&\le\frac{1}{b-a}\int_a^b(\varphi\circ h)(x)dx\end{aligned}$$

Proof .

(i) By the convexity of φ and Jensen's inequality we have

$$\begin{aligned}\varphi(\frac{1}{b-a}\int_a^b h(x)dx)&=\varphi(\sum_{i=1}^{n}\frac{1}{n}\cdot\frac{n}{b-a}\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}h(x)dx)\\&\le\frac{1}{n}\sum_{i=1}^{n}\varphi(\frac{n}{b-a}\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}h(x)dx)\\&\le\frac{1}{n}\sum_{i=1}^{n}\frac{n}{b-a}\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}(\varphi\circ h)(x)dx\\&=\frac{1}{b-a}\int_a^b(\varphi\circ h)(x)dx\end{aligned}$$

(ii) Since φ is convex and $\sum_{k=0}^{m}\binom{m}{k}(\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{m-k}=1$, we have

$$\begin{aligned}\varphi(\frac{1}{b-a}\int_a^b h(x)dx)&=\varphi(\frac{1}{b-a}\int_a^b\sum_{k=0}^{m}\binom{m}{k}(\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{m-k}h(x)dx)\\&=\varphi(\sum_{k=0}^{m}\binom{m}{k}\int_a^b(\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{m-k}h(x)\frac{dx}{b-a})\end{aligned}$$

By change of variable $t = \dfrac{x-a}{b-a}$, $dt = \dfrac{dx}{b-a}$ we obtain

$$
\begin{aligned}
&= \varphi(\sum_{k=0}^{m}\binom{m}{k}\int_0^1 t^k(1-t)^{m-k}h(a+t(b-a))dt) \\
&= \varphi(\sum_{k=0}^{m}\binom{m}{k}\int_0^1 t^k(1-t)^{m-k}dt\frac{\int_0^1 t^k(1-t)^{m-k}h(a+t(b-a))dt}{\int_0^1 t^k(1-t)^{m-k}dt})
\end{aligned}
$$

Since $\sum_{k=0}^{m}\binom{m}{k}\int_0^1 t^k(1-t)^{m-k}dt = \sum_{k=0}^{m}\binom{m}{k}\frac{k!(m-k)!}{(m+1)} = 1$, by the convexity of φ we get

$$
\begin{aligned}
&\leq \sum_{k=0}^{m}\binom{m}{k}\int_0^1 t^k(1-t)^{m-k}dt\,\varphi(\frac{\int_0^1 t^k(1-t)^{m-k}h(a+t(b-a))dt}{\int_0^1 t^k(1-t)^{m-k}dt}) \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\varphi(\frac{\int_0^1 t^k(1-t)^{m-k}h(a+t(b-a))dt}{\int_0^1 t^k(1-t)^{m-k}dt})
\end{aligned}
$$

Again by the convenity of φ and inequality 1.2 we deduce that

$$
\begin{aligned}
&\leq \frac{1}{m+1}\sum_{k=0}^{m}\frac{\int_0^1 t^k(1-t)^{m-k}(\varphi\circ h)(a+t(b-a))dt}{\int_0^1 t^k(1-t)^{m-k}dt} \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\frac{(m+1)!}{k!(m-k)!}\int_0^1 t^k(1-t)^{m-k}(\varphi\circ h)(a+t(b-a))dt \\
&= \sum_{k=0}^{m}\binom{m}{k}\int_0^1 t^k(1-t)^{m-k}(\varphi\circ h)(a+t(b-a))dt \\
&= \int_0^1\sum_{k=0}^{m}\binom{m}{k}t^k(1-t)^{m-k}(\varphi\circ h)(a+t(b-a))dt \\
&= \frac{1}{b-a}\int_a^b(\varphi\circ h)(x)dx
\end{aligned}
$$

Because $\sum_{k=0}^{m}\binom{m}{k}t^k(1-t)^{m-k} = 1$. Since

$$
\begin{aligned}
&\varphi(\frac{\int_0^1 t^k(1-t)^{m-k}h(a+t(b-a))dt}{\int_0^1 t^k(1-t)^{m-k}dt}) = \varphi(\frac{\int_0^1 t^k(1-t)^{m-k}h(a+t(b-a))dt}{B(k+1,m-k+1)}) \\
&= \varphi(\frac{\Gamma(m+2)}{\Gamma(k+1)\Gamma(m-k+1)}\int_0^1 t^k(1-t)^{m-k}h(a+t(b-a))dt \\
&= \varphi(\frac{\Gamma(m+2)}{\Gamma(k+1)\Gamma(m-k+1)(b-a)}\int_0^1(\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{m-k}dx) \\
&= \varphi(\frac{\Gamma(m+2)}{\Gamma(k+1)\Gamma(m-k+1)(b-a)^{m+1}}\int_0^1(x-a)^k(b-x)^{m-k}dx)
\end{aligned}
$$

The proof is complete.

□

Corollary 2.2. With the assumption of theorem 2.1 the following inequalities hold

$$\varphi(\frac{1}{b-a}\int_a^b h(x)dx) \leq \frac{1}{n}\sum_{i=1}^{n}\varphi(\frac{n}{b-a}\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)} h(x)dx)$$
$$\leq \frac{1}{n(m+1)}\sum_{i=1}^{n}\sum_{k=0}^{m}\varphi(\frac{n\Gamma(m+2)}{\Gamma(k+1)\Gamma(m-k+1)(b-a)^{m+1}}$$
$$\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}(nx-na-(i-1)(b-a))^k(na+i(b-a)-nx)^{m-k}h(x)dx)$$
$$\leq \frac{1}{b-a}\int_a^b(\varphi\circ h)(x)$$

Proof . By using the theorem 2.1 (ii) we have

$$\varphi(\frac{n}{b-a}\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)} h(x)dx) \leq \frac{1}{m+1}\sum_{k=0}^{m}\varphi(\frac{n\Gamma(m+2)}{(b-a)^m\Gamma(k+1)\Gamma(m-k+1)}$$
$$\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}(nx-na-(i-1)(b-a))^k(na+i(b-a)-nx)^{m-k}h(x)dx)$$
$$\leq \frac{n}{b-a}\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}(\varphi\circ h)(x)dx$$

The rest of assertion is obvious by theorem 2.1 (i) □

In the following theorem we obtain a new refinements of Hermite-Hadmard inequality.

Theorem 2.3. Let φ be a convex function on $[a,b]$. Then the following inequalites hold

$$\varphi(\frac{a+b}{2}) \leq \frac{1}{n}\sum_{i=1}^{n}\varphi(a+\frac{b-a}{n}(i-\frac{1}{2}))$$
$$\leq \frac{1}{n(m+1)}\sum_{i=1}^{n}\sum_{k=0}^{m}\varphi(a+\frac{b-a}{n}(i-1+\frac{k+1}{m+2}))$$
$$\leq \frac{1}{b-a}\int_a^b\varphi(x)dx$$

Proof . By putting $h(x)=x$ in Corollary 2.2 we have

$$\varphi(\frac{1}{b-a}\int_a^b xdx) \leq \frac{1}{n}\sum_{i=1}^{n}\varphi(\frac{n}{b-a}\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)} xdx)$$
$$\leq \frac{1}{n(m+1)}\sum_{i=1}^{n}\sum_{k=0}^{m}\varphi(\frac{n\Gamma(m+2)}{(b-a)^{m+1}\Gamma(k+1)\Gamma(m-k+1)}$$
$$\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}(nx-na-(i-1)(b-a))^k(na+i(b-a)-nx)^{m-k}xdx)$$
$$\leq \frac{1}{b-a}\int_a^b\varphi(x)dx$$

By change of variable $\dfrac{nx-na-(i-1)(b-a)}{b-a}=t$, $\dfrac{ndx}{b-a}=dt$ and Beta integral we get

$$\begin{aligned}
&\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}(nx-na-(i-1)(b-a))^k(na+i(b-a)-nx)^{m-k}xdx\\
&=\frac{(b-a)^{m+1}}{n}\int_0^1 t^k(1-t)^{m-k}(a+\frac{b-a}{n}(i-1)+\frac{b-a}{n}t)dt\\
&=\frac{(b-a)^{m+1}}{n}[(a+\frac{b-a}{n}(i-1))\int_0^1 t^k(1-t)^{m-k}dt+\frac{b-a}{n}\int_0^1 t^{k+1}(1-t)^{m-k}dt]\\
&=\frac{(b-a)^{m+1}}{n}[(a+\frac{b-a}{n}(i-1))\frac{k!(m-k)!}{(m+1)!}+\frac{b-a}{n}\frac{(k+1)!(m-k)!}{(m+2)!}]\\
&=\frac{(b-a)^{m+1}}{n}[\frac{k!(m-k)!}{(m+1)!}(a+\frac{b-a}{n}(i-1)+\frac{b-a}{n}\frac{k+1}{m+2})]\\
&=\frac{(b-a)^{m+1}}{n}\cdot\frac{\Gamma(k+1)\Gamma(m-k+1)}{\Gamma(m+2)}(a+\frac{b-a}{n}(i-1+\frac{k+1}{m+2}))
\end{aligned}$$

Hence

$$\begin{aligned}
\varphi(\frac{a+b}{2})&\le\frac{1}{n}\sum_{i=1}^{n}\varphi(a+\frac{b-a}{2}(i-\frac{1}{2}))\\
&\le\frac{1}{n(m+1)}\sum_{i=1}^{n}\sum_{k=0}^{m}\varphi(a+\frac{b-a}{n}(i-1+\frac{k+1}{m+2}))\\
&\le\frac{1}{b-a}\int_a^b\varphi(x)dx
\end{aligned}$$

□

In the following theorem we establish a new refinements of Holder's inequality.

Theorem 2.4. Let $p,q>1$ be such that $\dfrac{1}{p}+\dfrac{1}{q}=1$.

If f and g be non-negative functions such that $f\in L^p[a,b]$ and $g\in L^q[a,b]$, then

(i) $\|fg\|_1\le\frac{1}{2}n^{\frac{1}{q}}[\sum_{i=1}^{n}(\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}fgdt)^p]^{\frac{1}{p}}+\dfrac{1}{2}n^{\frac{1}{p}}[\sum_{i=1}^{n}(\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}fgdt)^q]^{\frac{1}{q}}\le\|f\|_p\|g\|_q$

(ii) $\|fg\|_1\le\dfrac{(m+1)^{\frac{1}{p}}}{2(b-a)^{m+1}}[\sum_{k=0}^{m}\binom{m}{k}^q(\int_a^b I(t)dt)^q]^{\frac{1}{q}}+\dfrac{(m+1)^{\frac{1}{q}}}{2(b-a)^{m+1}}[\sum_{k=0}^{m}\binom{m}{k}^p(\int_a^b I(t)dt)^p]^{\frac{1}{p}}$
$\le\|f\|_p\|g\|_q$, where $I(t)=(t-a)^k(b-t)^{m-k}fg$.

Proof . The inequalities is trivial if either, $f=0$ a.e. or $g=0$ a.e. So assume that $f>0$ a.e. and $g>0$ a.e. This gives that $\|f\|_p>0$ and $\|g\|_q>0$. Since $\varphi(x)=x^p$ $(p>1)$ is convex on $[a,b]$ $(b>a>0)$, by theorem 2.1 (i) we have

$$(\frac{1}{b-a}\int_a^b h(x)dx)^p\le\frac{1}{n}\sum_{i=1}^{n}(\frac{n}{b-a}\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}h(x)dx)^p\le\frac{1}{b-a}\int_a^b h^p(x)dx$$

$$\Rightarrow(\frac{1}{b-a}\int_a^b h(x)dx)^p\le\frac{n^{p-1}}{(b-a)^p}\sum_{i=1}^{n}(\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}h(x)dx)^p\le\frac{1}{b-a}\int_a^b h^p(x)dx$$

Put $h = fg^{1-q}$ and $dx = \dfrac{g^p(b-a)}{\int_a^b g^q dt}dt$, then $hdx = \dfrac{(b-a)fg}{\int_a^b g^q dt}dt$ and $h^p dx = \dfrac{(b-a)f^p}{\int_a^b g^q dt}dt$. So

$$\frac{(\int_a^b fgdt)^p}{(\int_a^b g^q dt)^p} \le \frac{n^{p-1}}{(b-a)^p}\sum_{i=1}^{n}\frac{1}{(\int_a^b g^q dt)^p}(\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)}(b-a)fgdt)^p \le \frac{1}{b-a}\frac{\int_a^b (b-a)f^p dt}{\int_a^b g^q dt}$$

Multiplying both sides by $(\int_a^b g^q dt)^p > 0$, we get

$$(\int_a^b fgdt)^p \le n^{p-1}\sum_{i=1}^{n}(\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)} fgdt)^p \le (\int_a^b f^p dt)(\int_a^b g^q dt)^{p-1}$$

Since $\frac{1}{p}+\frac{1}{q}=1$, it follows that

$$\int_a^b fgdt \le n^{\frac{1}{q}}[\sum_{i=1}^{n}(\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)} fgdt)^p]^{\frac{1}{p}} \le (\int_a^b f^p dt)^{\frac{1}{p}}(\int_a^b g^q dt)^{\frac{1}{q}}$$

$$\Rightarrow \|fg\|_1 \le n^{\frac{1}{q}}[\sum_{i=1}^{n}(\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)} fgdt)^p]^{\frac{1}{p}} \le \|f\|_p\|g\|_q \qquad (2.1)$$

By the similar way we obtain

$$\|fg\|_1 \le n^{\frac{1}{p}}[\sum_{i=1}^{n}(\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)} fgdt)^q]^{\frac{1}{q}} \le \|f\|_p\|g\|_q \qquad (2.2)$$

Finally by (2.1) and (2.2) we deduce that

$$\|fg\|_1 \le \frac{1}{2}n^{\frac{1}{q}}[\sum_{i=1}^{n}(\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)} fgdt)^p]^{\frac{1}{p}} + \frac{1}{2}n^{\frac{1}{p}}[\sum_{i=1}^{n}(\int_{a+\frac{i-1}{n}(b-a)}^{a+\frac{i}{n}(b-a)} fgdt)^q]^{\frac{1}{q}} \le \|f\|_p\|g\|_q$$

The proof of (i) is complete.

For the proof of (ii) by the convexity of $\varphi(x) = x^p$ $(p>1)$ and theorem 2.1 (ii) we have

$$(\frac{1}{b-a}\int_a^b h(x)dx)^p \le \frac{1}{m+1}\sum_{k=0}^{m}\frac{\Gamma^p(m+2)}{(b-a)^{p(m+1)}\Gamma^p(k+1)\Gamma^p(m-k+1)}$$
$$(\int_a^b (x-a)^k(b-x)^{m-k}h(x)dx)^p \le \frac{1}{b-a}\int_a^b h^p(x)dx$$

By the similar way and putting $h = fg^{1-q}$ and $dx = \dfrac{g^p(b-a)}{\int_a^b g^q dt}dt$ we get

$$\frac{(\int_a^b fgdt)^p}{(\int_a^b g^q dt)^p} \le \frac{(m+1)^{p-1}}{(b-a)^{p(m+1)}}\sum_{k=0}^{m}\frac{1}{(\int_a^b g^q dt)^p}\binom{m}{k}^p(\int_a^b (t-a)^k(b-t)^{m-k}fgdt)^p \le \frac{\int_a^b f^p dt}{\int_a^b g^q dt}$$

$$\Rightarrow (\int_a^b fgdt)^p \le \frac{(m+1)^{p-1}}{(b-a)^{p(m+1)}}\sum_{k=0}^{m}\binom{m}{k}^p(\int_a^b (t-a)^k(b-t)^{m-k}fgdt)^p \le (\int_a^b f^p dt)(\int_a^b g^q dt)^{p-1}$$

$$\Rightarrow \|fg\|_1 \le \frac{(m+1)^{\frac{1}{q}}}{(b-a)^{m+1}}[\sum_{k=0}^{m}\binom{m}{k}^p(\int_a^b (t-a)^k(b-t)^{m-k}fgdt)^p]^{\frac{1}{p}} \le \|p\|_p\|g\|_q \qquad (2.3)$$

By the same way we obtain

$$\|fg\|_1 \le \frac{(m+1)^{\frac{1}{p}}}{(b-a)^{m+1}}[\sum_{k=0}^{m}\binom{m}{k}^q(\int_a^b (t-a)^k(b-t)^{m-k}fgdt)^q]^{\frac{1}{q}} \le \|f\|_p\|g\|_q \qquad (2.4)$$

Finally by (2.3) and (2.4) we get (ii) □

3 Application on means

Theorem 3.1. Let $b > a > 0$ and $m, n \in \mathbb{N}$, then the following inequalities hold

$$\sqrt{ab} \leq \frac{\sqrt[2n]{ab}(a-b)}{n(\sqrt[n]{a} - \sqrt[n]{b})} \leq \frac{1}{n(m+1)} \cdot \frac{a^{\frac{m+1}{n(m+2)}} - b^{\frac{m+1}{n(m+2)}}}{a^{\frac{1}{n(m+2)}} - b^{\frac{1}{n(m+2)}}} \leq \frac{b-a}{\ln b - \ln a}$$

Proof . since $\varphi(x) = e^x$ is convex on $\mathbb{R}$, for $d > c > 0$, $m, n \in \mathbb{N}$ by using theorem 2.3 we have

$$\begin{aligned} e^{\frac{c+d}{2}} &\leq \frac{1}{n}\sum_{i=1}^{n} e^{c+\frac{d-c}{n}(i-\frac{1}{2})} \\ &\leq \frac{1}{n(m+1)}\sum_{i=1}^{n}\sum_{k=0}^{m} e^{c+\frac{d-c}{n}(i-1+\frac{k+1}{m+2})} \leq \frac{1}{d-c}\int_c^d e^x dx \end{aligned} \tag{3.1}$$

By easy calculations we see that

$$\sum_{i=1}^{n} e^{c+\frac{d-c}{n}(i-\frac{1}{2})} = e^{c-\frac{d-c}{2n}} \sum_{i=1}^{n} e^{\frac{d-c}{n}i} = e^{\frac{c+d}{2n}} (\frac{e^c - e^d}{e^{\frac{c}{n}} - e^{\frac{d}{n}}})$$

and

$$\begin{aligned} \sum_{i=1}^{n}\sum_{k=0}^{m} e^{c+\frac{d-c}{n}(i-1+\frac{k+1}{m+2})} &= e^{c-\frac{d-c}{n}+\frac{d-c}{n(m+2)}} \sum_{i=1}^{n} e^{\frac{d-c}{n}i} \sum_{k=0}^{m} e^{\frac{d-c}{n(m+2)}k} \\ &= e^{\frac{c+d}{n(m+2)}} \cdot \frac{e^c - e^d}{e^{\frac{c}{n}} - e^{\frac{d}{n}}} \cdot \frac{e^{\frac{c(m+1)}{n(m+2)}} - e^{\frac{d(m+1)}{n(m+2)}}}{e^{\frac{c}{n(m+2)}} - e^{\frac{d}{n(m+2)}}} \end{aligned}$$

Put $e^d = b$ and $e^c = a$, then (3.1) follows that

$$\sqrt{ab} \leq \frac{\sqrt[2n]{ab}(a-b)}{n(\sqrt[n]{a} - \sqrt[n]{b})} \leq \frac{1}{n(m+1)} \cdot \frac{a^{\frac{m+1}{n(m+2)}} - b^{\frac{m+1}{n(m+2)}}}{a^{\frac{1}{n(m+2)}} - b^{\frac{1}{n(m+2)}}} \leq \frac{b-a}{\ln b - \ln a}$$

□

Theorem 3.2. Let $b > a > 0$, $n \in \mathbb{N}$ and $p \in (1, \infty)$, then the following inequalities hold

$$\frac{b-a}{\ln b - \ln a} \leq \frac{n^{\frac{1}{q}}(b^{\frac{1}{n}} - a^{\frac{1}{n}})(b^p - a^p)^{\frac{1}{p}}}{(\ln b - \ln a)(b^{\frac{p}{n}} - a^{\frac{p}{n}})^{\frac{1}{p}}} \leq \frac{(b^p - a^p)^{\frac{1}{p}}}{p^{\frac{1}{p}}(\ln b - \ln a)^{\frac{1}{p}}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof . By putting $h(x) = e^x$ in theorem 2.1 (i) we have

$$\begin{aligned} &\varphi(\frac{1}{d-c}\int_c^d e^x dx) \leq \frac{1}{n}\sum_{i=1}^{n}\varphi(\frac{n}{d-c}\int_{c+\frac{i-1}{n}(d-c)}^{a+\frac{i}{n}(d-c)} e^x dx) \leq \frac{1}{d-c}\int_c^d \varphi(e^x)dx \\ &\Rightarrow \varphi(\frac{e^d - e^c}{d-c}) \leq \frac{1}{n}\sum_{i=1}^{n}\varphi(\frac{n}{d-c}(e^{c+\frac{i}{n}(d-c)} - e^{c+\frac{i-1}{n}(d-c)})) \leq \frac{1}{d-c}\int_c^d \varphi(e^x)dx \end{aligned}$$

Since $\varphi(x) = x^p$ $(p > 1)$ is Convex on $[c, d]$ $(d > c > 0)$, It follows that

$$(\frac{e^d - e^c}{d-c})^p \leq \frac{n^{p-1}}{(d-c)^p}\sum_{i=1}^{n}(e^{c+\frac{i}{n}(d-c)} - e^{c+\frac{i-1}{n}(d-c)})^p \leq \frac{e^{pd} - e^{pc}}{p(d-c)}$$

Put $e^d = b$ and $e^c = a$ then we get

$$(\frac{b-a}{\ln b - \ln a})^p \leq \frac{n^{p-1}}{(\ln b - \ln a)^p}\sum_{i=1}^{n}(a^{1-\frac{i}{n}}b^{\frac{i}{n}} - a^{1-\frac{i-1}{n}}b^{\frac{i-1}{n}})^p \leq \frac{b^p - a^p}{p(\ln b - \ln a)} \tag{3.2}$$

By easy calculation we see that

$$\sum_{i=1}^{n}(a^{1-\frac{i}{n}}b^{\frac{i}{n}}-a^{1-\frac{i-1}{n}}b^{\frac{i-1}{n}})^p=\sum_{i=1}^{n}[(\frac{b}{a})^{\frac{i-\frac{1}{2}}{n}}(a^{1-\frac{1}{2n}}b^{\frac{1}{2n}}-a^{1+\frac{1}{2n}}b^{-\frac{1}{2n}})]^p$$
$$=[a^{1-\frac{1}{2n}}b^{\frac{1}{2n}}-a^{1+\frac{1}{2n}}b^{-\frac{1}{2n}}]^p\sum_{i=1}^{n}(\frac{b}{a})^{\frac{p(i-\frac{1}{2})}{n}}$$
$$=[a^{1-\frac{1}{2n}}b^{\frac{1}{2n}}-a^{1+\frac{1}{2n}}b^{-\frac{1}{2n}}]^p(\frac{b}{a})^{-\frac{p}{2n}}\sum_{i=1}^{n}(\frac{b}{a})^{\frac{pi}{n}}$$
$$=[a^{1-\frac{1}{2n}}b^{\frac{1}{2n}}-a^{1+\frac{1}{2n}}b^{-\frac{1}{2n}}]^p(\frac{b}{a})^{-\frac{p}{2n}}\frac{1-(\frac{b}{a})^p}{1-(\frac{b}{a})^{\frac{p}{n}}}\cdot(\frac{b}{a})^{\frac{p}{n}}$$
$$=[a^{1-\frac{1}{2n}}b^{\frac{1}{2n}}-a^{1+\frac{1}{2n}}b^{-\frac{1}{2n}}]^p b^{\frac{p}{2n}}\cdot a^{p(\frac{1}{2n}-1)}(\frac{a^p-b^p}{a^{\frac{p}{n}}-b^{\frac{p}{n}}})$$
$$=(b^{\frac{1}{n}}-a^{\frac{1}{n}})^p(\frac{a^p-b^p}{a^{\frac{p}{n}}-b^{\frac{p}{n}}})$$

Hence (3.2) becomes

$$(\frac{b-a}{\ln b-\ln a})^p\le\frac{n^{p-1}(b^{\frac{1}{n}}-a^{\frac{1}{n}})^p(b^p-a^p)}{(\ln b-\ln a)^p(b^{\frac{p}{n}}-a^{\frac{p}{n}})}\le\frac{b^p-a^p}{p(\ln b-\ln a)}$$
$$\Rightarrow\frac{b-a}{\ln b-\ln a}\le\frac{n^{\frac{1}{q}}(b^{\frac{1}{n}}-a^{\frac{1}{n}})(b^p-a^p)^{\frac{1}{p}}}{(\ln b-\ln a)(b^{\frac{p}{n}}-a^{\frac{p}{n}})^{\frac{1}{p}}}\le\frac{(b^p-a^p)^{\frac{1}{p}}}{p^{\frac{1}{p}}(\ln b-\ln a)^{\frac{1}{p}}}$$

□

Corollary 3.3. Let $b>a>0$, $m,n\in\mathbb{N}$ and $p\in(1,\infty)$, then

$$\sqrt{ab}\le\frac{\sqrt[2n]{ab}(a-b)}{n(\sqrt[n]{a}-\sqrt[n]{b})}\le\frac{a^{\frac{m+1}{n(m+2)}}-b^{\frac{m+1}{n(m+2)}}}{a^{\frac{1}{n(m+2)}}-b^{\frac{1}{n(m+2)}}}$$
$$\le\frac{b-a}{\ln b-\ln a}$$
$$\le\frac{n^{\frac{1}{q}}(b^{\frac{1}{n}}-a^{\frac{1}{n}})(b^p-a^p)^{\frac{1}{p}}}{(\ln b-\ln a)(b^{\frac{p}{n}}-a^{\frac{p}{n}})^{\frac{1}{p}}}$$
$$\le(\frac{b-a}{p(\ln b-\ln a)})^{\frac{1}{p}}$$

and with means notations

$$G(a,b)\le\frac{\sqrt[2n]{ab}(a-b)}{n(\sqrt[n]{a}-\sqrt[n]{b})}\le\frac{a^{\frac{m+1}{n(m+2)}}-b^{\frac{m+1}{n(m+2)}}}{n(m+1)(a^{\frac{1}{n(m+2)}}-b^{\frac{1}{n(m+2)}})}$$
$$\le L(a,b)\le\frac{n^{\frac{1}{q}}(b^{\frac{1}{n}}-a^{\frac{1}{n}})(b^p-a^p)^{\frac{1}{p}}}{(\ln b-\ln a)(b^{\frac{p}{n}}-a^{\frac{p}{n}})}\le T_p(a,b)$$

where $T_p(a,b)=(\frac{b-a}{p(\ln b-\ln a)})^{\frac{1}{p}}$

Proof . It is clear by theorems 3.1 and 3.2. □

Remark 3.4. By putting

$$X_n(a,b)=\frac{\sqrt[2n]{ab}(a-b)}{n(\sqrt[n]{a}-\sqrt[n]{a})}\ ,\ Y_{mn}(a,b)=\frac{a^{\frac{m+1}{n(m+2)}}-b^{\frac{m+1}{n(m+2)}}}{a^{\frac{1}{n(m+2)}}-b^{\frac{1}{n(m+2)}}}$$

and

$$Z(a,b) = \frac{n^{\frac{1}{q}}(b^{\frac{1}{n}} - a^{\frac{1}{n}})(b^p - a^p)^{\frac{1}{p}}}{(\ln b - \ln a)(b^{\frac{p}{n}} - a^{\frac{p}{n}})^{\frac{1}{p}}}$$

and easy calculations we see that $X_n(a,b)$, $Y_{mn}(a,b)$ and $Z_{mp}(a,b)$ are means (see [13]). Infact we have proved that

$$G(a,b) \leq X_n(a,b) \leq Y_{mn}(a,b) \leq L(a,b) \leq Z_{pn}(a,b) \leq T_p(a,b)$$

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