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On generalized Jordan $*$-derivations with associated Hochschild $*$-2-cocycles

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Abstract

In this paper, we introduce the notions of generalized $*$-derivations, generalized Jordan $*$-derivations and Jordan triple $*$-derivations with the associated Hochschild $*$-2-cocycles and then it is proved that if $\mathcal{R}$ is a prime $*$-ring and $f : \mathcal{R} \to \mathcal{R}$ is a nonzero generalized $*$-derivation with an associated Hochschild $*$-2-cocycle β, then $\mathcal{R}$ is commutative. Some other results regarding generalized Jordan $*$-derivations are also established.

Keywords: $*$-derivation, generalized Jordan $*$-derivation, Hochschild $*$-2-cocycle, $*$-ring, prime (semiprime) ring
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1 Introduction and preliminaries

Throughout the present paper, $\mathcal{R}$ represents an associative ring with center $Z(\mathcal{R})$. First of all, let us recall some basic definitions and set the notations which are used in what follows. A ring $\mathcal{R}$ is said to be n-torsion free, where $n > 1$ is an integer, if for $x \in \mathcal{R}$, $nx = 0$ implies that $x = 0$. Recall that a ring $\mathcal{R}$ is called prime if for $x, y \in \mathcal{R}$, $x\mathcal{R}y = \{0\}$ implies that $x = 0$ or $y = 0$, and is semiprime if for $x \in \mathcal{R}$, $x\mathcal{R}x = \{0\}$ implies that $x = 0$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. An involution over $\mathcal{R}$ is a map $* : \mathcal{R} \to \mathcal{R}$ satisfying the following conditions for all $x, y \in \mathcal{R}$:
(i) $(x^*)^* = x$,
(ii) $(xy)^* = y^*x^*$,
(iii) $(x + y)^* = x^* + y^*$.
A ring equipped with an involution is called ring with involution or $*$-ring and usually is denoted, as an ordered pair, by $(\mathcal{R}, *)$. An element x in an $*$-ring is called Hermitian (self-adjoint) if $x^* = x$ and is said to be skew-Hermitian if $x^* = -x$. The sets of all Hermitian and skew-Hermitian elements of an $*$-ring $\mathcal{R}$ are denoted by $H(\mathcal{R})$ and $S(\mathcal{R})$, respectively. The involution is said to be of the first kind if $Z(\mathcal{R}) \subseteq H(\mathcal{R})$, otherwise it is said to be of the second kind. In this case, $S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$. If $\mathcal{R}$ is 2-torsion free then every $x \in \mathcal{R}$ can be uniquely represented in the form $2x = h + k$ where $h \in H(\mathcal{R})$ and $k \in S(\mathcal{R})$. An element $x \in \mathcal{R}$ is normal if $xx^* = x^*x$ and in this case the mentioned elements h and k commute with each other. If all elements in $\mathcal{R}$ are normal, then $\mathcal{R}$ is called a normal ring.

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An example in this regard is the ring of quaternion. The reader is referred to [10] for more details and descriptions of such rings.

Let $\mathcal{R}$ be an $*$-ring. An additive mapping $d:\mathcal{R}\to\ \mathcal{R}$ is called an $*$-derivation (resp. Jordan $*$-derivation) whenever $d(xy)=d(x)y^*+xd(y)$ (resp. $d(x^2)=d(x)x^*+xd(x)$) holds for all $x,y\in\mathcal{R}$. Note that the mapping $x\mapsto ax^*-xa$ of $\mathcal{R}$ into itself, where a is a fixed element in $\mathcal{R}$, is a Jordan $*$-derivation; such Jordan $*$-derivations are said to be inner. Moreover, if $a[x,y]^*=0$ for all $x,y\in\mathcal{R}$, then the mapping $x\mapsto ax^*-xa$ is an $*$-derivation. The concepts of $*$-derivation and Jordan $*$-derivation were first introduced in [5]. In an interesting article, Zalar and Bresar [6] studied the structure of Jordan $*$-derivations and also they presented a characterization of these mappings on complex $*$-algebras. The innerness of Jordan $*$-derivations has also been investigated, see, e.g. [17].

The motivation for studying Jordan $*$-derivation is that these mappings appear naturally in the theory of the representability of quadratic forms by bilinear forms. For the results concerning this theory, the reader is referred to [9, 15, 16, 17, 19], where further references can be found. Similar to what was stated above, an $*$-derivation can also be defined from an $*$-ring $\mathcal{R}$ into an $\mathcal{R}$-bimodule $\mathcal{M}$. Let $\mathcal{R}$ be an $*$-ring and let $\mathcal{M}$ be an $\mathcal{R}$-bimodule. An additive mapping $f:\mathcal{R}\to\mathcal{M}$ is called a generalized $*$-derivation (resp. generalized Jordan $*$-derivation) if there exists an $*$-derivation (resp. Jordan $*$-derivation) $d:\mathcal{R}\to\mathcal{M}$ such that $f(xy)=f(x)y^*+xd(y)$ (resp. $f(x^2)=f(x)x^*+xd(x)$) for all $x,y\in\mathcal{R}$.

In 2006, Nakajima [13] introduced a new type of generalized derivations as follows. Let $\mathcal{R}$ be a ring and let $\mathcal{M}$ be an $\mathcal{R}$-bimodule. A biadditive mapping $\beta:\mathcal{R}\times\mathcal{R}\to\mathcal{M}$ is called a Hochschild 2-cocycle if

$$x\beta(y,z)-\beta(xy,z)+\beta(x,yz)-\beta(x,y)z=0$$

for all $x,y,z\in\mathcal{R}$. The mapping β is called *symmetric* (resp. *skew symmetric*) if $\beta(x,y)=\beta(y,x)$ (resp. $\beta(x,y)=-\beta(y,x)$). An additive mapping $f:\mathcal{R}\to\mathcal{R}$ is called a generalized derivation (resp. generalized Jordan derivation) with an associated Hochschild 2-cocycle β if $f(xy)=f(x)y+xf(y)+\beta(x,y)$ (resp. $f(x^2)=f(x)x+xf(x)+\beta(x,x)$) for all $x,y\in\mathcal{R}$. If $\beta=0$, then we reach an ordinary derivation (resp. Jordan derivation). For more examples and details, see, e.g. [13].

There are many of works dealing with the commutativity of prime and semiprime rings admitting certain types of derivations, see, e.g. [1, 2, 3, 4, 5, 7, 11] and references therein. Motivated by the above notions, we introduce the notions of generalized $*$-derivations, generalized Jordan $*$-derivations and generalized Jordan triple $*$-derivations with the associated Hochschild $*$-2-cocycles and it is proved that if $\mathcal{R}$ is a prime $*$-ring and $f:\mathcal{R}\to\mathcal{R}$ is a nonzero generalized $*$-derivation with an associated Hochshcild $*$-2-cocycle β, then $\mathcal{R}$ is commutative. Furthermore, we present some characterizations of generalized $*$-derivations. For instance, we prove the following result:

Let $\mathcal{R}$ be a $*$-ring having the unit element $\mathbf{1}$, containing the element $\frac{\mathbf{1}}{2}$, and containing an invertible skew-Hermitian $\xi\in Z(\mathcal{R})$. If $f:\mathcal{R}\to\mathcal{R}$ is a generalized $*$-Jordan derivation with an associate Hochschild $*$-2-cocycle β, then there exists $\mathfrak{a},\mathfrak{b}\in\mathcal{R}$ such that

$$f(x)=x\mathfrak{a}-\mathfrak{b}x^*+\frac{\xi^{-1}\big(\beta(x,\xi)-\beta(\xi,x)\big)}{2},$$

for all $x\in\mathcal{R}$.

Moreover, we show that every generalized Jordan $*$-derivations and generalized Jordan triple $*$-derivations with an associated Hochshcild $*$-2-cocycle β are equivalent. Some other results are also presented.

2 Definitions and examples

Let $\mathcal{R}$ be an $*$-ring and let $\mathcal{M}$ be an $\mathcal{R}$-bimodule. Let $\beta:\mathcal{R}\times\mathcal{R}\longrightarrow\mathcal{M}$ be a biadditive map, that is, an additive map on each components. The biadditive map β is called a **Hochschild $*$-2-cocycle** if

$$x\beta(y,z)-\beta(xy,z)+\beta(x,yz)-\beta(x,y)z^*=0, \tag{2.1}$$

for all $x,y,z\in\mathcal{R}$. An $*$-2-cocycle β is called symmetric (resp. skew symmetric) if $\beta(x,y)=\beta(y,x)$ (resp. $\beta(x,y)=-\beta(y,x)$).

An additive map $f:\mathcal{R}\to\mathcal{M}$ is called a generalized $*$-derivation with an associated Hochschild $*$-2-cocycle β if

$$f(xy)=f(x)y^*+xf(y)+\beta(x,y),\qquad (x,y\in\mathcal{R}) \tag{2.2}$$

and f is called a generalized Jordan $*$-derivation with an associated Hochschild $*$-2-cocycle β if

$$f(x^2)=f(x)x^*+xf(x)+\beta(x,x),\qquad (x\in\mathcal{R}). \tag{2.3}$$

If $\beta = 0$, then we get the usual notions of $*$-derivations and Jordan $*$-derivations, respectively. Also, a generalized Jordan triple $*$-derivation with an associated Hochschild $*$-2-cocycle β is an additive mapping $f : \mathcal{R} \longrightarrow \mathcal{R}$ satisfying

$$f(xyx) = f(x)y^*x^* + xf(y)x^* + xyf(x) + x\beta(y, x) + \beta(x, yx) \tag{2.4}$$

for all $x, y \in \mathcal{R}$. In the following, we present some examples of such generalized $*$-derivations.

Example 2.1. Let $\mathcal{R}$ be an $*$-ring and let $\mathcal{M}$ be an $\mathcal{R}$-bimodule.

(1) Let $f : \mathcal{R} \to \mathcal{M}$ be a generalized $*$-derivation associated with a $*$-derivation d. Then the mapping $\beta : \mathcal{R}\times\mathcal{R} \to \mathcal{M}$ defined by $\beta(x, y) = x(d - f)(y)$ is a Hochschild $*$-2-cocycle and also f is a generalized $*$-derivation with the associated mapping β.

(2) Let $f : \mathcal{R} \longrightarrow \mathcal{M}$ is left $*$-centralizer, that is, f is additive and $f(xy) = f(x)y^*$. We can write $f(xy) = f(x)y^* + xf(y) - xf(y)$ for all $x, y \in \mathcal{R}$. If we define a mapping $\beta : \mathcal{R} \times \mathcal{R} \to \mathcal{M}$ by $\beta(x, y) = -xf(y)$. So, f is a generalized $*$-derivation with the associated Hochschild $*$-2-cocycle β.

(3) Let $f : \mathcal{R} \to \mathcal{M}$ be an $* - (I, \tau)$ derivation, that is, $\tau : \mathcal{R} \to \mathcal{R}$ is a ring homomorphism of $\mathcal{R}$ and $f(xy) = f(x)y^* + \tau(x)f(y)$, where I is the identity mapping on $\mathcal{R}$. Then the map $\beta : \mathcal{R} \times \mathcal{R} \to \mathcal{M}$ defined by $\beta(x, y) = (\tau(x) - x)f(y)$ is a Hochschild $*$-2-cocycle. Hence, we have

$$f(xy) = f(x)y^* + xf(y) + \beta(x, y),$$

for all $x, y \in \mathcal{R}$, then f is a generalized $*$-derivation with the associated mapping β.

(4) Let $d : \mathcal{R} \to \mathcal{R}$ be an $*$-derivation and $T : \mathcal{R} \to \mathcal{R}$ be a left centralizer, that is, T is additive and $T(xy) = T(x)y$, then Td is a generalized $*$-derivation associated with the Hochschild $*$-2-cocycle $\beta : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ defined by

$$\beta(x, y) = T(x)d(y) - xT(y), \qquad (x, y \in \mathcal{R}).$$

3 Main Results

We begin our results with the following proposition that states the biadditivity of β is obtained from the additivity of f.

Proposition 3.1. Let $\mathcal{R}$ be an $*$-ring, let $f : \mathcal{R} \to \mathcal{R}$ be an additive mapping and let $\beta : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be a mapping. If f and β satisfy $f(xy) = f(x)y^* + xf(y) + \beta(x, y)$ for all $x, y \in \mathcal{R}$, then β is a biadditive mapping.

Proof . For each $x, y, z \in \mathcal{R}$, we have

$$\begin{aligned} f(x(y + z)) &= f(x)(y + z)^* + xf(y + z) + \beta(x, y + z) \\ &= f(x)y^* + f(x)z^* + xf(y) + xf(z) + \beta(x, y + z), \end{aligned}$$

which means that

$$f(x(y + z)) = f(x)y^* + f(x)z^* + xf(y) + xf(z) + \beta(x, y + z).$$

On the other hand, since f is an additive mapping, we have the following expressions:

$$\begin{aligned} f(x(y + z)) &= f(xy) + f(xz) \\ &= f(x)y^* + xf(y) + \beta(x, y) + f(x)z^* + xf(z) + \beta(x, z). \end{aligned}$$

Comparing the last two equations regarding $f(x(y + z))$, we get that

$$\beta(x, y + z) = \beta(x, y) + \beta(x, z).$$

Similarly, we can prove that $\beta(x + y, z) = \beta(x, z) + \beta(y, z)$. It means that β is a biadditive mapping on $\mathcal{R}$, as desired. □

As observed, biadditivity of the mapping β depends on additivity of the mapping f.

Lemma 3.2. [20, Lemma 1.3] Let $\mathcal{R}$ be a semiprime ring and let $a[x,y]=0$ for all $x,y\in\mathcal{R}$ and for some $a\in\mathcal{R}$. Then $a\in Z(\mathcal{R})$.

In the next theorem, we are going to prove that if $\mathcal{R}$ is a semiprime $*$-ring and f is a generalized $*$-derivation with an associated Hochschild $*$-2-cocycle β, then f maps $\mathcal{R}$ into $Z(\mathcal{R})$.

Theorem 3.3. Let $\mathcal{R}$ be a semiprime $*$-ring. If $f:\mathcal{R}\to\mathcal{R}$ is a generalized $*$-derivation associated with a Hochschild $*$-2-cocycle β, then $f(\mathcal{R})\subseteq Z(\mathcal{R})$.

Proof . For all $x,y,z\in\mathcal{R}$, we have

$$\begin{aligned} f(xyz) &= f((xy)z) \\ &= f(xy)z^*+xyf(z)+\beta(xy,z) \\ &= f(x)y^*z^*+xf(y)z^*+\beta(x,y)z^*+xyf(z)+\beta(xy,z) \end{aligned} \tag{3.1}$$

On the other hand, we have

$$\begin{aligned} f(xyz) &= f(x(yz)) \\ &= f(x)z^*y^*+xf(yz)+\beta(x,yz) \\ &= f(x)z^*y^*+xf(y)z^*+xyf(z)+x\beta(y,z)+\beta(x,yz) \end{aligned} \tag{3.2}$$

Comparing (3.1) and (3.2) with the fact that β ia a Hochshcild $*$-2-cocycle, we get that $f(x)[y^*,z^*]=0$ and so $f(x)[y,z]=0$ for all $x,y,z\in\mathcal{R}$. Using the above lemma, we get that $[f(x),z]=0$ for all $x,z\in\mathcal{R}$. This means that f maps $\mathcal{R}$ into $Z(\mathcal{R})$, as desired. □

An immediate consequence of the above theorem is as follows:

Corollary 3.4. Let $\mathcal{A}$ be a C^*-algebra. If $f:\mathcal{A}\to\mathcal{A}$ is a generalized $*$-derivation associated with a Hochshcild $*$-2-cocycle β, then $f(\mathcal{A})\subseteq Z(\mathcal{A})$.

Proof . It is evident that every C^*-algebra is semisimple and hence it is semprime. see, e.g. [8]. □

Here, we present another result of this paper.

Theorem 3.5. Let $\mathcal{R}$ be a prime $*$-ring. If $f:\mathcal{R}\to\mathcal{R}$ is a nonzero generalized $*$-derivation associated with a Hochshcild $*$-2-cocycle β, then $\mathcal{R}$ is commutative.

Proof . Since f is nonzero, there exists $x_0\in\mathcal{R}$ such that $f(x_0)\neq 0$. According to the proof of Theorem 3.3, we have $f(x_0)[y,z]=0$ for all $y,z\in\mathcal{R}$. Replacing y by yt in the previous equation and the using it, we arrive at

$$f(x_0)y[t,z]=0$$

for all $y,t,z\in\mathcal{R}$. The primeness of $\mathcal{R}$ forces that $[t,z]=0$ for all $t,z\in\mathcal{R}$ which means that $\mathcal{R}$ is commutative, as required. □

Corollary 3.6. Let $\mathcal{R}$ be a prime $*$-ring. If $\mathcal{R}$ admits a nonzero $*$-derivation or a nonzero $*$-left centralizer or a nonzero $*$-(I,τ)-derivation (as in Example 2.1), then $\mathcal{R}$ is commutative.

Remark 3.7. We can define a generalized reverse $*$-derivation $f:\mathcal{R}\to\mathcal{R}$ associated with a revers Hochschild $*$-2-cocycle $\beta:\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ as an additive mapping satisfying

$$f(xy)=f(y)x^*+yf(x)+\beta(x,y),$$

for all $x,y\in\mathcal{R}$, where β is a biadditive mapping satisfying the following revers Hochschild $*$-2-cocycles property:

$$\beta(xy,z)-\beta(y,z)x^*+\beta(x,yz)-y\beta(x,z)=0$$

for all $x,y,z\in\mathcal{R}$. We can establish Theorems 3.3 and 3.5 for the above-mentioned generalized reverse $*$-derivations and we leave it to the interested reader.

In the following, we present some consequences about the commutativity of algebras. Let $\mathcal{R}$ be an $*$-ring. For every $a, b \in \mathcal{R}$, $ab^* - ba$ is denoted by $[a, b]_*$. Indeed, we have $[a, b]_* = ab^* - ba$

Theorem 3.8. Let $\mathcal{A}$ be a semiprime Banach $*$-algebra such that $dim(rad(\mathcal{A})) \leq 1$. If there exists an element $\mathfrak{z} \in \mathcal{A}$ such that $[\mathfrak{z}, a]_* \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$, then there is an ideal $\mathfrak{I}$ of $\mathcal{A}$ such that $\mathfrak{z} \in \mathfrak{I} \subseteq Z(\mathcal{A})$.

Proof . Using $[\mathfrak{z}, a]_* \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$, we get that $\mathfrak{z}a - a\mathfrak{z} \in Z(\mathcal{A})$ for all self-adjoint (Hermitian) elements $a \in \mathcal{A}$. Let a be an arbitrary element of $\mathcal{A}$. We know that there are two self-adjoint elements $a_1, a_2 \in \mathcal{A}$ such that $a = a_1 + ia_2$. Hence, we have

$$\mathfrak{z}a - a\mathfrak{z} = \mathfrak{z}(a_1 + ia_2) - (a_1 + ia_2)\mathfrak{z} = (\mathfrak{z}a_1 - a_1\mathfrak{z}) + i(a_2\mathfrak{z} - \mathfrak{z}a_2) \in Z(\mathcal{A}),$$

which means that $[\mathfrak{z}, a] \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$. It is evident that the linear mapping $d_{\mathfrak{z}} : \mathcal{A} \to \mathcal{A}$ defined by $d_{\mathfrak{z}}(a) = [\mathfrak{z}, a] = \mathfrak{z}a - a\mathfrak{z}$ is a derivation which maps into $Z(\mathcal{A})$. It follows from [12, Theorem 7] that $d_{\mathfrak{z}}(\mathcal{A}) \subseteq rad(\mathcal{A})$. By hypothesis, $dim(rad(A)) \leq 1$ and it follows from [14, Proposition 2.1] that $d_{\mathfrak{z}} = 0$. Therefore, we get that $\mathfrak{z} \in Z(\mathcal{A})$. Using this fact and the assumption that $[\mathfrak{z}, a]_* = \mathfrak{z}a^* - a\mathfrak{z} \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$, we get that $\mathfrak{z}(a^* - a) \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$. Let a be an arbitrary element of $\mathcal{A}$. So, there are two self-adjoint elements $a_1, a_2 \in \mathcal{A}$ such that $a = a_1 + ia_2$. Since $\mathfrak{z}(a^* - a) \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$, we obtain that $\mathfrak{z}a_2 \in Z(\mathcal{A})$ for all $a_2 \in \mathcal{S}_{\mathcal{A}}$. This yields that $\mathfrak{z}\mathcal{A} \subseteq Z(\mathcal{A})$. Since $\mathfrak{z} \in Z(\mathcal{A})$ and also $\mathfrak{z}\mathcal{A} \subseteq Z(\mathcal{A})$, we can thus deduce that there exists an ideal $\mathfrak{I}$ of $\mathcal{A}$ such that $\mathfrak{z} \in \mathfrak{I} \subseteq Z(\mathcal{A})$, as desired. □

An immediate corollary reads as follows:

Corollary 3.9. Let $\mathcal{A}$ be a C^*-algebra. If there exists an element $\mathfrak{z} \in \mathcal{A}$ such that $[\mathfrak{z}, a]_* \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$, then there is an ideal $\mathfrak{I}$ of $\mathcal{A}$ such that $\mathfrak{z} \in \mathfrak{I} \subseteq Z(\mathcal{A})$.

Proof . It is a well-known fact that every C^*-algebra is semisimple. □

Theorem 3.10. Let $\mathcal{A}$ be a unital semiprime $*$-algebra such that $dim(Z(\mathcal{A})) \leq 1$. If there exists an element $\mathfrak{z} \in \mathcal{A}$ such that $[\mathfrak{z}, a]_* \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$, then $\mathfrak{z} = 0$ or $\mathcal{A}$ is commutative and $dim(\mathcal{A}) = 1$.

Proof . According to the proof of Theorem 3.8, the linear mapping $d_{\mathfrak{z}} : \mathcal{A} \to \mathcal{A}$ defined by $d_{\mathfrak{z}}(a) = [\mathfrak{z}, a] = \mathfrak{z}a - a\mathfrak{z}$ is a derivation mapping into $Z(\mathcal{A})$. We are assuming that $dim(Z(\mathcal{A}) \leq 1$ and it follows from [14, Proposition 2.1] that $d_{\mathfrak{z}} = 0$ and therefore, $\mathfrak{z} \in Z(\mathcal{A})$. Reusing Theorem 3.8, we know that there exists an ideal $\mathfrak{I}$ of $\mathcal{A}$ such that $\mathfrak{z} \in \mathfrak{I} \subseteq Z(\mathcal{A})$. If $dim(Z(\mathcal{A})) = 0$, then $\mathfrak{z} = 0$. Now, suppose that $dim(Z(\mathcal{A})) = 1$. Since $\mathfrak{I}$ is an ideal of $\mathcal{A}$ and is a subset of $Z(\mathcal{A})$, $dim(\mathfrak{I}) = 0$ or $dim(\mathfrak{I}) = 1$. If $dim(\mathfrak{I}) = 0$, then $\mathfrak{z} = 0$. If $dim(\mathfrak{I}) = 1$, then $\mathfrak{I} = Z(\mathcal{A})$. Since $\mathcal{A}$ is unital, $\mathcal{A} = \mathfrak{I}$ and consequently, $\mathcal{A}$ is commutative and $dim(\mathcal{A}) = 1$, as desired. □

Corollary 3.11. Let $\mathcal{A}$ be a semiprime $*$-algebra. If there exists an element $\mathfrak{z} \in \mathcal{A}$ such that $[\mathfrak{z}, a]_* = 0$ for all $a \in \mathcal{A}$, then $\mathfrak{z} = 0$.

Theorem 3.12. Let $\mathcal{A}$ be a Banach algebra such that $dim(rad(\mathcal{A})) \leq 1$. If $[[[[b, a], a], a]a] \in rad(\mathcal{A})$ for all $a, b \in \mathcal{A}$, then $\mathcal{A}$ is commutative.

Proof . Suppose that $\mathcal{A}$ is a noncommutative Banach algebra. For any $b \in \mathcal{A}$, the linear mapping $d_b : \mathcal{A} \to \mathcal{A}$ defined by $d_b(a) = [b, a]$ is a continuous derivation. It follows from [18, Theorem 2] that $d_b(\mathcal{A}) \subseteq rad(\mathcal{A})$ and since we are assuming that $dim(rad(\mathcal{A})) \leq 1$, [14, Proposition 2.1] implies that $d_b(a) = 0$ for all $a \in \mathcal{A}$. Since b is an arbitrary element of $\mathcal{A}$, the algebra $\mathcal{A}$ is commutative, a contradiction. Hence, $\mathcal{A}$ must be commutative. □

There is a consequence of the previous theorem as follows:

Theorem 3.13. Let $\mathcal{A}$ be a noncommutative Banach algebra. If $[[[[b, a], a], a]a] \in rad(\mathcal{A})$ for all $a, b \in \mathcal{A}$, then $dim(rad(\mathcal{A})) > 1$. In this case, $\mathcal{A}$ is not a semisimple Banach algebra.

In [5, Lemma 2], Brešar and Vukman proved that any Jordan $*$-derivation on a 2-torsion free $*$-ring is a Jordan triple $*$-derivation. The following lemma presents some properties of the new notion of generalized Jordan $*$-derivations and it is especially an extension for [5, Lemma 2].

Lemma 3.14. Let $f:\mathcal{R}\longrightarrow\mathcal{M}$ be a generalized Jordan $*$-derivation with an associated Hochschild $*$-2-cocycle β and let $\mathcal{M}$ be a 2-torsion free $\mathcal{R}$-bimodule. Then the following relations hold for all $x,y,z\in\mathcal{R}$:

$$(i) f(xy+yx)=f(x)y^*+xf(y)+\beta(x,y)+f(y)x^*+yf(x)+\beta(y,x);$$
$$(ii) f(xyx)=f(x)y^*x^*+xf(y)x^*+xyf(x)+x\beta(y,x)+\beta(x,yx);$$
$$(iii) f(xyz+zyx)=f(x)y^*z^*+xf(y)z^*+xyf(z)+x\beta(y,z)+\beta(x,yz)$$
$$+f(z)y^*x^*+zf(y)x^*+zyf(x)+z\beta(y,x)+\beta(z,yx).$$

Proof .

(i) We know that $f(x^2)=f(x)x^*+xf(x)+\beta(x,x)$ holds for all $x\in\mathcal{R}$. So, we have

$$\begin{aligned}
f(xy+yx) &= f((x+y)^2)-f(x^2)-f(y^2)\\
&= f(x+y)(x+y)^*+(x+y)f(x+y)+\beta(x+y,x+y)\\
&\quad -f(x)x^*-xf(x)-\beta(x,x)-f(y)y^*-yf(y)-\beta(y,y)\\
&= f(x)x^*+f(x)y^*+f(y)x^*+f(y)y^*+xf(x)+yf(x)\\
&\quad +xf(y)+yf(y)+\beta(x,y)+\beta(x,x)+\beta(y,x)+\beta(y,y)\\
&\quad -f(x)x^*-xf(x)-\beta(x,x)-f(y)y^*-yf(y)-\beta(y,y)\\
&= f(x)y^*+xf(y)+\beta(x,y)+f(y)x^*+yf(x)+\beta(y,x).
\end{aligned}$$

(ii) Replacing y by $xy+yx$ in (i) and using the assumption that β is a Hochschild $*$-2-cocycle (see (2.1)), we have

$$\begin{aligned}
2f(xyx) &= f(x(xy+yx)+(xy+yx)x)-f(x^2y+yx^2)\\
&= f(x)(xy+yx)^*+xf(xy+yx)+\beta(x,xy+yx)+f(xy+yx)x^*\\
&\quad +(xy+yx)f(x)+\beta(xy+yx,x)-f(x^2)y^*-x^2f(y)-\beta(x^2,y)\\
&\quad -f(y)x^{*2}-yf(x^2)-\beta(y,x^2)\\
&= f(x)y^*x^*+f(x)x^*y^*+x\Big[f(x)y^*+xf(y)+\beta(x,y)+f(y)x^*\\
&\quad +yf(x)+\beta(y,x)\Big]+\beta(x,xy)+\beta(x,yx)+\Big[f(x)y^*+xf(y)\\
&\quad +\beta(x,y)+f(y)x^*+yf(x)+\beta(y,x)\Big]x^*+xyf(x)+yxf(x)\\
&\quad +\beta(xy,x)+\beta(yx,x)-\Big[f(x)x^*+xf(x)+\beta(x,x)\Big]y^*-x^2f(y)\\
&\quad -\beta(x^2,y)-f(y)x^{*2}-y\Big[f(x)x^*+xf(x)+\beta(x,x)\Big]-\beta(y,x^2)\\
&= 2f(x)y^*x^*+2xf(y)x^*+2xyf(x)\\
&\quad +\Big[x\beta(x,y)-\beta(x^2,y)+\beta(x,xy)-\beta(x,x)y^*\Big]\\
&\quad -\Big[y\beta(x,x)-\beta(yx,x)+\beta(y,x^2)-\beta(y,x)x^*\Big]\\
&\quad +\Big[x\beta(y,x)+\beta(x,yx)+\beta(xy,x)+\beta(x,y)x^*\Big].
\end{aligned}$$

Since $x\beta(y,x)+\beta(x,yx)=\beta(xy,x)+\beta(x,y)x^*$ and $\mathcal{M}$ is 2-torsion free, we obtain equation (2).

(iii) Replacing x by $x+z$ in (ii), we have

$$\begin{aligned}
f((x+z)y(x+z)) &= f(x+z)y^*(x+z)^*+(x+z)f(y)(x+z)^*+(x+z)yf(x+z)\\
&\quad +(x+z)\beta(y,x+z)+\beta(x+z,y(x+z))
\end{aligned}$$

and so,

$$\begin{aligned} f(xyx)+f(xyz+zyx)+f(zyz) &= f(x)y^*x^*+f(z)y^*x^*+f(x)y^*z^*+f(z)y^*z^* \\ &+xf(y)x^*+zf(y)x^*+xf(y)z^*+zf(y)z^* \\ &+xyf(x)+zyf(x)+xyf(z)+zyf(z) \\ &+x\beta(y,x)+z\beta(y,x)+x\beta(y,z)+z\beta(y,z) \\ &+\beta(x,yx)+\beta(x,yz)+\beta(z,yx)+\beta(z,yz) \end{aligned}$$

Using equation (ii), we get the required result. □

In the following theorem, we present a characterization of a generalized $*$-Jordan derivation with an associated Hochschild $*$-2-cocycle.

Theorem 3.15. Let $\mathcal{R}$ be a unital $*$-ring containing the element $\frac{\mathbf{1}}{2}$, and let ξ be an invertible skew-Hermitian element of $Z(\mathcal{R})$. If $f:\mathcal{R}\to\mathcal{R}$ is a generalized $*$-Jordan derivation with an associated Hochschild $*$-2-cocycle β, then there exist the elements $\mathfrak{a},\mathfrak{b}\in\mathcal{R}$ such that

$$f(x)=x\mathfrak{a}-\mathfrak{b}x^*+\frac{\xi^{-1}\big(\beta(x,\xi)-\beta(\xi,x)\big)}{2},$$

for all $x\in\mathcal{R}$.

Proof . Using Lemma 3.14(ii), we have

$$\begin{aligned} f(\xi) &= f(\xi\xi^{-1}\xi) \\ &= f(\xi)-\xi^2f(\xi^{-1})+f(\xi)+\xi\beta(\xi^{-1},\xi)+\beta(\xi,\mathbf{1}). \end{aligned}$$

Thus

$$f(\xi^{-1})=\xi^{-2}f(\xi)+\xi^{-1}\beta(\xi^{-1},\xi)+\xi^{-2}\beta(\xi,\mathbf{1}). \tag{3.3}$$

According to Lemma 3.14(iii) and equation (3.3), we have

$$\begin{aligned} 2f(x) &= f(\xi x\xi^{-1}+\xi^{-1}x\xi) \\ &= -\xi^{-1}f(\xi)x^*-f(x)+x\xi^{-1}f(\xi)+x\beta(\xi^{-1},\xi)+x\xi^{-1}\beta(\xi,\mathbf{1}) \\ &+\xi\beta(x,\xi^{-1})+\beta(\xi,x\xi^{-1})-\xi^{-1}f(\xi)x^*-\beta(\xi^{-1},\xi)x^*-\xi^{-1}\beta(\xi,\mathbf{1})x^* \\ &-f(x)+x\xi^{-1}f(\xi)+\xi^{-1}\beta(x,\xi)+\beta(\xi^{-1},x\xi) \end{aligned} \tag{3.4}$$

Since β is a Hochschild $*$-2-cocycle mapping, we have

$$\xi\beta(x,\xi^{-1})+\beta(\xi,x\xi^{-1})=\beta(\xi x,\xi^{-1})-\xi^{-1}\beta(\xi,x) \tag{3.5}$$

$$\begin{aligned} \beta(\xi x,\xi^{-1}) &= \beta(x\xi,\xi^{-1}) \\ &= x\beta(\xi,\xi^{-1})+\beta(x,\mathbf{1})+\xi^{-1}\beta(x,\xi) \end{aligned} \tag{3.6}$$

$$\begin{aligned} \beta(\xi^{-1},x\xi) &= \beta(\xi^{-1},\xi x) \\ &= \beta(\xi^{-1},\xi)x^*+\beta(\mathbf{1},x)-\xi^{-1}\beta(\xi,x) \end{aligned} \tag{3.7}$$

Using (3.5) ,(3.6) and (3.7) in the above relation (3.4), we have

$$\begin{aligned} 2f(x) &= -\xi^{-1}f(\xi)x^*-f(x)+x\xi^{-1}f(\xi)+x\beta(\xi^{-1},\xi)+x\xi^{-1}\beta(\xi,\mathbf{1}) \\ &+x\beta(\xi,\xi^{-1})+\beta(x,\mathbf{1})+\xi^{-1}\beta(x,\xi)-\xi^{-1}\beta(\xi,x)-\xi^{-1}f(\xi)x^* \\ &-\beta(\xi^{-1},\xi)x^*-\xi^{-1}\beta(\xi,\mathbf{1})x^*-f(x)+x\xi^{-1}f(\xi)+\xi^{-1}\beta(x,\xi) \\ &+\beta(\xi^{-1},\xi)x^*+\beta(\mathbf{1},x)-\xi^{-1}\beta(\xi,x) \end{aligned}$$

By relations $\xi^{-1}\beta(\xi,\mathbf{1})=\beta(\mathbf{1},\mathbf{1})$ and $\beta(x,\mathbf{1})=x\beta(\mathbf{1},\mathbf{1})$ and so $\beta(\mathbf{1},x)=\beta(\mathbf{1},\mathbf{1})x^*$, we have

$$\begin{aligned}4f(x) &= x\big(2\xi^{-1}f(\xi)+\beta(\xi^{-1},\xi)+\beta(\xi,\xi^{-1})+2\beta(\mathbf{1},\mathbf{1})\big)\\ &\quad -(2\xi^{-1}f(\xi))x^*+2\xi^{-1}\big(\beta(x,\xi)-\beta(\xi,x)\big),\end{aligned}$$

Considering $\mathfrak{a}=\frac{2\xi^{-1}f(\xi)+\beta(\xi^{-1},\xi)+\beta(\xi,\xi^{-1})+2\beta(\mathbf{1},\mathbf{1})}{4}$ and $\mathfrak{b}=\frac{\xi^{-1}f(\xi)}{2}$, we see that

$$f(x)=x\mathfrak{a}-\mathfrak{b}x^*+\frac{\xi^{-1}\big(\beta(x,\xi)-\beta(\xi,x)\big)}{2},$$

as desired. □

An immediate consequence of the previous theorem is as follows:

Corollary 3.16. Suppose that all the conditions of Theorem 3.15 are fulfilled and additionally β is a symmetric mapping. Then $f(x)=x\mathfrak{a}-\mathfrak{b}x^*$ for all $x\in\mathcal{R}$ and for some $a,b\in\mathcal{R}$.

Lemma 3.17. [5, Lemma 3] Let $\mathcal{R}$ be a noncommutative prime $*$-ring. If $a\in\mathcal{R}$ is such that $ax^*=xa$ for all $x\in\mathcal{R}$, then $a=0$.

Theorem 3.18. Let $\mathcal{R}$ be a noncommutative prime $*$-ring. If $f:\mathcal{R}\to\mathcal{R}$ is a generalized $*$-Jordan derivation with an associated Hochschild $*$-2-cocycle β, then $[f(c),x]_*=\beta(x,c)-\beta(c,x)$ for all $c\in Z(\mathcal{R})\cap H(\mathcal{R})$ and all $x\in\mathcal{R}$.

Proof . Let $c\in Z(\mathcal{R})\cap H(\mathcal{R})$. According to Lemma (3.7)(iii), we have

$$\begin{aligned}f(xcy+ycx) &= f(x)cy^*+xf(c)y^*+xcf(y)+x\beta(c,y)+\beta(x,cy)\\ &\quad +f(y)cx^*+yf(c)x^*+ycf(x)+y\beta(c,x)+\beta(y,cx)\end{aligned}\tag{3.8}$$

Also, since $c\in Z(\mathcal{R})$, we have

$$\begin{aligned}f(cxy+yxc) &= f(c)x^*y^*+cf(x)y^*+cxf(y)+c\beta(x,y)+\beta(c,xy)\\ &\quad +f(y)x^*c+yf(x)c+yxf(c)+y\beta(x,c)+\beta(y,xc)\end{aligned}\tag{3.9}$$

By Hochschild $*$-2-cocycle property, we have

$$x\beta(c,y)+\beta(x,cy)=\beta(xc,y)+\beta(x,c)y^*\tag{3.10}$$

$$c\beta(x,y)+\beta(c,xy)=\beta(cx,y)+\beta(c,x)y^*\tag{3.11}$$

Comparing the expressions (3.8) and (3.9) and using relations (3.10) and (3.11), we get that

$$(f(c)x^*-xf(c)+\beta(c,x)-\beta(c,x))y^*=y(f(c)x^*-xf(c)+\beta(c,x)-\beta(x,c)),$$

for all $x,y\in\mathcal{R}$. Now, using Lemma 3.17, we arrive at

$$f(c)x^*-xf(c)+\beta(c,x)-\beta(x,c)=0$$

Therefore, $[f(c),x]_*=\beta(x,c)-\beta(c,x)$. □

In [16, Theorem 2.1], Semrl proved that the notions of Jordan $*$-derivations and Jordan triple $*$-derivations on a real Banach $*$-algebra are equivalent. In the following theorem, we obtain a generalization for this theorem.

Theorem 3.19. Let $\mathcal{A}$ be a real Banach $*$-algebra, let $f:\mathcal{A}\longrightarrow\mathcal{A}$ be an additive mapping and let $\beta:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A}$ be a Hochschild $*-2-$cocycle. The following statements are equivalent:

(i) f is a generalized Jordan $*$-derivation with an associated mapping β,

(ii) For any invertible element $a\in\mathcal{A}$,

$$f(a)=-af(a^{-1})a^*-a\beta(a^{-1},a)-\beta(a,\mathbf{1}),\tag{3.12}$$

(iii) For all $a, b \in \mathcal{A}$,
$$f(aba) = f(a)b^*a^* + af(b)a^* + abf(a) + a\beta(b,a) + \beta(a,ba). \tag{3.13}$$

Proof . $(ii) \Rightarrow (i)$.

If a is invertible and $\|a\| < \mathbf{1}$, then we know that $\mathbf{1}+a, \mathbf{1}-a$ and $\mathbf{1}-a^2$ are invertible. We show that for such an a we have
$$f(a^2) = f(a)a^* + af(a) + \beta(a,a).$$

Indeed,

$$\begin{aligned} f(a) + a^{-1}f(a)a^{*-1} &= f(a) - f(a^{-1}) - a^{-1}\beta(a,a^{-1}) - \beta(a^{-1},\mathbf{1}) \\ &= f(a^{-1}(a^2-\mathbf{1})) - a^{-1}\beta(a,a^{-1}) - \beta(a^{-1},\mathbf{1}) \\ &= -a^{-1}(a^2-\mathbf{1})f((a^2-\mathbf{1})^{-1}a)(a^{*2}-\mathbf{1})a^{*-1} \\ &\quad -a^{-1}(a^2-\mathbf{1})\beta((a^2-\mathbf{1})^{-1}a, a^{-1}(a^2-\mathbf{1})) \\ &\quad -\beta(a^{-1}(a^2-\mathbf{1}),\mathbf{1}) - a^{-1}\beta(a,a^{-1}) - \beta(a^{-1},\mathbf{1}) \end{aligned}$$

We use equation (3.12) and also the following equation
$$(a^2-\mathbf{1})^{-1}a = (a-\mathbf{1})^{-1} - (a^2-\mathbf{1})^{-1}, \tag{3.14}$$
to calculate $f((a-\mathbf{1})^{-1})$ and $f((a^2-\mathbf{1})^{-1})$. So, we have

$$\begin{aligned} f(a) + a^{-1}f(a)a^{*-1} &= -a^{-1}(a^2-\mathbf{1})f((a-\mathbf{1})^{-1} - (a^2-\mathbf{1})^{-1})(a^{*2}-\mathbf{1})a^{*-1} \\ &\quad -a^{-1}(a^2-\mathbf{1})\beta((a^2-\mathbf{1})^{-1}a, a^{-1}(a^2-\mathbf{1})) \\ &\quad -\beta(a^{-1}(a^2-\mathbf{1}),\mathbf{1}) - a^{-1}\beta(a,a^{-1}) - \beta(a^{-1},\mathbf{1}) \\ &= a^{-1}(a+\mathbf{1})f(a-\mathbf{1})(a^*+\mathbf{1})a^{*-1} \\ &\quad +a^{-1}(a+\mathbf{1})\beta((a-\mathbf{1}),(a-\mathbf{1})^{-1})(a^{*2}-\mathbf{1})a^{*-1} \\ &\quad +a^{-1}(a^2-\mathbf{1})\beta((a-\mathbf{1})^{-1},\mathbf{1})(a^{*2}-\mathbf{1})a^{*-1} \\ &\quad -a^{-1}f(a^2-\mathbf{1})a^{*-1} \\ &\quad -a^{-1}\beta((a^2-\mathbf{1}),(a^2-\mathbf{1})^{-1})(a^{*2}-\mathbf{1})a^{*-1} \\ &\quad -a^{-1}(a^2-\mathbf{1})\beta((a^2-\mathbf{1})^{-1},\mathbf{1})(a^{*2}-\mathbf{1})a^{*-1} \\ &\quad -a^{-1}(a^2-\mathbf{1})\beta((a^2-\mathbf{1})^{-1}a, a^{-1}(a^2-\mathbf{1})) \\ &\quad -\beta(a^{-1}(a^2-\mathbf{1}),\mathbf{1}) - a^{-1}\beta(a,a^{-1}) - \beta(a^{-1},\mathbf{1}) \end{aligned}$$

It follows from (3.12) that $f(\mathbf{1}) = -\beta(\mathbf{1},\mathbf{1})$ and so we have

$$\begin{aligned} f(a) + a^{-1}f(a)a^{*-1} &= a^{-1}(a+\mathbf{1})\left(f(a)+\beta(\mathbf{1},\mathbf{1})\right)(a^*+\mathbf{1})a^{*-1} \\ &\quad +a^{-1}(a+\mathbf{1})\beta((a-\mathbf{1}),(a-\mathbf{1})^{-1})(a^{*2}-\mathbf{1})a^{*-1} \\ &\quad +a^{-1}(a^2-\mathbf{1})\beta((a-\mathbf{1})^{-1},\mathbf{1})(a^{*2}-\mathbf{1})a^{*-1} \\ &\quad -a^{-1}\left(f(a^2)+\beta(\mathbf{1},\mathbf{1})\right)a^{*-1} \\ &\quad -a^{-1}\beta((a^2-\mathbf{1}),(a^2-\mathbf{1})^{-1})(a^{*2}-\mathbf{1})a^{*-1} \\ &\quad -a^{-1}(a^2-\mathbf{1})\beta((a^2-\mathbf{1})^{-1},\mathbf{1})(a^{*2}-\mathbf{1})a^{*-1} \\ &\quad -a^{-1}(a^2-\mathbf{1})\beta((a^2-\mathbf{1})^{-1}a, a^{-1}(a^2-\mathbf{1})) \\ &\quad -\beta(a^{-1}(a^2-\mathbf{1}),\mathbf{1}) - a^{-1}\beta(a,a^{-1}) - \beta(a^{-1},\mathbf{1}) \end{aligned} \tag{3.15}$$

Multiplying (3.15) from the left by a and the right by a^*, we get that

$$\begin{aligned} af(a)a^* + f(a) &= (a+\mathbf{1})\left(f(a)+\beta(\mathbf{1},\mathbf{1})\right)(a^*+\mathbf{1}) \\ &+(a+\mathbf{1})\beta((a-\mathbf{1}),(a-\mathbf{1})^{-1})(a^{*2}-\mathbf{1}) \\ &+(a^2-\mathbf{1})\beta((a-\mathbf{1})^{-1},\mathbf{1})(a^{*2}-\mathbf{1}) - (f(a^2)+\beta(\mathbf{1},\mathbf{1})) \\ &-\beta((a^2-\mathbf{1}),(a^2-\mathbf{1})^{-1})(a^{*2}-\mathbf{1}) \\ &-(a^2-\mathbf{1})\beta((a^2-\mathbf{1})^{-1},\mathbf{1})(a^{*2}-\mathbf{1}) \\ &-(a^2-\mathbf{1})\beta((a^2-\mathbf{1})^{-1}a, a^{-1}(a^2-\mathbf{1}))a^* \\ &-a\beta(a^{-1}(a^2-\mathbf{1}),\mathbf{1})a^* - \beta(a,a^{-1})a^* - a\beta(a^{-1},\mathbf{1})a^* \end{aligned} \tag{3.16}$$

Putting $z=\mathbf{1}$ in the Hochschild $*$-2-cocycle property (see (2.1)), we have

$$x\beta(y,\mathbf{1}) = \beta(xy,\mathbf{1}) \tag{3.17}$$

and so,

$$\begin{aligned} &(a^2-\mathbf{1})\beta((a^2-\mathbf{1})^{-1}a, a^{-1}(a^2-\mathbf{1})) + \beta(a^2-\mathbf{1},\mathbf{1}) \\ &= \beta(a^2-\mathbf{1},(a^2-\mathbf{1})^{-1}a)(a^{*2}-\mathbf{1})a^{*-1} + \beta(a,a^{-1}(a^2-\mathbf{1})). \end{aligned} \tag{3.18}$$

Substituting (3.17) and (3.18) in (3.16), we get that

$$\begin{aligned} af(a)a^* + f(a) &= af(a)a^* + af(a) + f(a)a^* + f(a) + (a+\mathbf{1})\beta(\mathbf{1},\mathbf{1})(a^*+\mathbf{1}) \\ &+(a+\mathbf{1})\beta((a-\mathbf{1}),(a-\mathbf{1})^{-1})(a^{*2}-\mathbf{1}) + \beta((a+\mathbf{1}),\mathbf{1})(a^{*2}-\mathbf{1}) \\ &-f(a^2) - \beta(\mathbf{1},\mathbf{1}) - \beta((a^2-\mathbf{1}),(a^2-\mathbf{1})^{-1})(a^{*2}-\mathbf{1}) \\ &-\beta(\mathbf{1},\mathbf{1})(a^{*2}-\mathbf{1}) - (\beta(a,a^{-1}(a^2-\mathbf{1})) \\ &+\beta(a^2-\mathbf{1},(a^2-\mathbf{1})^{-1}a)(a^{*2}-\mathbf{1})a^{*-1})a^* \\ &-\beta(a,a^{-1})a^* - a\beta(a^{-1},\mathbf{1})a^* \end{aligned}$$

Using (3.14) and relation above, we deduce that

$$\begin{aligned} 0 &= af(a) + f(a)a^* + (a+\mathbf{1})\beta(\mathbf{1},\mathbf{1})(a^*+\mathbf{1}) + (a+\mathbf{1})\beta(a-\mathbf{1},(a-\mathbf{1})^{-1})(a^{*2}-\mathbf{1}) \\ &+\beta(a,\mathbf{1})(a^{*2}-\mathbf{1}) + \beta(\mathbf{1},\mathbf{1})(a^{*2}-\mathbf{1}) - f(a^2) - \beta(\mathbf{1},\mathbf{1}) \\ &-\beta((a^2-\mathbf{1}),(a^2-\mathbf{1})^{-1})(a^{*2}-\mathbf{1}) - \beta(\mathbf{1},\mathbf{1})(a^{*2}-\mathbf{1}) - \beta(a,a^{-1}(a^2-\mathbf{1}))a^* \\ &-\beta(a^2-\mathbf{1},(a-\mathbf{1})^{-1}-(a^2-\mathbf{1})^{-1})(a^{*2}-\mathbf{1}) - \beta(a,a^{-1})a^* - a\beta(a^{-1},\mathbf{1})a^* \\ &= af(a) + f(a)a^* - f(a^2) + (a+\mathbf{1})\beta(\mathbf{1},\mathbf{1})(a^*+\mathbf{1}) + \beta(a,\mathbf{1})(a^{*2}-\mathbf{1}) \\ &+\beta(\mathbf{1},\mathbf{1})(a^{*2}-\mathbf{1}) + \left((a+\mathbf{1})\beta(a-\mathbf{1},(a-\mathbf{1})^{-1}) - \beta(a^2-\mathbf{1},(a-\mathbf{1})^{-1})\right)(a^{*2}-\mathbf{1}) \\ &-\beta(\mathbf{1},\mathbf{1}) - \beta(a,a-a^{-1})a^* - \beta(a,a^{-1})a^* - a\beta(a^{-1},\mathbf{1})a^* \end{aligned} \tag{3.19}$$

Since β is a Hochschild $*$-2-cocycle, we have

$$(a+\mathbf{1})\beta(a-\mathbf{1},(a-\mathbf{1})^{-1}) - \beta(a^2-\mathbf{1},(a-\mathbf{1})^{-1}) = \beta(a+\mathbf{1},a-\mathbf{1})(a^*-\mathbf{1})^{-1} - \beta(a+\mathbf{1},\mathbf{1}) \tag{3.20}$$

and also

$$\beta(\mathbf{1},a) = \beta(\mathbf{1},\mathbf{1})a^* \tag{3.21}$$

Substituting (3.20) and (3.21) in (3.19), we have

$$\begin{aligned} 0 &= af(a)+f(a)a^*-f(a^2)+a\beta(\mathbf{1},\mathbf{1})(a^*+\mathbf{1})+\beta(\mathbf{1},\mathbf{1})(a^*+\mathbf{1}) \\ &\quad +\beta(a,\mathbf{1})(a^{*2}-\mathbf{1})+\beta(\mathbf{1},\mathbf{1})(a^{*2}-\mathbf{1})+\beta(a,a)(a^*+\mathbf{1})+\beta(\mathbf{1},a)(a^*+\mathbf{1}) \\ &\quad -\beta(a,\mathbf{1})(a^*+\mathbf{1})-\beta(\mathbf{1},\mathbf{1})(a^*+\mathbf{1})-\beta(a,\mathbf{1})(a^{*2}-\mathbf{1})-\beta(\mathbf{1},\mathbf{1})(a^{*2}-\mathbf{1}) \\ &\quad -\beta(\mathbf{1},\mathbf{1})-\beta(a,a)a^*+\beta(a,a^{-1})a^*-\beta(a,a^{-1})a^*-a\beta(a^{-1},\mathbf{1})a^* \\ &= af(a)+f(a)a^*-f(a^2)+\beta(a,a)a^*+\beta(a,a)+\beta(\mathbf{1},a)a^*+\beta(\mathbf{1},a) \\ &\quad -\beta(\mathbf{1},\mathbf{1})a^{*2}+\beta(\mathbf{1},\mathbf{1})-\beta(\mathbf{1},\mathbf{1})-\beta(a,a)a^*-\beta(\mathbf{1},\mathbf{1})a^* \end{aligned}$$

Then

$$f(a^2)=f(a)a^*+af(a)+\beta(a,a). \tag{3.22}$$

Now, let a be an invertible element with $\|a\|>\mathbf{1}$ and let n be a positive number such that $\|\frac{a}{n}\|<\mathbf{1}$. It follows from (3.22) that

$$f(\frac{a^2}{n^2})=f(\frac{a}{n})(\frac{a^*}{n})+\frac{a}{n}f(\frac{a}{n})+\beta(\frac{a}{n},\frac{a}{n})$$

and hence,

$$f(a^2)=f(a)a^*+af(a)+\beta(a,a).$$

Finally, let a be an arbitrary element of $\mathcal{A}$. Then, there exists a positive number n such that $\|\frac{a}{n}\|<\mathbf{1}$. It follows that $n^{-1}(n\mathbf{1}-a)=\mathbf{1}-\frac{a}{n}$ is invertible and so is $n\mathbf{1}-a$. We have the following expressions:

$$\begin{aligned} f(a^2)-2nf(a)-n^2\beta(\mathbf{1},\mathbf{1}) &= f(a-n)^2 \\ &= f(a-n)(a-n)^*+(a-n)f(a-n)+\beta(a-n,a-n) \\ &= (f(a)+n\beta(\mathbf{1},\mathbf{1}))(a-n)^*+(a-n)(f(a)+n\beta(\mathbf{1},\mathbf{1})) \\ &\quad +\beta(a,a)-\beta(n,a)-\beta(a,n)+\beta(n,n) \\ &= f(a)a^*-f(a)n+n\beta(\mathbf{1},\mathbf{1})a^*-n^2\beta(\mathbf{1},\mathbf{1})+af(a) \\ &\quad -nf(a)+na\beta(\mathbf{1},\mathbf{1})-n^2\beta(\mathbf{1},\mathbf{1})+\beta(a,a)-\beta(n,a) \\ &\quad -\beta(a,n)+\beta(n,n) \\ &= f(a)a^*-2nf(a)+n\beta(\mathbf{1},a)+af(a)+n\beta(a,\mathbf{1}) \\ &\quad +\beta(a,a)-\beta(n,a)-n\beta(a,\mathbf{1})-n^2\beta(\mathbf{1},\mathbf{1}) \end{aligned}$$

Therefore, we have

$$f(a^2)=f(a)a^*+af(a)+\beta(a,a), \qquad (a\in\mathcal{A}),$$

which means that f is a generalized Jordan $*$-derivation with an associated mapping β, as desired.

(i) $\Rightarrow$ *(iii)*.

Replacing a by $a+b$ in $f(a^2)=f(a)a^*+af(a)+\beta(a,a)$, we get that

$$f(ab)+f(ba)=f(a)b^*+af(b)+\beta(a,b)+f(b)a^*+bf(a),+\beta(b,a) \tag{3.23}$$

for all $a,b\in\mathcal{A}$. Considering $\mu=f(a(ab+ba)+(ab+ba)a)$ and using (3.23), we arrive at

$$\begin{aligned} \mu &= f(a)(b^*a^*+a^*b^*)+af(ab+ba) \\ &\quad +f(ab+ba)a^*+(ab+ba)f(a) \\ &\quad +\beta(a,ab+ba)+\beta(ab+ba,a) \\ &= 2abf(a)+a^2f(b)+af(a)b^*+2af(b)a^* \\ &\quad +baf(a)+bf(a)a^*+2f(a)b^*a^*+f(b)a^{*2} \\ &\quad +f(a)a^*b^*+a\beta(a,b)+a\beta(b,a)+\beta(a,b)a^* \\ &\quad +\beta(b,a)a^*+\beta(ab,a)+\beta(ba,a)+\beta(a,ab)+\beta(a,ba). \end{aligned}$$

On the other hand,

$$\begin{aligned}\mu &= 2f(aba)+f(a^2b+ba^2)\\ &= 2f(aba)+f(a^2)b^*+a^2f(b)+\beta(a^2,b)\\ &\quad +f(b)a^{*2}+bf(a^2)+\beta(b,a^2)\\ &= 2f(aba)+f(a)a^*b^*+af(a)b^*+\beta(a,a)b^*\\ &\quad +a^2f(b)+\beta(a^2,b)+f(b)a^{*2}\\ &\quad +bf(a)a^*+baf(a)+b\beta(a,a)+\beta(b,a^2)\end{aligned}$$

Comparing the two expressions obtained for μ and using the assumption that β is a Hochschild $*-2-$cocycle, we arrive at (3.13).

$(iii) \Rightarrow (ii)$. Taking $b = a^{-1}$ in (3.13), we achieve the required result. □

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