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Energy decay rate of solutions for a plate equation with nonlocal source and singular nonlocal damping terms

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Abstract

The initial-boundary value problem for a plate equation with a nonlocal source and singular nonlocal damping terms is considered. By using the multiplier method and weighted integral inequalities, we prove that the energy decays exponentially when the damping term has a certain singular nonlinearity. The results of this paper improve the earlier results.

Keywords: plate equation, energy decay, singular nonlocal damping, nonlocal nonlinear source
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1 Introduction

Analysis of the dynamic behavior of structures under nonlocal source and singular nonlocal damping terms is of great importance, both from the point of view of fundamental research and engineering applications, because almost all structures and their elements are subjected to it at various stages during manufacturing and installation, when used in normal and extreme conditions. Physical phenomena arising in the event of these impact, are diverse and include structural changes in materials, contact effects and propagation of nonstationary waves. To solve the problems of dynamic interaction in the scientific literature, various approaches and methods have been proposed, a review of which can be found.

In this paper, we consider the initial boundary value problem for the plate equation with singular nonlocal damping and nonlocal nonlinear source terms

$$u_{tt} + \Delta_x^2 u + \alpha(t) h\Big(\int_\Omega |\nabla_x u|^2 dx\Big) g(u_t) + f\Big(\int_\Omega |u|^p dx\Big)|u|^{p-2}u = 0 \ in\ \Omega \times (0, +\infty), \tag{1.1}$$

$$u(x,t) = 0 \ on\ x \in \partial\Omega, \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x),\ x \in \Omega, \tag{1.3}$$

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where $\Omega \subset R^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $h(s) = s^{-l}$, $l \geq 0$, $f(s) = s^\rho$, $\rho \geq 0$, α and g are given functions to be specified later.

Recently, Narciso[17] consider the well-posedness, stability and long-time behavior of solutions to the following nonlocal nonlinear plate equation

$$u_{tt} + \Delta_x^2 u + M(\int_\Omega |\nabla_x u|^2 dx)|u_t|^\gamma u_t + f(\int_\Omega |u|^{\rho+2} dx)|u|^\rho u = h(x), \tag{1.4}$$

where $\gamma, \rho > 0$, $f(s) \geq 0$ is nonlinear source coefficient and $M(s) > 0$ corresponds to a nonlocal damping coefficient. This kind of nonlocal dissipative effect, namely $M(\int_\Omega |\nabla_x u|^2 dx)u_t$, was introduced by Lange and Perla Menzala [9] for the beam equation

$$u_{tt} + \Delta_x^2 u + M(||\nabla_x u||_2^2)u_t = 0 \ in \ R^n \tag{1.5}$$

and subsequently studied by Cavalcanti et al. [2] in another context. When the Kirchhoff function M is nondegenerate, that is, there exists $m_0 > 0$ such that $M(s) \geq m_0$ for all $s \geq 0$, then a full damping is in place and the exponential decay follows trivially [3]. For the related study on this topic, one can see also [5, 6, 7, 10, 11, 14, 18, 25, 24, 23, 27] and reference therein, where global existence and long-time behavior of solutions of the problem were proved..

When the Kirchhoff function M is degenerate, that is, M can be zero at zero, in [3] the authors considered the case $M(s) = s$ for $s \geq 0$, where well-posedness results are discussed, as well as the exponential stability of the solutions. Pucci and Saldi [19] extended the results in [3]. They considered the question of the asymptotic stability of solutions of Kirchhoff systems, governed by the fractional p-Laplacian operator, with an external force and nonlinear degenerate nonlocal damping terms. Very recently, by potential well theory, Zhang et.al [26] showed the asymptotic stability of energy in presence of a degenerate damping of polynomial type $M(s) = s^l$ when the initial energy is small. Also, They firstly derive some sufficient conditions on initial data which lead to finite time blow-up.

As far as we know, another kind of nonlocal fractional damping is given by

$$M(||\nabla_x u||^2)(-\Delta_x u_t)^\theta, 0 \leq \theta \leq 1. \tag{1.6}$$

There exists a huge literature regarding hyperbolic equations which involves nonlocal fractional damping term $M > 0$. For the related study on this topic, one can see [4, 12, 13, 20, 21, 22] and references therein. The majority of these papers deal with the global existence and long-time behavior of solutions for the following equation

$$u_{tt} + a\Delta_x^2 u - \phi(||\nabla_x u||_2^2)\Delta_x u + M(||\nabla_x u||_2^2)(-\Delta_x)^\theta u_t + f(u) = h. \tag{1.7}$$

As a matter of fact, to our best knowledge, there is no stability result for wave models with degenerate nonlocal damping such as $||\nabla_x u||_2^2(-\Delta_x)^\theta u_t$ since the traditional multipliers do not work for this kind of degenerate nonlocal damping term.

Our goal in this paper is to examine the asymptotic stability of the energy for problem (1.1)-(1.3). The nonlocal damping term given by $||\nabla_x u||^{-l} g(u_t)$ are different from those in above mentioned papers. Motivated by a method introduced by Martinez [15] to study the decay rate of solutions to the wave equation $u_{tt} - \Delta_x u + g(u_t) = 0$, we give an explicit energy decay estimates of the solutions to the problem (1.1)-(1.3).

It is also worth mentioning that the initial boundary value problem of the wave equation with singular nonlinearities of the form

$$u_{tt} - u_{xx} + |u|^{-r} g(u_t) + |u|^{-\alpha} u = 0 \tag{1.8}$$

was studied by [1, 16].

The contents of this paper is organized as follows. In Sect. 2, we prepare some material needed in our proof and state the energy functional. In Sect. 3, we state and prove our main result.

2 Preliminaries

First assume the following hypotheses:

(A1) $\alpha : R^+ \longrightarrow R^+$ is a nonincreasing function of class $C^1(R^+)$ satisfying

$$\int_0^{+\infty} \alpha(\tau) d\tau = +\infty. \tag{2.1}$$

(A2) $h(s) = s^{-l}$, $l \geq 0$.

(A3) $f(s) = s^{\rho}$, $\rho \geq 0$.

(A4) $g : R \longrightarrow R$ is a nondecreasing function of class $C(R)$ such that: there exist positive constants $c_i, i = 1, 2, 3, 4$ such that $g(y)y \geq 0$, $c_1|y|^m \leq |g(y)| \leq c_2|y|^{\theta}$, $m \geq 1, m \geq \theta \geq \frac{1}{m}$, for $|y| \leq 1$, and $c_1'|y| \leq |g(y)| \leq c_2'|y|^r$, $1 \leq r \leq \frac{n}{n-2}$, for $|y| \geq 1$.

Now, we introduce the energy functional

$$E(t) = \frac{1}{2}||u_t(t)||^2 + \frac{1}{2}||\Delta_x u(t)||^2 + \frac{1}{p(\rho+1)}||u||_p^{p(\rho+1)}. \tag{2.2}$$

Multiplying (1.1) by u_t and integrating over Ω, we get

$$\frac{d}{dt}E(t) = -\sigma(t)||\nabla_x u(t)||^{-2l}\int_\Omega g(u_t)u_t dx \leq 0. \tag{2.3}$$

We denote C_0 the best constant of Poincar'e inequality $\int_\Omega |u|^2 dx \leq C_0 \int_\Omega |\nabla_x u|^2 dx$.

Finally, we present the following lemma by Martinez [15] which plays important role in our proof.

Lemma 2.1 [8, 15] Let $E : R^+ \to R^+$ be a non-increasing function and $\phi : R^+ \to R^+$ a strictly increasing function of class C^2 such that $\phi(0) = 0$ and $\phi(t) \to +\infty$ as $t \to +\infty$. Assume that there exist $\sigma \geq 0$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{\sigma+1}(s)\phi'(s)ds \leq \frac{1}{\omega}E^{\sigma}(0)E(S), \forall S \geq 0.$$

Then E has the following decay properties

$$E(t) \leq E(0)exp(1 - \omega\phi(t)), \quad \forall t \geq 0, \quad if \quad \sigma = 0,$$
$$E(t) \leq E(0)(\frac{1+\sigma}{1+\omega\sigma\phi(t)})^{\frac{1}{\sigma}}, \quad \forall t \geq 0 \quad if \quad \sigma > 0.$$

3 Energy decay rate of solutions

In this section, we give our main result.

Theorem 3.1 Assume that $(u_0, u_1) \in V \times L^2(\Omega)$ and (A1)-(A4) hold, then there exists $K > 0$ such that the solution energy of the problems (1.1)-(1.3) satisfies the following decay rates

$$E(t) \leq C(E(0))exp(1 - \int_0^t \alpha(s)ds), \quad \forall t \geq 0, \quad if \quad m = 1,$$
$$E(t) \leq (\frac{C(E(0))}{\int_0^t \alpha(s)ds})^{\frac{2}{m-1}}, \quad \forall t \geq 0 \quad if \quad m > 1.$$

Proof Denote $\phi(t) = \int_0^t \alpha(s)ds$, then $\phi(t) \in C^2(R^+, R^+)$ is a strictly increasing function such that $\phi(0) = 0$ and $\phi(t) \to +\infty$ as $t \to +\infty$. Multiplying equation (1.1) by $\phi'(t)E(t)^q(t)u(t)$ and integrating the equation over $\Omega \times (S,T)$ for $0 < S \leq T < +\infty$, where q is a positive constant which will be specified later, we have

$$\begin{aligned}
0 = &[\phi'(t)E^q \int_\Omega uu_t dx]|_S^T - \int_S^T [\phi'(t)qE^{q-1}E' + \phi''(t)E^q]\int_\Omega uu_t dxdt \\
&- \int_S^T \phi'(t)E^q \int_\Omega |u_t|^2 dxdt + \int_S^T \phi'(t)E^q \int_\Omega |\Delta_x u|^2 dxdt \\
&+ \int_S^T \phi'(t)\alpha(t)E^q \int_\Omega u||\nabla_x u||^{-2l}g(u_t)dxdt + \int_S^T \phi'(t)E^q||u||_p^{p\rho}\int_\Omega |u|^p dxdt.
\end{aligned}$$

Then, we deduce that

$$
\begin{aligned}
&2\int_S^T \phi'(t)E^{q+1}dt = \int_S^T \phi'(t)E^q(||u_t||^2 + ||\Delta_x u||^2 + \frac{2}{p(\rho+1)}||u||_p^{p(\rho+1)})dt \\
&= -[\phi'(t)E(t)^q \int_\Omega uu_t dx]|_S^T + 2\int_S^T E(t)^q\phi'(t)||u_t||^2 dt \\
&+ \int_S^T [\phi'(t)qE(t)^{q-1}E' + \phi''(t)E(t)^q]\int_\Omega uu_t dxdt \\
&- \int_S^T \phi'(t)\alpha(t)E(t)^q||\nabla_x u||^{-2l}\int_\Omega ug(u_t)dxdt - \frac{p(\rho+1)-2}{p(\rho+1)}\int_S^T \phi'(t)E(t)^q||u||_p^{p(\rho+1)}dt \\
&\leq -[\phi'(t)E(t)^q \int_\Omega uu_t dx]|_S^T + 2\int_S^T E(t)^q\phi'(t)||u_t||^2 dt \\
&+ \int_S^T [\phi'(t)qE(t)^{q-1}E' + \phi''(t)E(t)^q]\int_\Omega uu_t dxdt \\
&- \int_S^T \phi'(t)\alpha(t)E(t)^q||\nabla_x u||^{-2l}\int_\Omega ug(u_t)dxdt. \qquad (3.1)
\end{aligned}
$$

Using Holder's inequality, Poincare's inequality, the fact that $E(t)$ is non-increasing and non-negative function on R^+ and $\phi'(t)$ is a bounded nonnegative function on R^+ (and we denote by α_0 its maximum), we have

$$
\begin{aligned}
&|-\phi'(t)[E^q\int_\Omega uu_t dx]|_S^T| \leq E(S)^q[\phi'(t)||u||||u_t|||_{t=S} + \phi'(t)||u||||u_t|||_{t=T}] \\
&\leq \alpha_0 E(S)^q(E(T)+E(S)) \leq C_1 E(S)^{q+1}, \qquad (3.2)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\int_S^T [(\phi'(t)qE^{q-1}E' + \phi''(t)E^q)\int_\Omega uu_t dx]dt| \\
&\leq \int_S^T |(\phi'(t)qE^{q-1}E' + \phi''(t)E^q)|(||u||^2 + ||u_t||^2)dt \\
&\leq C_2\int_S^T E^q(-E'(t))dt + C_3\int_S^T E^{q+1}(-\phi''(t))dt \\
&\leq C_2 E(S)^q\int_S^T (-E')dt + C_3 E(S)^{q+1}\int_S^T (-\phi''(t))dt \leq C_4 E(S)^{q+1}. \qquad (3.3)
\end{aligned}
$$

Then, we conclude from (3.1), using the estimations (3.2) and (3.3), that

$$
\begin{aligned}
&2\int_S^T \phi'(t)E^{q+1}dt \leq C_5 E(S)^{q+1} + 2\int_S^T \phi'(t)E(t)^q||u_t||^2 dt \\
&- \int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_\Omega ug(u_t)dxdt. \qquad (3.4)
\end{aligned}
$$

Now, we denote

$$\Omega_1 = \{x\in\Omega, |u_t|\leq 1\}, \Omega_2 = \{x\in\Omega, |u_t|\geq 1\},$$

then we have

$$2\int_S^T \phi'(t)E^q||u_t||^2 dt = 2\int_S^T \phi'(t)E^q\int_{\Omega_1}|u_t|^2 dxdt + 2\int_S^T \phi'(t)E^q\int_{\Omega_2}|u_t|^2 dxdt, \qquad (3.5)$$

and

$$
\begin{aligned}
&\int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_{\Omega_1} ug(u_t)dxdt \\
&= \int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_{\Omega_1} ug(u_t)dxdt + \int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_{\Omega_2} ug(u_t)dxdt. \qquad (3.6)
\end{aligned}
$$

Using assumption (A3), Holder's inequality and (2.3), we get

$$
\begin{aligned}
&2\int_S^T \phi'(t)E^q\int_{\Omega_1}|u_t|^2dxdt = 2\int_S^T \phi'(t)E^q\int_{\Omega_1} u_t^{\frac{2}{m+1}}(u_t^m)^{\frac{2}{m+1}}dxdt\\
&\le 2\int_S^T \phi'(t)E^q\int_{\Omega_1}(u_tg(u_t))^{\frac{2}{m+1}}dxdt\\
&\le 2\int_S^T \phi'(t)^{\frac{m-1}{m+1}}E^q||\nabla_x u||^{\frac{4l}{m+1}}\int_\Omega(\alpha(t)||\nabla_x u||^{-2l}u_tg(u_t))^{\frac{2}{m+1}}dxdt\\
&\le 2\int_S^T \phi'(t)^{\frac{m-1}{m+1}}E^q||\nabla_x u||^{\frac{4l}{m+1}}(-E')^{\frac{2}{m+1}}dt\\
&\le C_6\int_S^T \phi'(t)^{\frac{m-1}{m+1}}E^qE^{\frac{2l}{m+1}}(-E')^{\frac{2}{m+1}}dt\\
&\le C_7\epsilon_1^{-\frac{m+1}{2}}\int_S^T(-E')dt + C_8\epsilon_1^{\frac{m+1}{m-1}}\int_S^T\phi'(t)E^{q\frac{m+1}{m-1}+\frac{2l}{m-1}}dt, \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
&2\int_S^T \phi'(t)E^q\int_{\Omega_2}|u_t|^2dxdt \le 2\int_S^T \phi'(t)E^q\int_{\Omega_2}u_tg(u_t)dxdt\\
&\le C_9\int_S^T E^q||\nabla_x u||^{2l}\alpha(t)||\nabla_x u||^{-2l}\int_\Omega u_tg(u_t)dxdt\\
&\le C_{10}\int_S^T E^{q+l}(-E')dt. \qquad (3.8)
\end{aligned}
$$

By the Holder's inequality, the Sobolev embedding, the assumptions (A3) and (A4), and the expressions of $E(t)$, we get

$$
\begin{aligned}
&\int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_{\Omega_1}ug(u_t)dxdt\\
&\le \int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_{\Omega_1}ug(u_t)dxdt\\
&= \int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_{\Omega_1}u(g(u_t))^{\frac{\theta}{\theta+1}}(g(u_t))^{\frac{1}{\theta+1}}dxdt\\
&\le C_{11}\int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_{\Omega_1}u(g(u_t)u_t)^{\frac{\theta}{\theta+1}}dxdt\\
&\le C_{12}\int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}||u||_{\theta+1}(\int_\Omega g(u_t)u_tdx)^{\frac{\theta}{\theta+1}}dt\\
&\le C_{13}\int_S^T \phi'(t)\alpha^{\frac{1}{\theta+1}}(t)E^q||\nabla_x u||^{-2l}||\nabla_x u||^{1+\frac{2l\theta}{\theta+1}}(\int_\Omega \alpha(t)||\nabla_x u||^{-2l}g(u_t)u_tdx)^{\frac{\theta}{\theta+1}}dt\\
&\le C_{14}\int_S^T \phi'(t)\alpha^{\frac{1}{\theta+1}}(t)E^q||\nabla_x u||^{-2l}||\nabla_x u||^{1+\frac{2l\theta}{\theta+1}}(\int_\Omega \alpha(t)||\nabla_x u||^{-2l}g(u_t)u_tdx)^{\frac{\theta}{\theta+1}}dt\\
&\le C_{14}\int_S^T \phi'(t)\alpha^{\frac{1}{\theta+1}}(t)E^qE^{\frac{\theta+1-2l}{2(\theta+1)}}(-E')^{\frac{\theta}{\theta+1}}dt\\
&\le C_{15}\epsilon_2^{-\frac{\theta+1}{\theta}}\int_S^T(-E')dt + C_{16}\epsilon_2^{\theta+1}\int_S^T E^{q(1+\theta)+\frac{\theta+1-2l}{2}}\phi'(t)dt, \qquad (3.9)
\end{aligned}
$$

and

$$\begin{aligned}
&\int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_{\Omega_2} ug(u_t)dxdt\\
&=\int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_{\Omega_2} u(g(u_t))^{\frac{r}{r+1}}(g(u_t))^{\frac{1}{r+1}}dxdt\\
&\le C_{16}\int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}\int_{\Omega_2} u(g(u_t)u_t)^{\frac{r}{r+1}}dxdt\\
&\le C_{17}\int_S^T \phi'(t)\alpha(t)E^q||\nabla_x u||^{-2l}||u||_{r+1}(\int_{\Omega} g(u_t)u_t dx)^{\frac{r}{r+1}}dt\\
&\le C_{18}\int_S^T \phi'(t)\alpha^{\frac{1}{r+1}}(t)E^q||\nabla_x u||^{-2l}||\nabla_x u||^{1+\frac{2lr}{r+1}}(\int_{\Omega} \alpha(t)||\nabla_x u||^{-2l}g(u_t)u_t dx)^{\frac{r}{r+1}}dt\\
&\le C_{19}\int_S^T \phi'(t)\alpha^{\frac{1}{r+1}}(t)E^q E^{\frac{r+1-2l}{2(r+1)}}(-E')^{\frac{r}{r+1}}dt\\
&\le C_{20}\epsilon_3^{-\frac{r+1}{r}}\int_S^T(-E')dt + C_{21}\epsilon_3^{r+1}\int_S^T E^{q(r+1)+\frac{r+1-2l}{2}}\phi'(t)dt. \qquad (3.10)
\end{aligned}$$

Substituting the estimates (3.7)- (3.8) into (3.5) and the estimates (3.9)- (3.10) into (3.6), and then (3.4) can be rewritten

$$\begin{aligned}
&2\int_S^T \phi'(t)E^{q+1}dt \le C_5E(S)^{q+1} + (C_7\epsilon_1^{-\frac{m+1}{2}} + C_{15}\epsilon_2^{-\frac{\theta+1}{\theta}} + C_{20}\epsilon_3^{-\frac{r+1}{r}})\int_S^T(-E')dt\\
&+C_8\epsilon_1^{\frac{m+1}{m-1}}\int_S^T \phi'(t)E^{q\frac{m+1}{m-1}+\frac{2l}{m-1}}dt + C_{10}\int_S^T E^{q+l}(-E')dt\\
&+C_{16}\epsilon_2^{\theta+1}\int_S^T E^{q(1+\theta)+\frac{\theta+1-2l}{2}}\phi'(t)dt + C_{21}\epsilon_3^{r+1}\int_S^T E(t)^{q(r+1)+\frac{r+1-2l}{2}}\phi'(t)dt. \qquad (3.11)
\end{aligned}$$

If $r=\theta=1+2l$, then we take $q=0$ and choose ϵ_1,ϵ_2 and ϵ_3 small enough, by the monotonicity of $E(t)$, we deduce from (3.11) that

$$\int_S^T \phi'(t)Edt \le C_{22}E(S). \qquad (3.12)$$

If $0 < min\{r,\theta\} < 1+2l$, then we take $q = min\{\frac{1+2l-r}{2r}, \frac{1+2l-\theta}{2\theta}\}$ and choose ϵ_1,ϵ_2 and ϵ_3 small enough, by the monotonicity of $E(t)$, we deduce from (3.11) that

$$\int_S^T \phi'(t)Edt \le C_{23}E(S). \qquad (3.13)$$

By (3.12) and (3.13), letting $T\to+\infty$, then, by Lemma 2.1, we get the result.

4 Conclusion

In this paper, we consider with the initial-boundary value problem for a plate equation with nonlocal source and singular nonlocal damping terms. By using multiplier method and weighted integral inequalities, we give an explicit energy decay estimates of the solutions when the damping term has a certain singular nonlinearity. The results of this paper improves the earlier results. This method can also be applied to a system of plate equations with nonlocal source and singular nonlocal damping terms.

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