Int. J. Nonlinear Anal. Appl. 14 (2023) 4, 297–312
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23396.2530

# A novel approach for convergence and stability of Jungck-Kirk-Type algorithms for common fixed point problems in Hilbert spaces

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*(Communicated by Abasalt Bodaghi)*

**Abstract**

In this paper, two novel iteration algorithms called Jungck-DI-Noor-multistep and Jungck-DI-SP-multistep iterative schemes are introduced and studied. Using their strong convergence, a common fixed point of nonself mappings was achieved without any imposition of 'sum conditions' on the control sequences. Further, we studied and proved the stability results of our new iterative schemes in the setting of a real Hilbert space. Our results improve, generalize and unify several known results currently in the literature.

Keywords: Strong convergence, Jungck-DI-Noor-multistep iterative scheme, Jungck-DI-SP-multistep iterative scheme, Stability, Contractive operator, fixed point, Real Hilbert space
2020 MSC: 47H09, 47H10, 47J05, 65J15

## 1 Introduction

Several real life problems of the form

$$\Gamma(x) = z \tag{1.1}$$

arising from physical formulations can equivalently be transformed into a fixed point problem of the form

$$\Gamma(x) = x. \tag{1.2}$$

The solution of (1.2) can be achieved using approximate fixed point theorem which, among other things, unlock the information on existence or existence and uniqueness of fixed point of the original equation.

Let $(Y, \rho)$ be a complete metric space and $\Gamma : Y \longrightarrow Y$ a selfmap of $Y$. Suppose that $F_\Gamma = \{q \in Y : \Gamma q = q\}$ is the set of fixed points of $\Gamma$. Over the years, a lot of iterative schemes for which the fixed point of (1.2) could be approximated has been developed and implemented in the current literature, see for example, [1], [2], [3], [6], [10], [11], [17], [22], [25], [27], [38] and the references therein.

In [17], Jungck introduced and studied the following iterative scheme: Let $Z$ be a Banach space, $Y$ an arbitrary set and $S, \Gamma : Y \longrightarrow Z$ such that $\Gamma(Y) \subseteq S(Y)$. For arbitrary $x_0 \in Y$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$Sx_{n+1} = \Gamma x_n, n = 1, 2, \cdots \tag{1.3}$$

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*Received:* March 2021 *Accepted:* May 2021

ia called Jungck iterative scheme. Subsequently, different authors have generalised (1.3) in different spaces. For instance, Olaeru and Akewe [22] introduced and studied the following iteration algorithm for the approximation of fixed points of a pair of generalised contractive-like operators without any assumption of injectivity on the operator (their results were obtained using a pair of weakly compartible maps, $S, \Gamma$) in the setting of a real Banach space:

Let $Z$ be a real Banch space, $Y$ an arbitrary set and $S, \Gamma : Y \longrightarrow Z$ two nonself mappings such that $\Gamma(Y) \subseteq S(Y)$. For $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty}$ as follows

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n \Gamma t_n^{1.}; \\ St_n^1 = (1-\gamma_n^i)Sx_n + \gamma_n^i \Gamma t_n^{i+1}, i = 1, 2, \cdots, k-1 \\ St_n^{k-1} = (1-\gamma_n^{k-1})Sx_n + \gamma_n^{k-1}\Gamma x_n, k \geq 2, n \geq 0, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\gamma_n^i\}_{n=0}^{\infty}, i = 1, 2, \cdots, k-1$ are real sequences in $[0,1]$ such that $\sum_{n=0}^{\infty} = \infty$. The iterative scheme (1.4) is called Jungck-multistep iterative scheme. Note that Jungck-multistep iterative scheme (1.4) includes, as special cases, the following iteration algorithms: Jungck-Noor [21], Jungck-lshikawa [25] and Jungck-Mann [38] iterative schemes.

Recently, Akewe and Mogbademu [4] introduced, studied and proved convergence and stability results of more general iterative schemes of the Jungck-Kirk-type in the following way: Let $Z$ be a real Banach space, $Y$ an arbitrary set and $S, \Gamma : Y \longrightarrow Z$ two nonself mappings such that $\Gamma(Y) \subseteq S(Y)$. For $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty}$ in the sense of Kirk [18] as follows:

$$\begin{cases} Sx_{n+1} = \alpha_{n,0}Sx_n + \sum_{i=1}^{\ell_1} \alpha_{n,i}\Gamma^i t_n^{1.}, \sum_{i=1}^{\ell_1} \alpha_{n,i} = 1; \\ St_n^j = \gamma_{n,0}^i Sx_n + \sum_{i=1}^{\ell_{1+1}} \gamma_{n,i}^j \Gamma t_n^{j+1}, \sum_{i=1}^{\ell_{1+1}} \gamma_{n,i}^j, j = 1, 2, \cdots, k-1 \\ St_n^{k-1} = \sum_{i=0}^{\ell_k} \gamma_n^{k-1}\Gamma^i x_n, \sum_{i=0}^{\ell_k} \gamma_n^{k-1} = 1, k \geq 2, n \geq 0 \end{cases} \tag{1.5}$$

and

$$\begin{cases} Sx_{n+1} = \alpha_{n,0}St_n^1 + \sum_{i=1}^{\ell_1} \alpha_{n,i}\Gamma^i t_n^{1.}, \sum_{i=1}^{\ell_1} \alpha_{n,i} = 1; \\ St_n^j = \gamma_{n,0}^i St_n^{j+1} + \sum_{i=1}^{\ell_{1+1}} \gamma_{n,i}^j \Gamma t_n^{j+1}, \sum_{i=1}^{\ell_{1+1}} \gamma_{n,i}^j, j = 1, 2, \cdots, k-1 \\ St_n^{k-1} = \sum_{i=0}^{\ell_k} \gamma_n^{k-1}\Gamma^i x_n, \sum_{i=0}^{\ell_k} \gamma_n^{k-1} = 1, k \geq 2, n \geq 0, \end{cases} \tag{1.6}$$

where $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k$, for each $j, \alpha_{n,i} \geq 0, \alpha_{n,0} \neq 0, \gamma_{n,i}^j \geq 0, \gamma_{n,0} \neq 0$, for each $j$, $\alpha_{n,i}, \gamma_{n,j}^i \in [0,1]$ for each $j$ and $\ell_i, \ell_k$ are fixed integers (for each $j$). They called (1.5) and (1.6) Jungck-Kirk-multistep-Noor and Jungck-Kirk-multistep-SP iterative schemes, respectively. Again, we note that (1.5) includes Jungck-Kirk-Noor, Jungck-Kirk-lshikawa and Jungck-Mann iterative schemes. Indeed, if $k = 3$ *in* (1.5), we get Jungck-Kirk-Noor [22]; if $k = 2$ *in* (1.5), we obtain Jungck-Kirk-lshikawa [23] and if $k = 2$ *and* $\ell_2 = 0$ *in* (1.5), we have Jungck-Kirk-Mann [23] iterative schemes.

Stability results on $\Gamma$-stable (which is paramount in practical sense) was initiated by Ostrowski [30], in which case he proved that Picard's iterative scheme is stable under Banach contractive condition. Afterwards, Osilike et al [28] improved this result on $S, \Gamma$-stable as follows:

Let $Z$ be a real Banach space, $Y$ an arbitrary set, $z$ a coincidence point of $S$ *and* $\Gamma$. Let $S, \Gamma : Y \longrightarrow Z$ such that $S(Y) \subseteq \Gamma(Y)$. For every $x_0 \in Y$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by

$$Sx_{n+1} = f(\Gamma x_n), n \geq 0 \tag{1.7}$$

converge to $q$. Suppose that $\{z_n\}_{n=0}^{\infty} \subset Z$ be an arbitrary sequence and put $\epsilon_n = d(Sz_n, f(\Gamma, x_n))$, $n = 1, 2, \cdots$. Then, the iterative sequence (1.7) will be called $(S, \Gamma)$-stable if and only if $\epsilon_n \to 0$ *as* $n \to \infty$ implies that $Sz_n \to q$ *as* $n \to \infty$. For more information on the stability of different iterative schemes in different spaces, interested readers should consult [6], [8], [10], [12], [15], [25], [26], [28], [30], [31], [32], [38] and the references therein.

**Remark 1.1.** Interesting and remarkable as the above results and their inclusions seem, one, however, wonders the implications of the sum conditions ($\sum_{i=1}^{\ell_1} \alpha_{n,i} = 1$ *and* $\sum_{i=1}^{\ell_{1+1}} \gamma_{n,i}^j = 1,$ *where* $j = 1, 2, \cdots, k-1, \ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k$, for each $j, \alpha_{n,i} \geq 0, \alpha_{n,0} \neq 0, \gamma_{n,i}^j \geq 0, \gamma_{n,0} \neq 0$, for each $j$, $\alpha_{n,i}, \gamma_{n,j}^i \in [0,1]$) . For instance, the sum condition implies that

1. for large $\ell_k, k \geq 1$, one has to choose different points of the sequences $\{\alpha_{n,i}\}_{n=0}^{\infty}$ *and* $\{\gamma_{n,i}^j\}_{n=0}^{\infty}$ that would guarantee instant generation of such a finite family of control sequences such that $\sum_{i=1}^{\ell_1} \alpha_{n,i} = 1$ *and* $\sum_{i=1}^{\ell_{1+1}} \gamma_{n,i}^j = 1$ which might be almost impossible and

2. one has to make adrquate provision of computing time and memory space for the computation and storage of the bulky, complex and windy task of generating $\sum_{i=1}^{\ell_1} \alpha_{n,i} = 1$ *and* $\sum_{i=1}^{\ell_1+1} \gamma_{n,i}^{j} = 1$, which invariably leads to enormous computational cost.

In an attempt to solve the above challenges enlisted in Remark 1.1, the following question ensued:

*Can one construct more efficient and cost effective iterative schemes that would guarantee the results in [4] without imposing the sum conditions ($\sum_{i=1}^{\ell_1} \alpha_{n,i} = 1$ and $\sum_{i=1}^{\ell_1+1} \gamma_{n,i}^{j} = 1$) on the control parameters?*

Inspired and moltivated by the above challenges raised in Remark 1.1, the aim of this paper is to provide an affirmative answer to Question 1.1 using the method of linear combination of products introduced in [16].

## 2 Preliminary

The following definitions, lemmas and propositions will be needed to prove our main results.

**Definition 2.1.** (see [30]) Let (Y, d) be a metric space and let $\Gamma : Y \longrightarrow Y$ be a self-map of $Y$. Let $\{x_n\}_{n=0}^{\infty} \subseteq Y$ be a sequence generated by an iteration scheme

$$x_{n+1} = g(\Gamma, x_n), \tag{2.1}$$

where $x_0 \in Y$ is the initial approximation and $g$ is some function. Suppeose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point $q$ of $\Gamma$. Let $\{t_n\}_{n=0}^{\infty} \subseteq Y$ be an arbitrary sequence and set $\epsilon_n = d(t_n, g(\Gamma, t_n)), n = 1, 2, \cdots$ Then, the iteration scheme (2.1) is called $\Gamma$-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = q$.

Note that in practice, the sequence $\{t_n\}_{n=0}^{\infty}$ could be obtained in the following manner: let $x_0 \in Y$. Set $x_{n+1} = g(\Gamma, x_n)$ and let $t_0 = x_0$. Now, $x_1 = g(\Gamma, x_0)$ because of rounding in the function $\Gamma$, and a new value $t_1$ (approximately equal to $x_1$) might be calculated to give $t_2$, an approximate value of $g(\Gamma, t_1)$. The procedure is continued to yield the sequence $\{t_n\}_{n=0}^{\infty}$, an approximate sequence of $\{x_n\}_{n=0}^{\infty}$.

**Definition 2.2.** Let $Z$ be a Banach space and $Y$ an arbitrary set. Let $\Gamma, S : Y \longrightarrow Z$ be two nonself mappings such that $S(Y) \subseteq \Gamma(Y)$. A point $q \in Z$ ia said to be a coincident point of a pair of self maps $\Gamma, S$ if there exists a point $p$(called a point of coincidence) in $Z$ such that $p = Sq = \Gamma q$. $\Gamma, S$ (considered as self maps) are weakly compartible if they commute at their coincident points; that is, if $Sq = \Gamma q$ for some $q \in Z$, then $S\Gamma q = \Gamma Sq$.

**Definition 2.3.** Let $Z$ be a Banach space and $Y$ an arbitrary set. Let $\Gamma, S : Y \longrightarrow Z$ be two nonself mappings such that $S(Y) \subseteq \Gamma(Y)$ and $S(Y)$ is a complete subspace of $Z$. For $z, t \in Y$ and $\gamma \in (0, 1)$, we get

$$\|\Gamma z - \Gamma t\| \leq \gamma max\left\{\|Sz - St\|, \frac{\|Sz - \Gamma z\| + \|St - \Gamma t\|}{2}, \frac{\|Sz - \Gamma t\| + \|St - \Gamma z\|}{2}\right\} \tag{2.2}$$

$$\|\Gamma x - \Gamma y\| \leq \gamma max\left\{\|Sz - St\|, \frac{\|Sxz\Gamma z\| + \|St - \Gamma t\|}{2}, \|Sz - \Gamma t\|, \|St - \Gamma z\|\right\} \tag{2.3}$$

There exists a real number $\delta \in [0, 1)$ and $L > 0$ such that for every $z, t \in Y$, the inequality

$$\|\Gamma z - \Gamma t\| \leq \delta\|Sz - St\| + L\|Sz - \Gamma z\| \tag{2.4}$$

holds.

There exists a real number $\delta \in [0, 1)$ and a monotone increasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\phi(0) = 0$ and for every $z, t \in Y$, the inequality

$$\|\Gamma z - \Gamma t\| \leq \frac{\delta\|Sz - St\| + \phi(\|Sz - \Gamma z\|)}{1 + M\|Sz - \Gamma t\|} \tag{2.5}$$

$$\|\Gamma z-\Gamma t\|\le\delta\|Sz-St\|+\phi(\|Sz-\Gamma z\|) \tag{2.6}$$

It is shown in (Proposition 1, [22]) that (2.2)-(2.6) are related in the following manner:

$$(2.2)\Rightarrow(2.3)\Rightarrow(2.4)\Rightarrow(2.5)\Rightarrow(2.6) \tag{2.7}$$

However, the converses of (2.7) are not true; see, for example, [22] for further details.

**Lemma 2.4.** (see, e.g., [6]) Let $\{\tau_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\tau_n\to 0\ as\ n\to\infty$. For $0\le\delta<1$, let $\{w_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $w_{n+1}\le\delta w_n+\tau_n, n=0,1,2,\cdots$ Then, $w_n\to 0\ as\ n\to\infty$.

**Lemma 2.5.** (see, e.g., [27]) Let (Y, $\|\ .\|$) be a normed space, the self-map $\Gamma:Y\longrightarrow Y$ satisfies (2.2) and $\psi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be a monotone increasing subadditive function such that $\psi(0)=0,\psi(Mt)=M\psi(t),M\ge 0,t\in\mathbb{R}^+$. Then, $\forall i\in\mathbb{N}\ and\ \forall s,t\in Y$, we have

$$\|\Gamma^j s-\Gamma^j t\|\le\rho^j\|s-t\|+\sum_{i=0}^{j}\binom{j}{i}\rho^{j-1}\phi(\|s-\Gamma s\|). \tag{2.8}$$

**Lemma 2.6.** (see, e.g, [22]) Let $(Z,\|.\|)$ be a normed linear space and $\Gamma,S:Y\longrightarrow Z$ be nonself maps of $Z$ satisfying (2.2) such tha $S(Y)\subseteq\Gamma(Y),\|S^2x-\Gamma(Sx)\|\le\|Sx-\Gamma y\|,\forall x\in Y$ and $\forall x,y\in Y,\|S^2x-Sy\|\le\|Sx-Sy\|$. Let $\phi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be a sublinear, monotone increasing function such that $\phi(0)=0$. Let $z$ be the coincident point of $S,\Gamma,S^i,\Gamma^i(i.e,Sz=\Gamma z=p\ and\ S^iz=\Gamma^iz=p)$. Then, $\forall j\in\mathbb{N},L\ge 0,\ and\ \forall x,y\in Y$, the inequality

$$\|S^ix-S^iy\|\le\nu^j\|\Gamma x-\Gamma y\|+\sum_{i=0}^{j}\binom{j}{i}\nu^{j-i}\phi(\|Sx-\Gamma y\|) \tag{2.9}$$

holds.

**Proposition 2.7.** (see, e.g., [16]) Let $\{\alpha_i\}_{i=1}^{\infty}\subseteq\mathbb{N}$ be a countable subset of the set of real numbers $\mathbb{R}$, where $k$ is a fixed nonnegative integer and $N\mathbb{N}$ is any integer with $k+1\le N$. Then, the following holds:

$$\alpha_k+\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)+\prod_{j=k}^{N}(1-\alpha_j)=1. \tag{2.10}$$

**Proposition 2.8.** (see, e.g., [16]) Let $t,u$ and $v$ be arbitrary elements of a real Hilbert space $H$. Let $k$ be any fixed nonnegetive integer and $N\in\mathbb{N}$ be such that $k+1\le N$. Let $\{v_i\}_{i=1}^{N-1}\subseteq H$ and $\{\alpha_i\}_{i=1}^{N}\subseteq[0,1]$ be a countable finite subset of $H$ and $\mathbb{R}$, respectively. Define

$$y=\alpha_kt+\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1}+\prod_{j=k}^{N}(1-\alpha_j)v.$$

Then,

$$\begin{aligned}\|y-u\|^2 &= \alpha_k\|t-u\|^2+\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2+\prod_{j=k}^{N}(1-\alpha_j)\|v-u\|^2\\ &-\alpha_k\Big[\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2+\prod_{j=k}^{i-1}(1-\alpha_j)\|t-v\|^2\Big]\\ &-(1-\alpha_k)\Big[\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-(\alpha_{i+1}+w_{i+1})\|^2\\ &+\alpha_N\prod_{j=k}^{i-1}(1-\alpha_j)\|v-v_{N-1}\|^2\Big],\end{aligned}$$

where $w_k=\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1}+\prod_{j=k}^{i-1}(1-\alpha_j)v,k=1,2,\cdots,N$ and $w_n=(1-c_n)v$.

## 3 Main Results I

Let $D$ be a nonempty subset of a real Banach space $E$, $S, \Gamma : D \longrightarrow E$ nonself commuting maps of $D$ with $\Gamma(D) \subseteq S(D)$ and $x_0 \in D$- Then, the sequence $\{\Gamma x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} \Gamma x_{n+1} = \delta_{n,1}\Gamma x_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1-\delta_{n,i})S^{j-1}y_n^1 + \prod_{i=1}^{\ell_1}(1-\delta_{n,p})S^{\ell_1}y_n^1; \\ \Gamma y_n^s = \Gamma\alpha_{n,1}^s x_n + \sum_{j=2}^{\ell_{s+1}} \gamma_{n,t} \prod_{i=1}^{j-1}(1-\alpha_{n,i}^s)S^{j-1}y_n^{s+1} + \prod_{i=1}^{\ell_{s+1}}(1-\alpha_{n,i})S^{\ell_{s+1}}y_n^{s+1}; \\ \Gamma y_n^{k-1} = \sum_{j=1}^{\ell_k} \alpha_{n,j}^{k-1} \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-1})S^{j-1}x_n + \prod_{i=1}^{\ell_k}(1-\alpha_{n,i}^{k-1})S^{\ell_k}x_n, n \geq 0, k \geq 2, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \Gamma x_{n+1} = \delta_{n,1}\Gamma y_n^1 + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1-\delta_{n,i})S^{j-1}y_n^1 + \prod_{i=1}^{\ell_1}(1-\delta_{n,p})S^{\ell_1}y_n^1; \\ \Gamma y_n^s = \Gamma\alpha_{n,1}^s y_n^{s+1} + \sum_{j=2}^{\ell_{s+1}} \gamma_{n,t} \prod_{i=1}^{j-1}(1-\alpha_{n,i}^s)S^{j-1}y_n^{s+1} + \prod_{i=1}^{\ell_{s+1}}(1-\alpha_{n,i})S^{\ell_{s+1}}y_n^{s+1}; \\ \Gamma y_n^{k-1} = \sum_{j=1}^{\ell_k} \alpha_{n,j}^{k-1} \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-1})S^{j-1}x_n + \prod_{i=1}^{\ell_k}(1-\alpha_{n,i}^{k-1})S^{\ell_k}x_n, n \geq 0, k \geq 2, \end{cases} \tag{3.2}$$

where $\{\{\delta_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{\ell_k}, \{\{\alpha_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{\ell_k}$ are countable finite family of real sequences in $[0,1]$ for each $i$, $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k$ and $\ell_1, \ell_2, \cdots, \ell_k \in \mathbb{N}$, for each $k$ shall be called the Jungck-IH-multistep-Noor iterative scheme and Jungck-DI-multistep-SP iterative scheme respectively.

**Remark 3.1.** Jungck-IH-multistep-Noor iterative scheme (3.1) and Jungck-DI-multistep-SP iterative scheme (3.2) are generalisations of Jungck-IH-Noor ( Jungck-IH-Ishikawa and Jungck-IH-Mann) and Jungck-DI-SP iterative schemes. Indeed, if $k = 3$ in (3.1), we obtain Jungck-IH-Noor iterative scheme. If $k = 2$ in (3.1), we get Jungck-IH-Ishikawa iterative scheme and if $k = 2$ and $\ell_2 = 0$ in (3.1), we get Jungck-IH-Mann iterative scheme. Again, if $k = 3$ in (3.2), we obtain Jungck-DI-SP iterative scheme.

**Theorem 3.2.** Let $H$ be a real Hilbert space and $S, \Gamma : D \longrightarrow H$ nonself commuting mappings for an arbitrary set $D$ satisfying the contractive condition

$$\|S^{j-1}x - S^{j-1}y\| \leq \nu^j\|\Gamma x - \Gamma y\| + \sum_{t=0}^{j}\binom{j}{t}\rho^{j-t}\phi(\|Sx - \Gamma x\|), \tag{3.3}$$

with $\Gamma(D) \subseteq S(D)$, where $2 \leq j \in \mathbb{N}, x, y \in D, 0 \leq \nu^i < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. Let $z$ be a coincidence point of $\Gamma, S, \Gamma^i, S^i$(*i.e*, $\Gamma z = Sz = q$ *and* $\Gamma^i z = S^i z = q$). For arbitrary $x_0 \in D$, the Jungck-IH-multistep-Noor iterative scheme defined by (3.1) converges strongly to $q$. Further, if $D = H$ and $\Gamma, S$ commute at $q$ (that is, $\Gamma$ *and* $S$ are weakly compartible), then $q$ is the unique common fixed point of $\Gamma$ *and* $S$.

**Proof .** Using (3.1), Lemma 2.1 and Proposition 2.4 with $\Gamma x_{n+1} = y, u = q, \Gamma x_n = t, k = 1, S^{j-1}y_n^1 = v_{j-1}$ *and* $S^{\ell_1}y_n^1$, we get

$$\begin{aligned} \|\Gamma x_{n+1} - q\|^2 \quad &\leq \quad \delta_{n,1}\|\Gamma x_n - q\|^2 + \sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\|S^{j-1}y_n^1 - S^{j-1}q\|^2 \\ &\quad + \prod_{i=1}^{\ell_1}(1-\delta_{n,i})\|S^{\ell_1}y_n^1 - S^{\ell_1}y_n^1\|^2 \end{aligned} \tag{3.4}$$

Using (3.3), with $y_n^1 = y$, we get

$$\begin{aligned} \|S^{j-1}y_n^1 - S^{j-1}q\| \quad &\leq \quad \nu^j\|\Gamma y_n^1 - \Gamma z\| + \sum_{t=0}^{j}\binom{j}{t}\rho^{j-t}\phi(\|Sz - \Gamma z\|) \\ &= \quad \nu^j\|\Gamma y_n^1 - \Gamma z\| \end{aligned} \tag{3.5}$$

(3.4) and (3.5) imply that

$$\|\Gamma x_{n+1}-q\|^2 \leq \delta_{n,1}\|\Gamma x_n-q\|^2+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})(\nu^j)^2\|\Gamma y_n^1-\Gamma z\|^2+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\nu^j)^2\|\Gamma y_n^1-\Gamma z\|^2,$$

which by Proposition 2.3 yields

$$\begin{aligned}\|\Gamma x_{n+1}-q\|^2 &\leq \delta_{n,1}\|\Gamma x_n-q\|^2+(1-\delta_{n,1}-\prod_{i=1}^{j-1}(1-\delta_{n,i})(\nu^j)^2)\|\Gamma y_n^1-q\|^2+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\nu^j)^2\|\Gamma y_n^1-q\|^2\\ &= \delta_{n,1}\|\Gamma x_n-q\|^2+(1-\delta_{n,1})\|\Gamma y_n^1-q\|^2 \qquad (3.6)\end{aligned}$$

In view of the fact that $\ell_1,\ell_2,\cdots,\ell_k$ are fixed integers and $\alpha_{n,i}^s\in[0,1]$ for each $s$, we obtain the following estimates for $n=1,2,\cdots$ and $1\leq s\leq k-1$ using Propositions [ 2.3 and 2.4]:

$$\begin{aligned}\|\Gamma y_n^1-q\|^2 &\leq \alpha_{n,1}^1\|\Gamma x_n-q\|^2+\sum_{i=1}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\|S^{j-1}y_n^2-S^{j-1}z\|^2+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\|S^{\ell_2}y_n^2-S^{\ell_2}z\|^2\\
&\leq \alpha_{n,1}^1\|\Gamma x_n-q\|^2+\sum_{i=1}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2\|\Gamma y_n^2-q\|^2\\
&\quad+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\|\Gamma y_n^2-q\|^2 \ (by\ (3.4)\ with\ y_n^1=y_n^2)\\
&= \alpha_{n,1}^1\|\Gamma x_n-q\|^2+\Big(\sum_{i=1}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\|\Gamma y_n^2-q\|^2\\
&\leq \alpha_{n,1}^1\|\Gamma x_n-q\|^2+\Big(\sum_{i=1}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\\
&\quad\times\Big[\alpha_{n,1}^2\|\Gamma x_n-q\|^2+\sum_{i=1}^{\ell_3}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\|S^{j-1}y_n^3-S^{j-1}z\|^2+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\|S^{\ell_3}y_n^3-S^{\ell_3}z\|^2\Big]\\
&\leq \alpha_{n,1}^1\|\Gamma x_n-q\|^2+\Big(\sum_{i=1}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^2\|\Gamma x_n-q\|^2+\sum_{i=1}^{\ell_3}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\nu^j)^2\Big)\\
&\quad\times\|\Gamma y_n^3-q\|^2 \ (by\ (3.4)\ with\ y_n^1=y_n^3)\\
&\leq \alpha_{n,1}^1\|\Gamma x_n-q\|^2+\Big(\sum_{i=1}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\alpha_{n,1}^2\\
&\quad\times\|\Gamma x_n-q\|^2+\Big(\sum_{i=1}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\\
&\quad\times\Big(\sum_{i=1}^{\ell_3}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\nu^j)^2\Big)\\
&\quad\times\Big[\alpha_{n,1}^3\|\Gamma x_n-q\|^2+\sum_{i=1}^{\ell_4}\alpha_{n,j}^3\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\|S^{j-1}y_n^4-S^{j-1}z\|^2\\
&\quad+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)\|S^{\ell_4}y_n^4-S^{\ell_4}z\|^2\Big]\\
&\leq \alpha_{n,1}^1\|\Gamma x_n-q\|^2+\Big(\sum_{i=1}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\alpha_{n,1}^2\end{aligned}$$

$$
\begin{aligned}
&\times\|\Gamma x_n-q\|^2+\Big(\sum_{i=1}^{\ell_2}\alpha^1_{n,j}\prod_{i=1}^{j-1}(1-\alpha^1_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha^1_{n,i})(\nu^j)^2\Big)\\
&\times\Big(\sum_{i=1}^{\ell_3}\alpha^2_{n,j}\prod_{i=1}^{j-1}(1-\alpha^2_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha^2_{n,i})(\nu^j)^2\Big)\alpha^3_{n,1}\|\Gamma x_n-q\|^2\\
&+\Big(\sum_{i=1}^{\ell_2}\alpha^1_{n,j}\prod_{i=1}^{j-1}(1-\alpha^1_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha^1_{n,i})(\nu^j)^2\Big)\\
&\times\Big(\sum_{i=1}^{\ell_3}\alpha^2_{n,j}\prod_{i=1}^{j-1}(1-\alpha^2_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha^2_{n,i})(\nu^j)^2\Big)\\
&\times\Big(\sum_{i=1}^{\ell_4}\alpha^3_{n,j}\prod_{i=1}^{j-1}(1-\alpha^3_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_4}(1-\alpha^3_{n,i})(\nu^j)^2\Big)\|\Gamma y^4_n-q\|^2\ (by\ (3.4)\ with\ y^1_n=y^4_n)\\
\le\ &\alpha^1_{n,1}\|\Gamma x_n-q\|^2+\Big(\sum_{i=1}^{\ell_2}\alpha^1_{n,j}\prod_{i=1}^{j-1}(1-\alpha^1_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha^1_{n,i})(\nu^j)^2\Big)\alpha^2_{n,1}\\
&\times\|\Gamma x_n-q\|^2+\Big(\sum_{i=1}^{\ell_2}\alpha^1_{n,j}\prod_{i=1}^{j-1}(1-\alpha^1_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha^1_{n,i})(\nu^j)^2\Big)\\
&\times\Big(\sum_{i=1}^{\ell_3}\alpha^2_{n,j}\prod_{i=1}^{j-1}(1-\alpha^2_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha^2_{n,i})(\nu^j)^2\Big)\alpha^3_{n,1}\|\Gamma x_n-q\|^2\\
&+\Big(\sum_{i=1}^{\ell_2}\alpha^1_{n,j}\prod_{i=1}^{j-1}(1-\alpha^1_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha^1_{n,i})(\nu^j)^2\Big)\\
&\times\Big(\sum_{i=1}^{\ell_3}\alpha^2_{n,j}\prod_{i=1}^{j-1}(1-\alpha^2_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha^2_{n,i})(\nu^j)^2\Big)\\
&\times\Big(\sum_{i=1}^{\ell_4}\alpha^3_{n,j}\prod_{i=1}^{j-1}(1-\alpha^3_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_4}(1-\alpha^3_{n,i})(\nu^j)^2\Big)\|\Gamma x_n-q\|^2+\cdots\\
&+\Big(\sum_{i=1}^{\ell_2}\alpha^1_{n,j}\prod_{i=1}^{j-1}(1-\alpha^1_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha^1_{n,i})(\nu^j)^2\Big)\\
&\times\Big(\sum_{i=1}^{\ell_3}\alpha^2_{n,j}\prod_{i=1}^{j-1}(1-\alpha^2_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha^2_{n,i})(\nu^j)^2\Big)\\
&\times\Big(\sum_{i=1}^{\ell_4}\alpha^3_{n,j}\prod_{i=1}^{j-1}(1-\alpha^3_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_4}(1-\alpha^3_{n,i})(\nu^j)^2\Big)\times\cdots\\
&\times\Big(\sum_{i=1}^{\ell_{s-1}}\alpha^{\ell_{s-2}}_{n,j}\prod_{i=1}^{j-1}(1-\alpha^{\ell_{s-2}}_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_{s-1}}(1-\alpha^{\ell_{s-2}}_{n,i})(\nu^j)^2\Big)\\
&\times\Big(\sum_{i=1}^{\ell_s}\alpha^{\ell_{s-1}}_{n,j}\prod_{i=1}^{j-1}(1-\alpha^{\ell_{s-1}}_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_s}(1-\alpha^{\ell_{s-1}}_{n,i})(\nu^j)^2\Big)\alpha^s_{n,1}\|\Gamma x_n-q\|^2\\
\le\ &\alpha^1_{n,1}\|\Gamma x_n-q\|^2+\alpha^2_{n,1}(1-\alpha^1_{n,i})(\nu^j)^2\|\Gamma x_n-q\|^2+\alpha^3_{n,1}(1-\alpha^1_{n,i})(1-\alpha^2_{n,i})(\nu^j)^4\|\Gamma x_n-q\|^2\\
&+\alpha^4_{n,1}(1-\alpha^1_{n,i})(1-\alpha^2_{n,i})(1-\alpha^3_{n,i})(\nu^j)^6\|\Gamma x_n-q\|^2+\cdots+\alpha^s_{n,1}(1-\alpha^1_{n,i})(1-\alpha^2_{n,i})(1-\alpha^3_{n,i})\\
&\times\cdots\times(1-\alpha^{\ell_{s-2}}_{n,i})(1-\alpha^{\ell_{s-1}}_{n,i})(\nu^j)^{2m}\|\Gamma x_n-q\|^2\\
<\ &\Big[\alpha^1_{n,1}+\alpha^2_{n,1}(1-\alpha^1_{n,i})+\alpha^3_{n,1}(1-\alpha^1_{n,i})(1-\alpha^2_{n,i})+\alpha^4_{n,1}(1-\alpha^1_{n,i})(1-\alpha^2_{n,i})(1-\alpha^3_{n,i})\\
&+\cdots+\alpha^s_{n,1}(1-\alpha^1_{n,i})(1-\alpha^2_{n,i})(1-\alpha^3_{n,i})\times\cdots\times(1-\alpha^{\ell_{s-2}}_{n,i})(1-\alpha^{\ell_{s-1}}_{n,i})\Big]\\
&\times\|\Gamma x_n-q\|^2
\end{aligned}
\tag{3.7}
$$

(3.7) is valid since $\Gamma z = Sz = q, \nu^j \in (0,1]$ *and* $\phi(0) = 0$.

From (3.6) and (3.7), we obtain

$$\begin{aligned} \|\Gamma x_{n+1} - q\|^2 &\leq \Big\{ \delta_{n,1} + (1-\delta_{n,1})\Big[\alpha^1_{n,1} + \alpha^2_{n,1}(1-\alpha^1_{n,i}) + \alpha^3_{n,1}(1-\alpha^1_{n,i})(1-\alpha^2_{n,i}) \\ &\quad +\alpha^4_{n,1}(1-\alpha^1_{n,i})(1-\alpha^2_{n,i})(1-\alpha^3_{n,i}) + \cdots + \alpha^s_{n,1}(1-\alpha^1_{n,i})(1-\alpha^2_{n,i})(1-\alpha^3_{n,i}) \\ &\quad \times \cdots \times (1-\alpha^{\ell_{s-2}}_{n,i})(1-\alpha^{\ell_{s-1}}_{n,i})\Big]\Big\} \|\Gamma x_n - q\|^2 \\ &< \Big[\delta_{n,1} + (1-\delta_{n,1})\alpha^1_{n,1} + (1-\delta_{n,1})(1-\alpha^1_{n,i}) + (1-\delta_{n,1})(1-\alpha^1_{n,i})(1-\alpha^2_{n,i}) \\ &\quad +(1-\delta_{n,1})(1-\alpha^1_{n,i})(1-\alpha^2_{n,i})(1-\alpha^3_{n,i}) + \cdots + (1-\delta_{n,1})(1-\alpha^1_{n,i})(1-\alpha^2_{n,i}) \\ &\quad \times(1-\alpha^3_{n,i}) \times \cdots \times (1-\alpha^{\ell_{s-2}}_{n,i})(1-\alpha^{\ell_{s-1}}_{n,i})\Big] \|\Gamma x_n - q\|^2 \end{aligned} \tag{3.8}$$

By Lemma 2.1, we get, using (3.8), that the sequence $\{\Gamma x_n\}_{n=0}^{\infty}$ converges strongly to $q$.

Next, we show that $q$ is unique. Assume, for contradiction, that there is another point of coincidence $q^\star$. Then, we can find a point $z^\star \in H$ such that $\Gamma z^\star = Sz^\star = q^\star$. Consequently, by (3.3), we obtain

$$\|z - z^\star\| = \|S^{j-1}z - S^{j-1}z^\star\| \leq \nu^j \|\Gamma z - \Gamma z^\star\| + \sum_{t=0}^{j} \binom{j}{t} \rho^{j-t}\phi(\|Sz - \Gamma z\|). \tag{3.9}$$

It follows from (3.9) that $(1-\nu^j)\|z - z^\star\| \leq 0$, so that $z = z^\star$(*since* $\nu^j \in (0,1]$). Hence, $q$ is unique.

Furthermore, since $\Gamma$ *and* $S$ are weakly compatible, it follows that $\Gamma S z = S\Gamma z$. Thus, $\Gamma q = Sq$ so that $q$ is a coincidence point of $\Gamma$ *and* $S$. Again, since the coincidence point is unique, it follows that $q = z$ and hence, $\Gamma q = Sq = q$. Thus, $q$ is the unique common fixed point of $\Gamma$ *and* $S$, and this proof is completed.

The corollary below immediately follows from Theorem 3.1. □

**Corollary 3.3.** Let $H$ be a real Hilbert space and $S, \Gamma : D \longrightarrow H$ nonself commuting mappings for an arbitrary set $D$ satisfying the contractive condition

$$\|S^{j-1}x - S^{j-1}y\| \leq \nu^j \|\Gamma x - \Gamma y\| + \sum_{t=0}^{j} \binom{j}{t} \rho^{j-t}\phi(\|Sx - \Gamma x\|), \tag{3.10}$$

with $\Gamma(D) \subseteq S(D)$, where $2 \leq j \in \mathbb{N}, x, y \in D, 0 \leq \nu^i < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. Let $z$ be a coincidence point of $\Gamma, S, \Gamma^i, S^i$(*i.e*, $\Gamma z = Sz = q$ *and* $\Gamma^i z = S^i z = q$). For arbitrary $x_0 \in D$,

(*i*) the Jungck-IH-Noor iterative scheme converges strongly to $q$;
(*ii*) the Jungck-IH-Ishikawa iterative scheme converges strongly to $q$;
(*iii*) the Jungck-IH-Mann iterative scheme converges strongly to $q$

In addition, if $D = H$ and $\Gamma, S$ commute at $q$ (that is, $\Gamma$ *and* $S$ are weakly compartible), then $q$ is the unique common fixed point of $\Gamma$ *and* $S$.

**Theorem 3.4.** Let $H$ be a real Hilbert space and $S, \Gamma : D \longrightarrow H$ nonself commuting mappings for an arbitrary set $D$ satisfying the contractive condition

$$\|S^{j-1}x - S^{j-1}y\| \leq \nu^j \|\Gamma x - \Gamma y\| + \sum_{t=0}^{j} \binom{j}{t} \rho^{j-t}\phi(\|Sx - \Gamma x\|), \tag{3.11}$$

with $\Gamma(D) \subseteq S(D)$, where $2 \leq j \in \mathbb{N}, x, y \in D, 0 \leq \nu^i < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. Let $z$ be a coincidence point of $\Gamma, S, \Gamma^i, S^i$(*i.e*, $\Gamma z = Sz = q$ *and* $\Gamma^i z = S^i z = q$). For arbitrary $x_0 \in D$, the Jungck-DI-multistep-SP iterative scheme defined by (3.2) converges strongly to $q$. If, in addition, $D = H$ and $\Gamma, S$ commute at $q$ (that is, $\Gamma$ *and* $S$ are weakly compartible), then $q$ is the unique common fixed point of $\Gamma$ *and* $S$.

**Proof .** Using similar approach as in the proof of Theorem 3.1, the result of Theorem 3.2 follows immeately. □

**Corollary 3.5.** Let $H$ be a real Hilbert space and $S, \Gamma : D \longrightarrow H$ nonself commuting mappings for an arbitrary set $D$ satisfying the contractive condition

$$\|S^{j-1}x - S^{j-1}y\| \leq \nu^j \|\Gamma x - \Gamma y\| + \sum_{t=0}^{j} \binom{j}{t} \rho^{j-t} \phi(\|Sx - \Gamma x\|), \tag{3.12}$$

with $\Gamma(D) \subseteq S(D)$, where $2 \leq j \in \mathbb{N}, x, y \in D, 0 \leq \nu^i < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. Let $z$ be a coincidence point of $\Gamma, S, \Gamma^i, S^i (i.e,\ \Gamma z = Sz = q\ and\ \Gamma^i z = S^i z = q)$. For arbitrary $x_0 \in D$, the Jungck-DI-SP iterative scheme defined by (3.2) converges strongly to $q$. If, in addition, $D = H$ and $\Gamma, S$ commute at $q$ (that is, $\Gamma$ *and* $S$ are weakly compartible), then $q$ is the unique common fixed point of $\Gamma$ *and* $S$.

## 4 Main Results II

**Theorem 4.1.** Let $H$ be a real Hilbert space and $S, \Gamma : D \longrightarrow H$ nonself commuting mappings for an arbitrary set $D$ satisfying the contractive condition

$$\|S^{j-1}x - S^{j-1}y\| \leq \nu^j \|\Gamma x - \Gamma y\| + \sum_{t=0}^{j} \binom{j}{t} \rho^{j-t} \phi(\|Sx - \Gamma x\|), \tag{4.1}$$

with $\Gamma(D) \subseteq S(D)$, where $2 \leq j \in \mathbb{N}, x, y \in D, 0 \leq \nu^i < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. Let $z$ be a coincidence point of $\Gamma, S, \Gamma^i, S^i (i.e,\ \Gamma z = Sz = q\ and\ \Gamma^i z = S^i z = q)$. For arbitrary $x_0 \in D$, let $\{\Gamma x_n\}_{n=0}^{\infty}$ be the Jungck-DI-multistep-SP iterative scheme (3.2) converging strongly to $q$ $(i.e, \Gamma q = Sq = q\ and\ \Gamma^{j-1} q =^{j-1} q = q)\ with\ 0 < \delta < \delta_{n,i}, 0 < \alpha < \alpha^s_{n,i},\ for\ i = 1, 2, \cdots, k-1$ and for all $n$. Then, the iterative scheme defined by (3.2) is $\Gamma, S$-stable.

**Proof .** Let $\{\Gamma z_n\}_{n=0}^{\infty}$ and $\{\Gamma t^1_n\}_{n=0}^{\infty}$ , for $i = 1, 2, \cdots, k-1$, be two arbitrary real sequences in $H$. Let

$$\epsilon_n = \|\Gamma z_{n+1} - \delta_{n,1}\Gamma t^1_n - \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1 - \delta_{n,i}) S^{j-1} t^1_n - \prod_{i=1}^{\ell_1}(1 - \delta_{n,i}) S^{\ell_1} t^1_n\|^2, n \geq 1, \tag{4.2}$$

where, for $s = 1, 2, \cdots, k-1$,

$$\Gamma t^s_n = \alpha^s_{n,1} \Gamma t^{s+1}_n + \sum_{j=2}^{\ell_{s+1}} \alpha^s_{n,j} \prod_{i=1}^{j-1}(1 - \alpha^s_{n,i}) S^{j-1} t^{s+1}_n + \prod_{i=1}^{\ell_{s+1}}(1 - \alpha^s_{n,i}) S^{\ell_{s+1}} t^{s+1}_n \tag{4.3}$$

and for $k \geq 2$,

$$\Gamma t^{k-1}_n = \sum_{j=1}^{\ell_k} \alpha^{k-1}_{n,j} \prod_{i=1}^{j-1}(1 - \alpha^{k-1}_{n,i}) S^{j-1} z_n + \prod_{i=1}^{\ell_k}(1 - \alpha^{k-1}_{n,i}) S^{\ell_k} z_n, n \geq 1, \tag{4.4}$$

and let $\epsilon_n \to 0\ as\ n \to \infty$. Then, we show that $\Gamma z_n \to q\ as\ n \to \infty$ using the contractive mapping for which (4.1) holds.

Now, from Proposition 2.4, with $u = q, \Gamma t^1_n = t, k = 1, \Gamma^{\ell_1} t^1_n = v, S^{j-1} t^1_n = v_{j-1}$, we have the following estimates:

$$\begin{aligned}
\|\Gamma z_{n+1} - q\|^2 &\leq \|\delta_{n,1}\Gamma t^1_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1 - \delta_{n,i}) S^{j-1} t^1_n + \prod_{i=1}^{\ell_1}(1 - \delta_{n,i}) S^{\ell_1} t^1_n - q \\
&\quad - \Big[\delta_{n,1}\Gamma t^1_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1 - \delta_{n,i}) S^{j-1} t^1_n + \prod_{i=1}^{\ell_1}(1 - \delta_{n,i}) S^{\ell_1} t^1_n - \Gamma z_{n+1}\Big]\|^2 \\
&\leq \|\delta_{n,1}\Gamma t^1_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1 - \delta_{n,i}) S^{j-1} t^1_n + \prod_{i=1}^{\ell_1}(1 - \delta_{n,i}) S^{\ell_1} t^1_n - q\|^2 \\
&\quad + \| - \Big[\Gamma z_{n+1} - \delta_{n,1}\Gamma t^1_n - \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1 - \delta_{n,i}) S^{j-1} t^1_n - \prod_{i=1}^{\ell_1}(1 - \delta_{n,i}) S^{\ell_1} t^1_n\Big]\|^2
\end{aligned}$$

$$\begin{aligned}
&= \|\delta_{n,1}\Gamma t_n^1 + \sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})S^{j-1}t_n^1 + \prod_{i=1}^{\ell_1}(1-\delta_{n,i})S^{\ell_1}t_n^1 - q\|^2 \\
&\quad +\|\Gamma z_{n+1} - \delta_{n,1}\Gamma t_n^1 - \sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})S^{j-1}t_n^1 - \prod_{i=1}^{\ell_1}(1-\delta_{n,i})S^{\ell_1}t_n^1\|^2 \\
&= \epsilon_n + \|\delta_{n,1}\Gamma t_n^1 + \sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})S^{j-1}t_n^1 + \prod_{i=1}^{\ell_1}(1-\delta_{n,i})S^{\ell_1}t_n^1 - q\|^2 \\
&\leq \epsilon_n + \delta_{n,1}\|\Gamma t_n^1 - q\|^2 + \sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\|S^{j-1}t_n^1 - q\|^2 + \prod_{i=1}^{\ell_1}(1-\delta_{n,i})\|S^{\ell_1}t_n^1 - q\|^2 \qquad (4.5)
\end{aligned}$$

But, from (4.1), with $t_n^1 = y$, we get

$$\begin{aligned}
\|S^{j-1}t_n^1 - S^{j-1}q\| &\leq \nu^j\|\Gamma t_n^1 - \Gamma z\| + \sum_{t=0}^{j}\binom{j}{t}\rho^{j-t}\phi(\|Sz - \Gamma z\|) \\
&= \nu^j\|\Gamma t_n^1 - q\| \qquad (4.6)
\end{aligned}$$

Since, from (4.6) with $t_n^1 = t_n^2$, $\|S^{j-1}t_n^2 - S^{j-1}q\| \leq \nu^j\|\Gamma t_n^2 - q\|$, it follows that

$$\begin{aligned}
\|\Gamma t_n^1 - q\|^2 &= \|\alpha_{n,1}^1\Gamma t_n^2 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)S^{j-1}t_n^2 + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)S^{\ell_2}t_n^2 - q\|^2 \\
&\leq \alpha_{n,1}^1\|\Gamma t_n^2 - q\|^2 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\|S^{j-1}t_n^2 - q\|^2 + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\|S^{\ell_2}t_n^2 - q\|^2 \\
&\leq \Big(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2 + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\|\Gamma t_n^2 - q\|^2 \\
&= \Big(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2 + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big) \\
&\quad \times\|\alpha_{n,1}^2\Gamma t_n^3 + \sum_{j=2}^{\ell_3}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)S^{j-1}t_n^3 + \prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^3)S^{\ell_3}t_n^3 - q\|^2 \\
&\leq \Big(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2 + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\Big[\alpha_{n,1}^2\|\Gamma t_n^3 - q\|^2 \\
&\quad + \sum_{j=2}^{\ell_3}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\|S^{j-1}t_n^3 - q\|^2 + \prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\|S^{\ell_3}t_n^3 - q\|^2\Big] \\
&\leq \Big(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2 + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big) \times \Big(\alpha_{n,1}^2 + \sum_{j=2}^{\ell_3}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)(\nu^j)^2 \\
&\quad + \prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\nu^j)^2\Big)\|t_n^3 - q\|^2 \quad (by\ (4.6)\ with\ t_n^1 = t_n^3) \\
&\leq \Big(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2 + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big) \\
&\quad \times\Big(\alpha_{n,1}^2 + \sum_{j=2}^{\ell_2}\alpha_{n,j}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)(\nu^j)^2 + \prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\nu^j)^2\Big)\Big[\alpha_{n,1}^3\|\Gamma t_n^4 - q\|^2 \\
&\quad + \sum_{j=2}^{\ell_4}\alpha_{n,j}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\|S^{j-1}t_n^4 - q\|^2 + \prod_{i=1}^{\ell_4}(1-\alpha_{n,i})\|S^{\ell_4}t_n^4 - q\|^2\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big(\alpha_{n,1}^1+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^3+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)(\nu^j)^2+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\nu^j)^2\Big)\\
&\quad\times\|t_n^4-q\|^2 \quad (by\ (4.6)\ with\ t_n^1=t_n^4)\\
&\leq \Big(\alpha_{n,1}^1+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^3+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)(\nu^j)^2+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\nu^j)^2\Big)\\
&\quad\times\cdots\times\Big(\alpha_{n,1}^{k-2}+\sum_{j=2}^{\ell_{k-1}}\alpha_{n,j}^{k-2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-2})(\nu^j)^2+\prod_{i=1}^{\ell_{k-1}}(1-\alpha_{n,i}^{k-2})(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^{k-1}+\sum_{j=2}^{\ell_k}\alpha_{n,j}^{k-1}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-1})(\nu^j)^2+\prod_{i=1}^{\ell_k}(1-\alpha_{n,i}^{k-1})(\nu^j)^2\Big)\\
&\quad\times\|\Gamma z_n-q\|^2 \qquad (4.7)
\end{aligned}
$$

Note that (4.7) is valid since $\Gamma q = Sq = q$ and $\phi(0) = 0$.

(4.5), (4.6) and (4.7) imply

$$
\begin{aligned}
\|\Gamma z_{n+1}-q\|^2 &\leq \epsilon_n+\Big(\delta_{n,1}^1+\sum_{j=2}^{\ell_1}\delta_{n,j}^1\prod_{i=1}^{j-1}(1-\delta_{n,i}^1)(\nu^j)^2+\prod_{i=1}^{\ell_1}(1-\delta_{n,i}^1)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^1+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^3+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)(\nu^j)^2+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\nu^j)^2\Big)\\
&\quad\times\cdots\times\Big(\alpha_{n,1}^{k-2}+\sum_{j=2}^{\ell_{k-1}}\alpha_{n,j}^{k-2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-2})(\nu^j)^2+\prod_{i=1}^{\ell_{k-1}}(1-\alpha_{n,i}^{k-2})(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^{k-1}+\sum_{j=2}^{\ell_k}\alpha_{n,j}^{k-1}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-1})(\nu^j)^2+\prod_{i=1}^{\ell_k}(1-\alpha_{n,i}^{k-1})(\nu^j)^2\Big)\\
&\quad\times\|\Gamma z_n-q\|^2 \qquad (4.8)
\end{aligned}
$$

Let

$$\begin{aligned}
\delta_n^{\star} &= \Big(\delta_{n,1}^1 + \sum_{j=2}^{\ell_1}\delta_{n,j}^1\prod_{i=1}^{j-1}(1-\delta_{n,i}^1)(\nu^j)^2 + \prod_{i=1}^{\ell_1}(1-\delta_{n,i}^1)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)(\nu^j)^2 + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^2 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)(\nu^j)^2 + \prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^3 + \sum_{j=2}^{\ell_4}\alpha_{n,j}^3\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)(\nu^j)^2 + \prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\nu^j)^2\Big)\\
&\quad\times\cdots\times\Big(\alpha_{n,1}^{k-2} + \sum_{j=2}^{\ell_{k-1}}\alpha_{n,j}^{k-2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-2})(\nu^j)^2 + \prod_{i=1}^{\ell_{k-1}}(1-\alpha_{n,i}^{k-2})(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha_{n,1}^{k-1} + \sum_{j=2}^{\ell_k}\alpha_{n,j}^{k-1}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-1})(\nu^j)^2 + \prod_{i=1}^{\ell_k}(1-\alpha_{n,i}^{k-1})(\nu^j)^2\Big)\\
&< \Big(\delta_{n,1}^1 + \sum_{j=2}^{\ell_1}\delta_{n,j}^1\prod_{i=1}^{j-1}(1-\delta_{n,i}^1) + \prod_{i=1}^{\ell_1}(1-\delta_{n,i}^1)\Big)\times\Big(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1) + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\Big)\\
&\quad\times\Big(\alpha_{n,1}^2 + \sum_{j=2}^{\ell_2}\alpha_{n,j}^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2) + \prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\Big)\times\Big(\alpha_{n,1}^3 + \sum_{j=2}^{\ell_4}\alpha_{n,j}^3\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3) + \prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)\Big)\\
&\quad\times\cdots\times\Big(\alpha_{n,1}^{k-2} + \sum_{j=2}^{\ell_{k-1}}\alpha_{n,j}^{k-2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-2}) + \prod_{i=1}^{\ell_{k-1}}(1-\alpha_{n,i}^{k-2})\Big)\\
&\quad\times\Big(\alpha_{n,1}^{k-1} + \sum_{j=2}^{\ell_k}\alpha_{n,j}^{k-1}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-1}) + \prod_{i=1}^{\ell_k}(1-\alpha_{n,i}^{k-1})\Big) = 1 \qquad (4.9)
\end{aligned}$$

(4.9) is true by virtue of Proposition 2.3 and the fact that $\nu^j \in [0,1)$. Hence, using (4.8) and (4.9), we get

$$\|\Gamma z_{n+1} - q\|^2 \le \delta_n^{\star}\|\Gamma z_n - q\|^2 + \epsilon_n, \qquad (4.10)$$

which by Lemma 2.1 and the fact that $\epsilon_n \to 0$ *as* $n \to \infty$ yields $\Gamma z_n \to q$ *as* $n \to \infty$. On the other hand, let $\Gamma z_n \to q$ *as* $n \to \infty$. Then, we show that $\epsilon_n \to 0$ *as* $n \to \infty$. From (4.2),

$$\begin{aligned}
\epsilon_n &= \|\Gamma z_{n+1} - \delta_{n,1}\Gamma t_n^1 - \sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})S^{j-1}t_n^1 - \prod_{i=1}^{\ell_1}(1-\delta_{n,i})S^{\ell_1}t_n^1\|^2\\
&= \|\Gamma z_{n+1} - q - \Big[\delta_{n,1}\Gamma t_n^1 + \sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})S^{j-1}t_n^1 + \prod(1-\delta_{n,i})S^{\ell_1}t_n^1 - q\Big]\|^2\\
&\le \|\Gamma z_{n+1} - q\|^2 + \|\delta_{n,1}\Gamma t_n^1 + \sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})S^{j-1}t_n^1 + \prod(1-\delta_{n,i})S^{\ell_1}t_n^1 - q\|^2\\
&\le \|\Gamma z_{n+1} - q\|^2 + \delta_{n,1}\|\Gamma t_n^1 - q\|^2 + \sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\|S^{j-1}t_n^1 - q\|^2\\
&\quad+\prod(1-\delta_{n,i})\|S^{\ell_1}t_n^1 - q\|^2
\end{aligned} \qquad (4.11)$$

$$
\begin{aligned}
&\leq \|\Gamma z_{n+1}-q\|^2+\Big(\delta_{n,1}+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})(\nu^j)^2+\prod(1-\delta_{n,i})(\nu^j)^2\Big)\times\|\Gamma t_n^1-q\|^2 \quad (by\ (4.6))\\
&\leq \|\Gamma z_{n+1}-q\|^2+\Big(\delta_{n,1}+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})(\nu^j)^2+\prod(1-\delta_{n,i})(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha^1_{n,1}+\sum_{j=2}^{\ell_2}\alpha^1_{n,j}\prod_{i=1}^{j-1}(1-\alpha^1_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_2}(1-\alpha^1_{n,i})(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha^2_{n,1}+\sum_{j=2}^{\ell_2}\alpha^2_{n,j}\prod_{i=1}^{j-1}(1-\alpha^2_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_3}(1-\alpha^2_{n,i})(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha^3_{n,1}+\sum_{j=2}^{\ell_4}\alpha^3_{n,j}\prod_{i=1}^{j-1}(1-\alpha^3_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_4}(1-\alpha^3_{n,i})(\nu^j)^2\Big)\\
&\quad\times\cdots\times\Big(\alpha^{k-2}_{n,1}+\sum_{j=2}^{\ell_{k-1}}\alpha^{k-2}_{n,j}\prod_{i=1}^{j-1}(1-\alpha^{k-2}_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_{k-1}}(1-\alpha^{k-2}_{n,i})(\nu^j)^2\Big)\\
&\quad\times\Big(\alpha^{k-1}_{n,1}+\sum_{j=2}^{\ell_k}\alpha^{k-1}_{n,j}\prod_{i=1}^{j-1}(1-\alpha^{k-1}_{n,i})(\nu^j)^2+\prod_{i=1}^{\ell_k}(1-\alpha^{k-1}_{n,i})(\nu^j)^2\Big)\times\|\Gamma z_n-q\|^2 \quad (by\ (4.7))\\
&\leq \|\Gamma z_{n+1}-q\|^2+\delta_n^\star\|\Gamma z_n-q\|^2 \quad (by\ (4.9))
\end{aligned}
\tag{4.12}
$$

Using the fact that $\Gamma z_n \to q\ as\ n\to\infty$, we obtain (from (4.12)) that $\epsilon_n \to 0\ as\ n\to\infty$. Hence, the Jungck-DI-multistep-SP iterative scheme (3.2) is $\Gamma, S$-stable. This completes the proof. □

The corollary below immediately follows from Theorem 4.1.

**Corollary 4.2.** Let $H$ be a real Hilbert space and $S,\Gamma : D \longrightarrow H$ nonself commuting mappings for an arbitrary set $D$ satisfying the contractive condition

$$
\|S^{j-1}x-S^{j-1}y\|\leq \nu^j\|\Gamma x-\Gamma y\|+\sum_{t=0}^{j}\binom{j}{t}\rho^{j-t}\phi(\|Sx-\Gamma x\|), \tag{4.13}
$$

with $\Gamma(D)\subseteq S(D)$, where $2\leq j\in\mathbb{N}, x,y\in D, 0\leq \nu^i<1$, and let $\phi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0)=0$ and $\phi(Mt)=M\phi(t), M\geq 0, t\in\mathbb{R}^+$. Let $z$ be a coincidence point of $\Gamma, S, \Gamma^i, S^i (i.e,\ \Gamma z = Sz = q\ and\ \Gamma^i z = S^i z = q)$. For arbitrary $x_0\in D$, let $\{\Gamma x_n\}_{n=0}^\infty$ be the Jungck-DI-SP iterative scheme converging strongly to $q\ (i.e, \Gamma q = Sq = q\ and\ \Gamma^{j-1}q =^{j-1} q = q)\ with\ 0<\delta<\delta_{n,i}, 0<\alpha<\alpha^s_{n,i},\ for\ i=1,2,\cdots,k-1$ and for all $n$. Then, Jungck-DI-SP iterative scheme is $\Gamma, S$-stable.

**Theorem 4.3.** Let $H$ be a real Hilbert space and $S,\Gamma : D \longrightarrow H$ nonself commuting mappings for an arbitrary set $D$ satisfying the contractive condition

$$
\|S^{j-1}x-S^{j-1}y\|\leq \nu^j\|\Gamma x-\Gamma y\|+\sum_{t=0}^{j}\binom{j}{t}\rho^{j-t}\phi(\|Sx-\Gamma x\|), \tag{4.14}
$$

with $\Gamma(D)\subseteq S(D)$, where $2\leq j\in\mathbb{N}, x,y\in D, 0\leq \nu^i<1$, and let $\phi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0)=0$ and $\phi(Mt)=M\phi(t), M\geq 0, t\in\mathbb{R}^+$. Let $z$ be a coincidence point of $\Gamma, S, \Gamma^i, S^i (i.e,\ \Gamma z = Sz = q\ and\ \Gamma^i z = S^i z = q)$. For arbitrary $x_0\in D$, let $\{\Gamma x_n\}_{n=0}^\infty$ be the Jungck-DI-multistep-Noor iterative scheme (3.1) converging strongly to $q\ (i.e, \Gamma q = Sq = q\ and\ \Gamma^{j-1}q =^{j-1} q = q)\ with\ 0<\delta<\delta_{n,i}, 0<\alpha<\alpha^s_{n,i},\ for\ i = 1,2,\cdots,k-1$ and for all $n$. Then, the iterative scheme defined by (3.1) is $\Gamma, S$-stable.

**Proof .** Using similar approach as in the proof of Theorem 4.1, the result of Theorem 4.3 follows immediately. □

The corollary below immediately follows from Theorem 4.1.

**Corollary 4.4.** Let $H$ be a real Hilbert space and $S, \Gamma : D \longrightarrow H$ nonself commuting mappings for an arbitrary set $D$ satisfying the contractive condition

$$\|S^{j-1}x - S^{j-1}y\| \leq \nu^j \|\Gamma x - \Gamma y\| + \sum_{t=0}^{j} \binom{j}{t} \rho^{j-t} \phi(\|Sx - \Gamma x\|), \tag{4.15}$$

with $\Gamma(D) \subseteq S(D)$, where $2 \leq j \in \mathbb{N}, x, y \in D, 0 \leq \nu^i < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. Let $z$ be a coincidence point of $\Gamma, S, \Gamma^i, S^i (i.e,\ \Gamma z = Sz = q \ and \ \Gamma^i z = S^i z = q)$. For arbitrary $x_0 \in D$, let $\{\Gamma x_n\}_{n=0}^{\infty}$ be the Jungck-IH-Noor iterative scheme, the Jungck-IH-Ishikawa iterative scheme and the Jungck-IH-Mann iterative scheme converging strongly to $q$, respectively $(i.e, \Gamma q = Sq = q \ and \ \Gamma^{j-1} q =^{j-1} q = q) \ with \ 0 < \delta < \delta_{n,i}, 0 < \alpha < \alpha_{n,i}^s$ and for all $n$. Then, For arbitrary $x_0 \in D$,

($i$) the Jungck-IH-Noor iterative scheme is $\Gamma, S$-stable;
($ii$) the Jungck-IH-Ishikawa iterative scheme is $\Gamma, S$-stable;
($iii$) the Jungck-IH-Mann iterative scheme is $\Gamma, S$-stable.

**Open problem**

Is it possible to prove Proposition 2.3 and Proposition 2.4 in arbitrary Banach spaces so as to generalise the results of this paper in such spaces?

**Conclusion**

An affirmative answer has been provided for Question 1.1. The results obtained in this paper improve the corresponding results in [4] and several others currently existing in literature.

## References


[1] I.K. Agwu and D.I. Igbokwe, *Convergence and stability of new approximation algorithms for certain contractive-type mappings*, Eur. J. Math. Anal. **2** (2022), 1–1.

[2] I.K. Agwu and D.I. Igbokwe, *Fixed points and stability of new approximation algorithms for contractive-type operators in Hilbert spaces*, Int. J. Nonlinear Anal. Appl. In Press, doi: 10.22075/ijnaa.2021.22923.2432

[3] I.K. Agwu and D.I. Igbokwe, *New iteration algorithm for equilibrium problems and fixed point problems of two finite families of asymptotically demicontractive multivalued mappings*, In Press.

[4] H. Akewe and A. Mogbademu, *Common fixed point of Jungck-Kirk-type iteration for nonself operators in normed linear spaces*, Fasciculi Math. **2016** (2016), 29–41.

[5] H. Akewe and H. Olaoluwa, *On the convergence of modified iteration process for generalise contractive-like operators*, Bull. Math. Anal. Appl. **4** (2012), no. 3, 78–86.

[6] H. Akewe, G.A. Okeeke and A. Olayiwola, *Strong convergence and stability of Kirk-multistep-type iterative schemes for contractive-type operators*, Fixed Point Theory Appl. **2014** (2014), 45.

[7] H. Akewe, *Approximation of fixed and common fixed points of generalised contractive-like operators*, PhD Thesis, University of Lagos, Nigeria, 2010.

[8] V. Berinde, *On the stability of some fixed point problems*, Bull. Stint. Univ. Bala Mare, Ser. B Fasc. Mat-inform. XVIII(1) **14** (2002), 7–14.

[9] V. Berinde, *Iterative approximation of fixed points for pseudo-contractive operators*, Seminar on Fixed Point Theory, 2002.

[10] R. Chugh and V. Kummar, *Stability of hybrid fixed point iterative algorithm of Kirk-Noor-type in nonlinear spaces for self and nonself operators*, Int. J. Contemp. Math. Sci. **7** (2012), no. 24, 1165–1184.

[11] R. Chugh and V. Kummar, *Strong convergence of SP iterative scheme for quasi-contractive operators*, Int. J. Comput. Appl. **31** (2011), no. 5, 21–27.

[12] A.M. Harder and T.L. Hicks, *Stability results for fixed point iterative procedures*, Math. Jpn. **33** (1988), no. 5, 693–706.

[13] N. Hussain, R. Chugh, V. Kummar and A. Rafig, *On the convergence of Kirk-type iterative schemes*, J. Appl. Math. **2012** (2012), Article ID 526503, 22 pages.

[14] S. Ishikawa, *Fixed points by a new iteration methods*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[15] C.O. Imoru and M.O. Olatinwo, *On the stability of Picard's and Mann's iteration*, Carpath. J. Math. **19** (2003), 155–160.

[16] F.O. Isiogugu., C. Izuchukwu and C.C. Okeke, *New iteration scheme for approximating a common fixed point of a finite family of mappings*, J. Math. **2020** (2020), Article ID 3287968.

[17] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Month. **83** (1976), no. 4, 261–263.

[18] W.A. Kick, *On successive approximations for nonexpansive mappings in Banach spaces*, Glasg. Math. J. **12** (1971), 6–9.

[19] W.R. Mann, *Mean value method in iteration*, Proc. Amer. Math. Soc. **44** (2000), 506–510.

[20] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. **251** (2000), 217–229.

[21] M.O. Olatinwo, *A generalization of some convergence results using a Jungck-Noor three-step iteration process in arbitrary Banach space*, Fasciculi Math. **40** (2008), 37–43.

[22] J.O. Olaleru and H. Akewe, *On the convergence of Jungck-type iterative schemes for generalized contractive-like operators*, Fasciculi Math. **45** (2010), 87–98.

[23] M.O. Olatinwo, *Stability results for Jungck-kirk-Mann and Jungck-kirk hybrid iterative algorithms*, Anal. Theory Appl. **29** (2013), 12–20.

[24] M.O. Olatinwo, *Convergence results for Jungck-type iterative process in convex metric spaces*, Acta Univ. Palacki Olomue, Fac. Rer. Nat. Math. **51** (2012), 79–87.

[25] M.O. Olatinwo, *Some stability and strong convergence results for the Jungck-Ishikawa iteration process*, Creative Math. Inf. **17** (2008), 33–42.

[26] J.O. Olaeru and H. Akewe, *An extension of Gregus fixed point theorem*, Fixed Point Theory Appl. **2007** (2007), Article ID 78628.

[27] M.O. Olutinwo, *Some stability results for two hybrid fixed point iterative algorithms in normed linear space*, Mat. vesnik **61** (2009), no. 4, 247–256.

[28] M.O. Osilike and A. Udoemene, *A short proof of stability resultsfor fixed point iteration procedures for a class of contractive-type mappings*, Indian J. Pure Appl. Math. **30** (1999), 1229–1234.

[29] M.O. Osilike, *Stability results for lshikawa fixed point iteration procedure*, Indian J. Pure Appl. Math. 26 (1996), no. 10, 937–941.

[30] A.M. Ostrowski, *The round off stability of iterations*, Z. Angew Math. Mech. **47** (1967), 77–81.

[31] B.E. Rhoade, *Fixed point theorems and stability results for fixed point iteration procedures*, Indian J. Pure Appl. Math. **24** (1993), no. 11, 691-703.

[32] B.E. Rhoade, *Fixed point theorems and stability results for fixed point iteration procedures*, Indian J. Pure Appl. Math. **21** (1990), 1–9.

[33] A. Ratiq, *A convergence theprem for Mann's iteration procedure*, Appl. Math. E-Note **6** (2006), 289–293.

[34] B.E. Rhoade, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **266** (1977), 257–290.

[35] B.E. Rhoade, *Comments on two fixed point iteration methods*, Trans. Am. Math. Soc. **56** (1976), 741–750.

[36] B.E. Rhoade, *Fixed point iteration using infinite matrices*, Trans. Am. Math. Soc. **196** (1974), 161–176.

[37] A. Ratiq, *On the convergence of the three step iteration process in the class of quasi-contractive operators*, Acta. Math. Acad. Paedagag Nayhazi **22** (2006), 300–309.

[38] S.L. Singh, C. Bhatnaga and S.N Mishra, *Stability of Jungck-type iteration procedures*, Int. J. Math. Math. Sci. **19** (2005), 3035–3043.

[39] T. Zamfirescu, *Fixed point theorems in metric spaces*, Arch. Math. **23** (1972), 292–298.