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A nilpotency criterion for finite groups by the sum of element orders

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Abstract

Let G be a finite group and $\psi(G) = \sum_{g\in G} o(g)$, where $o(g)$ denotes the order of $g \in G$. We give a criterion for nilpotency of finite groups G based on the sum of element orders of G. We prove that if $\psi(G) > \frac{13}{21}\psi(C_n)$ then G is a nilpotent group.

Keywords: Finite group, element orders sum, nilpotent group, simple group
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1 Introduction

In this article, all groups are finite. If G is a group of order n, then $\psi(G)$ denotes the sum of orders of all elements of G. More generally, if X is a subset of G, then $\psi(X)$ denotes the sum of the orders of all elements of X. Moreover, the cyclic group of order n is denoted by C_n.

In 2009, Amiri, Jafarian and Isaacs proved that, If G is a non-cylcic group of order n, then $\psi(G) < \psi(C_n)$ (see [3]). Thus C_n can be characterized by the order n and the value ψ. Following this publication, in several paper it is proved that certain classes of finite groups G can be characterized using the $\psi(G)$ (see [1, 2, 7, 8, 9, 12, 13, 14, 15]).

Second maximum value of ψ for groups of order n was discussed in several papers, for example [10, 11] and [16]. In [5], an exact upper bound for the ψ value in non-cyclic finite groups is given.

Theorem 1.1. [5] If G is a non-cyclic group of order n, then

$$\psi(G) \leq \frac{7}{11}\psi(C_n).$$

Moreover, the equality holds if $n = 4k$ and $(k, 2) = 1$, and $G = (C_2 \times C_2) \times C_2$.

In [6], the authors give two new criteria for solvability of finite groups and put forward a conjecture.

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Theorem 1.2. [6] If G is a finite group of order n and

$$\psi(G) > \frac{1}{6.68}\psi(C_n),$$

then G is solvable.

Conjecture 1.3. [6] If G is a group of order n, and

$$\psi(G) > \frac{211}{1617}\psi(C_n),$$

then G is solvable.

In [4], the authors proved this conjecture. The aim of this paper is to prove the following criteria for nilpotency of finite groups.

Theorem 1.4. If G is a finite group of order n and

$$\psi(G) > \frac{13}{21}\psi(C_n),$$

then G is a nilpotent group.

2 Basic preliminary results

Lemma 2.1. [3] Let $P \in Syl_p(G)$ and assume that $P \trianglelefteq G$ also P is cyclic. Then

$$\psi(G) \leq \psi(P)\psi(G/P).$$

Moreover, $\psi(G) = \psi(G/P)\psi(P)$ if and only if P is central in G.

Proposition 2.2. [1] If G and H are finite groups. then

$$\psi(G \times H) \leq \psi(G)\psi(H),$$

with equality if and only if $\gcd(|H|, |G|) = 1$.

Lemma 2.3. [4] Let G be a finite group of order $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where $p_1, p_2, \ldots, p_k$ are distinct primes. Let $\frac{r}{s}\psi(C_n) < \psi(G)$, for some integers r, s. Then there exists a cyclic subgroup $\langle x \rangle$ such that:

$$[G : \langle x \rangle] < \frac{s}{r}.\frac{p_1+1}{p_1} \ldots \frac{p_k+1}{p_k}$$

Proposition 2.4. [5] Let $p_1 < p_2 < \ldots < p_t = p$ be prime divisors of n and denote the corresponding Sylow subgroups of C_n by $P_1, P_2, \ldots, P_t$. Then

$$\psi(C_n) = \prod_{i=1}^{t} \psi(P_i) \geq \frac{2}{p+1}n^2.$$

Proposition 2.5. Let G be a finite group and suppose that there exists $x \in G$ such that $[G : \langle x \rangle] < p$, where p is the largest prime divisor of $|G|$. Then G has a normal cyclic Sylow p-subgroup.

Proof . Since $[G : \langle x \rangle] < p$, thus $\langle x \rangle$ contains a cyclic Sylow p-subgroup P of G. Since $\langle x \rangle \leq N_G(P)$, we have $|G : N_G(P)| < p$. It follows that $N_G(P) = G$. $\square$

Lemma 2.6. [5] Let G be a finite group of order n satisfying $G = P \rtimes F$, where P is a cyclic p-group for some prime p, $|F| > 1$ and $\gcd(p, |F|) = 1$. Then the following statements hold:

1. Each element of F acts on P trivially or fixed-point-freely.
2. If $x \in F$, $o(x) = m$ and $u \in P$, then m is the least positive integer satisfying $(ux)^m \in P$.
3. If $u \in P$ and $x \in C_F(P)$, then $o(ux) = o(u)o(x)$.
4. If $u \in P$ and $x \in F \setminus C_F(P)$, then $o(ux) = o(x)$.
5. Let $Z = C_F(P)$. Then

$$\psi(G) = \psi(P)\psi(Z) + |P|\psi(F \setminus Z) < \psi(P)\psi(Z) + |P|\psi(F).$$

Theorem 2.7. [5] If $H \leq C_n$, then $\psi(H) \leq \frac{\psi(C_n)}{3}$, if n is odd, then $\psi(H) \leq \frac{\psi(C_n)}{7}$.

Lemma 2.8. Let G be a group of order n satisfying $G = P \rtimes F$, where P is a cyclic p-group, $p > 3$ is the largest prime divisor of $|G|$, and F is a nilpotent subgroup of G. If G is not nilpotent, then

$$\psi(G) \leq \frac{13}{21}\psi(C_n).$$

Proof . Notice that $n = |P||F|$ and $\gcd(|P|, |F|) = 1$. Hence $\psi(C_n) = \psi(P)\psi(C_{|F|})$. If F is a cyclic subgroup of G then $\psi(P)\psi(F) = \psi(C_n)$, otherwise $\psi(P)\psi(F) \leq \psi(P)\left(\frac{7}{11}\psi(C_{|F|})\right) \leq \frac{7}{11}\psi(C_n)$, by Theorem 1.1. We suppose that $C_F(P) = Z$. We have $Z < F$. First suppose that F is a cyclic subgroup of G, then by Lemma 2.6(5), we have

$$\psi(G) = \psi(P)\psi(Z) + |P|\psi(F \setminus Z) < \psi(P)\psi(Z) + |P|\psi(F) = \psi(P)\psi(F)\left(\frac{\psi(Z)}{\psi(F)} + \frac{|P|}{\psi(P)}\right) = \psi(C_n)\left(\frac{\psi(Z)}{\psi(F)} + \frac{|P|}{\psi(P)}\right).$$

Notice that P is a cyclic p-subgroup and $p \geq 5$, hence

$$\frac{|P|}{\psi(P)} = \frac{|P|(p+1)}{p|P|^2 + 1} \leq \frac{p+1}{p^2} = \frac{1}{p} + \frac{1}{p^2} \leq \frac{6}{25} \leq \frac{1}{4}.$$

If F is cyclic, then $\frac{\psi(Z)}{\psi(F)} \leq \frac{1}{3}$, by Theorem 2.7. Therefore

$$\psi(G) \leq \psi(C_n)\left(\frac{1}{3} + \frac{1}{4}\right) \leq \frac{7}{12}\psi(C_n) < \frac{13}{21}\psi(C_n).$$

Now, we assume that F is not cyclic. Then Lemma 2.6(5) implies that

$$\begin{aligned}\psi(G) = \psi(P)\psi(Z) + |P|\psi(F \setminus Z) &< \psi(P)\psi(Z) + |P|\psi(F) \\ &\leq \psi(P)\psi(C_{|Z|}) + |P|\left(\frac{7}{11}\psi(C_{|F|})\right) \\ &\psi(P)\psi(C_{|F|})\left(\frac{\psi(C_{|Z|})}{\psi(C_{|F|})} + \frac{7}{11}\frac{|P|}{\psi(P)}\right) \\ &= \psi(C_n)\left(\frac{\psi(C_{|Z|})}{\psi(C_{|F|})} + \frac{7}{11}\frac{|P|}{\psi(P)}\right).\end{aligned}$$

Similarly,

$$\frac{\psi(C_{|Z|})}{\psi(C_{|F|})} \leq \frac{1}{3} \quad \text{and} \quad \frac{|P|}{\psi(P)} \leq \frac{1}{4},$$

which implies that

$$\psi(G) \leq \left(\frac{1}{3} + \frac{1}{4}\cdot\frac{7}{11}\right)\psi(C_n) < \frac{13}{21}\psi(C_n).$$

□

Theorem 2.9. [7] Let G be a finite group and p be the largest prime divisor of $|G|$. Suppose that G contains a cyclic subgroup X of index p. Then $G = P \rtimes K$, where P is the Sylow p-subgroup of G and $K \leq X$. Moreover, one of the following holds:

1. G is nilpotent,
2. P is cyclic,
3. $G = (\langle a\rangle \times \langle b\rangle) \rtimes \langle y\rangle$, where $|a| = p^{n-1}$ for some integer $n \geq 2$, $|b| = p$, $(|y|, p) = 1$, $a^y = a$ and $b^y = b^r$ for some integer which is not congruent to 1 modulo p.

Theorem 2.10. [7] Let G be a non-cyclic group of order $n = 2m$, with m odd integer. Then

$$\psi(G) \leq \frac{13}{21}\psi(C_n)$$

with equality if and only if $G = S_3 \times C_{\frac{n}{6}}$, where $n = 6m_1$, with $(m_1, 6) = 1$ and S_3 is the symmetric group on three letters.

Theorem 2.11. [7] Let $G = (\langle x\rangle \times \langle b\rangle) \rtimes \langle y\rangle$, where p is an odd prime number, $K = \langle y\rangle, P = \langle x\rangle \times \langle b\rangle, |x| = p^{n-1}$ for some integer $n \geq 2$, $|b| = p$, $(|y|, p) = 1$, $x^y = x$ and $b^y = b^r$ for some integer which is not congruent to 1 modulo p. Then

$$\psi(G) = (p-1)^2\psi(Z) + p\psi(\langle x\rangle)\psi(\langle y\rangle),$$

where $Z = C_{\langle y\rangle}(b)$. In particular,

$$\psi(\langle x\rangle \times \langle b\rangle) = \frac{p^{2n} + p^3 - p^2 + 1}{p+1}.$$

3 Proof of the main Theorem

Proof of the Theorem 1.4. We know that G is a solvable group, by Theorem 1.3. We prove by induction on $|\pi(G)|$ that if G is a group of order $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ where p_i are primes, $p_1 < p_2 < \ldots < p_k = p$ such that $\alpha_i > 0$, for each $1 \leq i \leq k$, and also $\psi(G) > \frac{13}{21}\psi(C_n)$, then G is nilpotent group. If $|\pi(G)| = 1$, then G is a p-group, therefore G is nilpotent. Assume that $|\pi(G)| \geq 2$ and the theorem holds for each group H such that $|\pi(H)| < |\pi(G)|$. Let p be the largest divisor of $|G|$, now we consider the two following cases:

Case 1. $p \geq 5$. By proposition 2.4, we have

$$\psi(C_n) \geq \frac{2}{p+1}n^2$$

and our assumption $\psi(G) > \frac{13}{21}\psi(C_n)$, implies that

$$\psi(G) > \frac{13}{21}\cdot\frac{2}{p+1}n^2.$$

Hence there exists $x \in G$ such that

$$o(x) > \frac{26}{21}\cdot\frac{n}{p+1}$$

and

$$[G : \langle x\rangle] < \frac{21}{26}(p+1) \leq \frac{21}{26}\cdot\frac{6}{5}p.$$

So that

$$[G : \langle x\rangle] < \frac{63}{65}p.$$

Thus $[G : \langle x\rangle] < p$ and $\langle x\rangle$ contains a normal cyclic Sylow p-subgroup P of G, by Proposition 2.5. By Lemma 2.1, we have:

$$\frac{13}{21},\ \psi(C_n) < \psi(G) \leq \psi(P)\psi(G/P).$$

On the other hand $\psi(C_n) = \psi(C_{|P|})\psi(C_{|G/P|})$. Since $\psi(C_{|P|}) = \psi(P)$ it follows that:

$$\frac{13}{21}\psi(C_{|G/P|}) < \psi(G/P)$$

Now inductive hypotheses implies that G/P is a nilpotent group and $G = P \rtimes F$, where $F \cong G/P$ also P is a cyclic normal Sylow $p-$group for some prime p, F is a nilpotent subgroup, $|F| > 1$ and $\gcd(p, |F|) = 1$.

Suppose that $C_F(P) = Z$. If $Z < F$, then by Lemma 2.8, $\psi(G) \leq \frac{13}{21}\psi(C_n)$, which is a contradiction. Hence $C_F(P) = F$ and $G = P \times F$ so that G is nilpotent as required.

Case 2. $p \leq 3$. If $p = 2$, then G is a 2-group and nilpotent. Hence we assume that $p = 3$. If G is a 3-group, then G is nilpotent. So we may assume that $n = 2^a 3^b$ for some positive integers a and b. If $a = 1$ then $n = 2m$ with m odd, thus by Theorem 2.10, $\psi(G) \leq \frac{13}{21}\psi(C_n)$, contrary to our assumption.

We assume that $a \geq 2$. Since $\psi(G) > \frac{13}{21}\psi(C_n)$, there exists a cyclic subgroup $\langle x \rangle$ such that $[G : \langle x \rangle] < \frac{42}{13}$, by Lemma 2.3. It follows that $[G : \langle x \rangle] \leq 3$, which implies that $[G : \langle x \rangle] = 2$ or $[G : \langle x \rangle] = 3$.

Suppose that $[G : \langle x \rangle] = 2$. Then $\langle x \rangle$ contains a normal cyclic Sylow 3-subgroup P of G, by Proposition 2.5. If there exists $y \in G \setminus \langle x \rangle$ with $[G : \langle y \rangle] = 2$, then $y \in C_G(P)$ and hence $P \leq Z(G)$, Thus $G = P \times F$, where F is a non-cyclic Sylow 2-subgroup of G and it follows that G is nilpotent. If $o(y) \leq \frac{n}{3}$ for all $y \in G \setminus \langle x \rangle$, we have

$$\psi(G) \leq \psi(C_{\frac{n}{2}}) + \frac{n}{2} \cdot \frac{n}{3}.$$

We show that $\psi(G) \leq \frac{13}{21}\psi(C_n)$ as follows:

$$\begin{aligned}
\psi(C_{\frac{n}{2}}) + \frac{n}{2} \cdot \frac{n}{3} &\leq \frac{13}{21}\psi(C_n) \\
\Leftrightarrow \psi(C_{2^{a-1}})\psi(C_{3^b}) + \frac{2^{2a}3^{3b}}{6} &\leq \frac{13}{21}\psi(C_{2^a})\psi(C_{3^b}) \\
\Leftrightarrow \left(\frac{2^{2a-1}+1}{3}\right)\left(\frac{3^{2b+1}+1}{4}\right) + 2^{2a-1}3^{2b-1} &< \frac{13}{21}\left(\frac{2^{2a+1}+1}{3}\right)\left(\frac{3^{2b+1}+1}{4}\right) \\
\Leftrightarrow 21\left(2^{2a-1}+1\right)\left(3^{2b+1}+1\right) + 7.2^{2a+1}3^{2b+1} &< 13\left(2^{2a+1}+1\right)\left(3^{2b+1}+1\right) \\
\Leftrightarrow 21.2^{2a-1}3^{2b+1} + 8.3^{2b+1} + 8 &< 6.2^{2a+1}3^{2b+1} + 31.2^{2a-1} + 96.3^{2b+1} \\
\Leftrightarrow 8.3^{2b+1} + 8 &< 3.2^{2a-1}3^{2b+1} + 31.2^{2a-1} \\
\Leftrightarrow 1 &< 2^{2a-4}3^{2b+2} + 31.2^{2a-4} - 3^{2b+1} \\
\Leftrightarrow 1 &< 3^{2b+1}\left(2^{2a-4}.3 - 1\right) + 31.2^{2a-4}.
\end{aligned}$$

For $a \geq 2$ is true which contradicts our assumption.

Finally, we suppose that $[G : \langle x \rangle] = 3$. By Theorem 2.9, $G = P \rtimes F$ where P is Sylow 3-subgroup of G and $F \leq \langle x \rangle$. Since $|G| = 2^a 3^b$, $|P| = 3^b$ and $F \leq \langle x \rangle$, we have $F \cong C_{2^a}$, and one of the following holds:

1. G is nilpotent,
2. $G \cong C_{3^b} \rtimes C_{2^a}$,
3. $G \cong (C_{3^{b-1}} \times C_3) \rtimes C_{2^a}$.

We show that, if $G \cong C_{3^b} \rtimes C_{2^a}$ or $G \cong (C_{3^{b-1}} \times C_3) \rtimes C_{2^a}$, then $\psi(G) \leq \frac{13}{21}\psi(C_n)$. First, Suppose that $G \cong P \rtimes F$ where $P = C_{3^b}$, $F = C_{2^a}$. Let $Z = C_F(P)$. Then $Z < F$ and Lemma 2.6(5) implies that

$$\psi(G) = \psi(P)\psi(Z) + |P|\psi(F \setminus Z).$$

If $b = 1$, then $G \cong C_3 \rtimes F$, $|G| = 3.2^a$, and $F = C_{2^a}$. Let $Z = C_F(C_3)$, then $\frac{\psi(Z)}{\psi(F)} \leq \frac{1}{3}$. Since $\psi(C_n) = \psi(C_3)\psi(F) = 7\psi(F)$, we have

$$\begin{aligned}
\psi(G) &= \psi(C_3)\psi(Z) + |C_3|\psi(F \setminus Z) \\
&= 7\psi(Z) + 3(\psi(F) - \psi(Z)) \\
&= 4\psi(Z) + 3\psi(F) \\
&\leq \frac{13}{3}\psi(F) \\
&= \frac{13}{21}\psi(C_n).
\end{aligned}$$

Now, we suppose that $b > 1$. Then $|P| \geq p^2$. Hence

$$\frac{|P|}{\psi(P)} = \frac{|P|(p+1)}{p|P|^2+1} \leq \frac{|P|(p+1)}{p|P|^2} \leq \frac{p+1}{p^3} = \frac{1}{p^2} + \frac{1}{p^3} \leq \frac{4}{27}.$$

By Theorem 2.11, $\frac{\psi(Z)}{\psi(F)} \leq \frac{1}{3}$. Therefore

$$\begin{aligned}\psi(G) &< \psi(C_n)\left(\frac{\psi(Z)}{\psi(F)} + \frac{|P|}{\psi(P)}\right)\\ &< \left(\frac{1}{3} + \frac{4}{27}\right)\psi(C_n)\\ &< \frac{13}{21}\psi(C_n).\end{aligned}$$

Now, we assume that $G = (\langle x\rangle \times \langle z\rangle) \rtimes \langle y\rangle$, where $\langle x\rangle = C_{3^{b-1}}$, $\langle z\rangle = C_3$, and $\langle y\rangle = C_{2^a}$. Then Theorem 2.11 implies that

$$\psi(G) = (p-1)^2\psi(Z) + p\psi(\langle x\rangle)\psi(\langle y\rangle),$$

where $Z = C_{\langle y\rangle}(z)$. Since $p = 3$, $b \geq 1$ and by Theorem 2.7, $\frac{\psi(Z)}{\psi(C_{2^a})} \leq \frac{1}{3}$, we have

$$\begin{aligned}\psi(G) &= \frac{4}{3}\psi(C_{2^a}) + 3\psi(C_{3^{b-1}})\psi(C_{2^a})\\ &= \left(\frac{4}{3} + 3\psi(C_{3^{b-1}})\right)\psi(C_{2^a})\\ &= \left(\frac{4}{3} + 3\frac{3^{2b-1}+1}{4}\right)\psi(C_{2^a})\\ &= \left(\frac{3^{2b+1}+25}{12}\right)\psi(C_{2^a})\\ &\leq \frac{13}{21}\left(\frac{3^{2b+1}+1}{4}\right)\psi(C_{2^a})\\ &\leq \frac{13}{21}\psi(C_{3^b})\psi(C_{2^a}) = \frac{13}{21}\psi(C_n).\end{aligned}$$

It follows that G is a nilpotent group. Thus the proof is complete.

Remark 3.1. We note that if $G = S_3$, then $|G| = 6$, and $\psi(G) = 13$. Since $\psi(C_6) = 21$, therefore

$$\frac{13}{21}\psi(C_{|G|}) = \psi(G).$$

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