Int. J. Nonlinear Anal. Appl. 14 (2023) 2, 11–22
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.24280.2708

Existence results for the σ-Hilfer fractional boundary value problem involving a generalized $(p_1(x), p_2(x), ..., p_n(x))$-Laplacian operator

Nadir Benkaci-Ali

University M'Hamed Bougara of Boumerdes, Algeria

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, we give the existence results of nontrivial positive solution to the integral-infinite point Hilfer-fractional boundary-value problem involving a generalized $(p_1(x), p_2(x), ..., p_n(x))$-Laplacian operator.

Keywords: Generalized $(p_1(x), p_2(x), ..., p_n(x))$-Laplacian operator; positive solution; fixed point index.)
2020 MSC: 34B15; 34B16; 34B18

1 Introduction

In this article, we give the existence results of a nontrivial positive solution to the following integral and infinite point Hilfer fractional boundary-value problem involving a weighted and generalized $(p_1(x), p_2(x), ..., p_n(x))$-Laplacian operator

$$\begin{cases} {}^H D_{0^+}^{\alpha,\omega,\sigma} \phi(x, p(x) D_{0^+}^{\beta} u(x)) + q(x) f(x, u(x)) = 0, \quad x \in (0,1), \\ u(0) = 0, \\ u(1) = \int_0^1 g(t)u(t)dt + \sum_{n=1}^{n=+\infty} \alpha_n u(\eta_n), \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha,\omega,\sigma}$ is the $\sigma-$Hilfer fractional derivative of order α and type $0 \leq \omega \leq 1$, $D_{0^+}^{\beta}$ is the Riemann-Liouville derivative of order β, σ is a function, $0 < \alpha < 1 < \beta \leq 2$ and $\alpha_n, \eta_n \in (0,1)$ for $n \geq 1$ such that

$$\sum_{n=1}^{n=+\infty} \alpha_n < \infty.$$

Throughout this paper, we assume that the following conditions are satisfied;

(A1) $\eta_0 = 0 < \eta_n < \eta_{n+1}$ for $n \in N$ with

$$\lim_{n \to +\infty} \eta_n = \eta < \infty.$$

Email address: radians_2005@yahoo.fr (Nadir Benkaci-Ali)

Received: July 2021 *Accepted:* September 2021

(A2) The functions $f : [0,1] \times R^+ \to R^+$ and $p, \sigma : [0,1] \to R^+$ are continuous such that $\sigma \in C^1([0,1])$ is increasing and for all $x \in [0,1]$

$$p(x) \neq 0 \text{ and } \sigma'(x) \neq 0$$

and $q, g : [0,1] \to R^+$ are measurable functions such that g is integrable and

$$0 < \sup_{t\in[0,1]} \int_0^t \sigma'(s)(\sigma(t) - \sigma(s))^{\alpha-1} q(s)ds < \infty.$$

(A3) $\int_0^1 t^{\beta-1} g(t)dt + \sum_{n=1}^{n=+\infty} \alpha_n \eta_n^{\beta-1} < 1$.

(A4) $\phi : [0,1] \times R \to R$ is continuous and for $t \in [0,1]$, the function $\phi(t,.)$ is odd and increasing , $\phi^{-1}(t,.)$ is the inverse function of $\phi(t,.)$ denoted by $\psi(t,.)$ where $\psi : [0,1] \times R \to R$ is continuous.

(A5) There exist $p^+, p^- \in \mathbb{R}$ with $p^+ \geq p^- > 1$ such that

$$\phi^-(x) \leq \phi(.,x) \leq \phi^+(x) \text{ for } (t,x) \in [0,1] \times \mathbb{R}, \tag{1.2}$$

with

$$\phi^-(x) = \begin{cases} \phi_{p^+}(x) \text{ if } x \in [0,1] \cup (-\infty,-1] \\ \phi_{p^-}(x) \text{ if } x \in [-1,0] \cup [1,+\infty) \end{cases} \tag{1.3}$$

and

$$\phi^+(x) = \begin{cases} \phi_{p^-}(x) \text{ if } x \in [0,1] \cup (-\infty,-1] \\ \phi_{p^+}(x) \text{ if } x \in [-1,0] \cup [1,+\infty) \end{cases} \tag{1.4}$$

Boundary value problems involving a p(t)-Laplacian operator have attracted a great deal of attention in the last ten years (see [3, 4, 6, 19]). At the same time, boundary value problems with fractional order and Hilfer fractional order differential equations involving p(t)- Laplacian are of great importance and are an interesting class of problems. Such kind of BVPs in Banach space has been studied by many authors, see, [1, 8, 10, 11, 12, 13, 15, 16, 17] and the references therein. Noting that the generalized ϕ-Laplacian operator can turn into the well known p(t)-Laplacian operator when we replace ϕ by $\phi_{p(t)}(x) = |x|^{p(t)-2}x$, so our results extend and enrich some existing papers.

The paper is organized as follows. In the first section, we recall some lemmas giving fxed point index calculations. In second section, we present a lemma making use of homotopical arguments of fixed point index in the first existence result (superlinear case). In third section, we give the related lemmas and a fixed point formulation for bvp (1.1). After that, we give our main results and their proofs. An example is given at the end of the paper to illustrate the main results.

2 Preliminaries

For sake of completeness let us recall some basic facts needed in this paper. Let E be a real Banach spce equipped with its norme noted $\|.\|$, $L(E)$ is the set of all linear continuous mapping from E into E. For $L \in L(E)$, $r(L) = \lim_{n\to\infty} \|L^n\|^{\frac{1}{n}}$ denotes the spectral radius of L. A nonempty closed convex subset K of E is said to be a cone if $K \cap (-K) = 0$ and $(tK) \subset K$ for all $t \geq 0$.

Let K be a cone in E. A cone K induces a partial ordering " $\leq$" , defined so that $x \leq y$ if and only if $y - x \in K$. K is said to be normal if there exists a positive constant N such that for all $u, v \in K$,

$$u \leq v \text{ implies } \|u\| \leq N\|v\|.$$

$L \in L(E)$ is said to be positive in K if $L(K) \subset K$, it is said to be strongly positive in K if $int(K) \neq \emptyset$ and $L(K \setminus \{0\}) \subset int(K)$, and it is said to be K - normal if for all $u, v \in K$,

$$u \leq v \text{ implies } \|Lu\| \leq \|Lv\|.$$

Let E be a real Banach space and let K be a cone. Let $R > 0$, $B(0,R)$ be the ball of radius R in E and $A : K_R \to K$ a completely continuous mapping, where $K_R = B(0,R) \cap K$. We will use the following lemmas concerning computations of the fixed point index, i, for a compact map A (See [7]).

Lemma 2.1. If $\|Ax\| < \|x\|$ for all $x \in \partial B(0,R) \cap K$, then $i(A, K_R, K) = 1$.

Lemma 2.2. If $\|Ax\| > \|x\|$ for all $x \in \partial B(0,R) \cap K$, then $i(A, K_R, K) = 0$.

Lemma 2.3. If $Ax \ngeq x$ for all $x \in \partial B(0,R) \cap K$, then $i(A, K_R, K) = 1$.

Lemma 2.4. If $Ax \nleq x$ for all $x \in \partial B(0,R) \cap K$, then $i(A, K_R, K) = 0$.

Lemma 2.5. If $Ax \neq \lambda x$ for all $x \in \partial B(0,R) \cap K$ and $\lambda > 1$, then $i(A, K_R, K) = 1$.

3 Fixed point index Lemma

Let $N : E \to E$ be an operator and K be a cone of a real Banach space E, and consider the partial ordering " $\leq$" in E, defined so that $x \leq y$ if and only if $y - x \in K$. Let $\rho \in K^*$, and consider the following cone $P = K(\rho) = \{u \in K : u \geq ||u||\rho\}$ and the positive value

$$\lambda_0(K) = \inf \Lambda^-(K)$$

where

$$\Lambda^-(K) = \{\lambda \geq 0 : \text{ there exists } u \in K \cap \partial B(0,1) \text{ such that } Nu \leq \lambda u\}.$$

Remark 3.1. If N is completely continuous, then by Lemmas 2.4 2.5, there exists $\lambda \geq 1$ such that $\lambda \in \Lambda^-(K)$.

Lemma 3.2. Assume that $N : E \to E$ is increasing, positively 1-homogeneous and completely continuous, such that $N(K \setminus \{0\}) \subset K \setminus \{0\}$.
If there exist $\rho \in K^*$ such that $NK \subset P = K(\rho)$, then $\lambda_0(K) = \lambda_0(P) > 0$.

Remark 3.3. Assume that $N : E \to E$is increasing, positively 1-homogeneous and completely continuous, such that $N(K\setminus \{0\}) \subset K\setminus \{0\}$. If there exist $\rho \in K$ such that $NK \subset P = K(\rho)$, then

$$\lambda_0(K) = \lambda_0(K) > 0,$$

Because, for $\lambda \geq 0, u \in K \cap \partial B(0,1)$ such that $Nu \leq \lambda u$. Since $NK \subset P$, N is strictly increasing and positively 1-homogeneous, we have

$$N(\frac{Nu}{||Nu||}) \leq \lambda \frac{Nu}{||Nu||}$$

then $\Lambda^-(K) \subset \Lambda^-(P)$ with $P \subset K$ we deduce $\Lambda^-(P) = \Lambda^-(K)$ and so $\lambda_0(K) = \lambda_0(P)$. Moreover, we have that $\lambda_0(P) > 0$. As, in the contrary, if we assume that there exist $(\lambda_n) \in R^+$ and $u_n \in P \cap \partial B(0,1)_n$ with $\lim_{n\to\infty} \lambda_n = 0$ such that $Nu_n \leq \lambda_n u_n$. For $n \in N$, $\lambda_n u_n \geq Nu_n \geq N(||u_n||\rho) = N(\rho)$ and so $\lambda_n u_n - N(\rho) \in K$. Then $\lim_{n\to\infty} \lambda_n u_n - N(\rho) = -N(\rho) \in K$, and we obtain $N(\rho) = 0$, which is a contradiction.

Remark 3.4. If K is a normal cone in a Banach space E, with the constant of normality $n = 1$ (i.e $||u|| \geq ||v||$ if $u \geq v \geq 0$), then $\lambda_0(K) \geq ||N(\rho)||$. Since for $\lambda \in \Lambda(P), u \in P \cap B(0,1)$,

$$\lambda u \geq Nu \geq N(||u||\rho) = N(\rho).$$

In the following lemma, we assume that N, $N_0 : E \to E$ are positively 1- homogeneous and completely continuous operators, with N is increasing such that $N(K \setminus \{0\}) \subset P \setminus \{0\}$, where $P = K(\rho)$, $\rho \in K^*$ and K is a normal cone in a Banach space E, (for simplicity, we can assume that the constant of normality $n = 1$).

Lemma 3.5. Let $Q, Q_0, G_2 : K \to K$ be continuous mappings with

$$\lim_{||u||\to+\infty} \frac{||Qu||}{||u||} < +\infty \text{ and } \lim_{||u||\to+\infty} \frac{||G_2 u||}{||u||} = 0 \leq \lim_{||u||\to+\infty} \frac{||Q_0 u||}{||u||} < +\infty, \tag{3.1}$$

such that

$$NQu - G_2 u \leq N_0 Q_0 u, \text{ for } u \in K \tag{3.2}$$

Suppose that there exist $\lambda_1 \in \mathbb{R}^+$ and $G_1 : K \to K$ with

$$\lim_{||u||\to+\infty} \frac{G_1 u}{||u||} = 0, \tag{3.3}$$

such that

$$Qu \geq \lambda_1 u - G_1(u), \text{ for } u \in K. \tag{3.4}$$

If

$$\lambda_1 > \lambda_0^{-1}(K), \tag{3.5}$$

then there exist $R_1 > 0$ such that for all $R \geq R_1$, $i(N_0 Q_0, K_R, K) = i(NQ, P_R, P)$. Moreover, if

$$\lambda_1 > ||N(\rho)||^{-1}, \tag{3.6}$$

then there exist $R_2 > 0$ such that for all $R \geq R_2$, $i(N_0 Q_0, K_R, K) = 0$.

Proof . First, we show that there exists $R_1 > 0$ such that for all $R \geq R_1$, $i(NQ, K_R, K) = i(N_0 Q_0, K_R, K)$. We consider the homotopy $H(t, u) = tN_0 Q_0 u + (1-t)NQu$. We show that there exists $R_1 > 0$ such that for all $R \geq R_1$ the equation $H(t, u) = u$ has not solutions in $[0, 1] \times (K \cap \partial B(0, R))$. In the contrary, we assume that for all $n \in N$, there exist $R_n \geq n$ and $(t_n, u_n) \in [0, 1] \times (K \cap \partial B(0, R_n))$ such that

$$u_n = H(t_n, u_n) = t_n N_0 Q_0 u_n + (1-t_n) NQu_n. \tag{3.7}$$

By dividing the above equation by $||u_n||$ we obtain

$$v_n = \frac{u_n}{||u_n||} = t_n N_0 \Big(\frac{Q_0 u_n}{||u_n||}\Big) + (1-t_n) N \Big(\frac{Qu_n}{||u_n||}\Big). \tag{3.8}$$

From 3.1,

$$\lim_{n\to\infty} \frac{||Q_0 u_n||}{||u_n||} < \infty \text{ and } \lim_{n\to\infty} \frac{||Qu_n||}{||u_n||} < \infty,$$

then the sequences $\left(\frac{Q_0 u_n}{||u_n||}\right)_n, \left(\frac{Qu_n}{||u_n||}\right)_n$ are bounded, and we deduce from the compactness of N and N_0, that $(v_n)_n$ admits a convergent sub-sequence also denoted by $(v_n)_n$. Let $v = \lim_{n\to\infty} v_n \in K \cap \partial B(0, 1)$ and $t = \lim_{n\to\infty} t_n$. By using the conditions 3.2 and 3.4, it follows from 3.8 that for all $n \in N$

$$v_n \geq N\left(\frac{Qu_n}{||u_n||}\right) - t_n \frac{G_2 u_n}{\|u_n\|} \geq N\left(\lambda_1 v_n - \frac{G_1 u_n}{\|u_n\|}\right) - t_n \frac{G_2 u_n}{\|u_n\|}.$$

With the fact that

$$\lim_{n\to+\infty} \frac{G_1 u_n}{||u_n||} = \lim_{n\to+\infty} \frac{G_2 u_n}{||u_n||} = 0$$

we have $v \geq \lambda_1 N v$, and so $\lambda_1^{-1} \in \Lambda^-(K)$ where

$$\Lambda^-(K) = \{\lambda \geq 0; \text{ there exists } u \in K \cap \partial B(0, 1) \text{ such that } Nu \leq \lambda u\}.$$

Then $\lambda_1^{-1} \geq \lambda_0(K)$, which contradicts 3.5. Then there exist $R_1 > 0$ such that for all $R \geq R_1$ the equation $H(t, u) = u$ has not the solutions in $[0, 1] \times (K \cap \partial B(0, R))$, and by invariance property of fixed point index we deduce that for all $R \geq R_1$ $i(NQ, K_R, K) = i(N_0 Q_0, K_R, K)$. By the fact that $NQ(K) \subset P$, we have from the excision property of the fixed point index that $i(NQ, K_R, K) = i(NQ, P_R, P)$. Then $i(N_0 Q_0, K_R, K) = i(NQ, P_R, P)$. Now, we assume that the condition 3.6 holds. By using Lemma 2.4, we prove that there exists $R_0 > 0$ such that, for all $R \geq R_0$, $i(NQ, P_R, P) = 0$. In the contrary, we assume that for all $n \in N$, there exist $R_n \geq n$and $u_n \in P \cap \partial B(0, R_n)$ such that $u_n \geq NQu_n$. By the condition 3.4 we have

$$v_n = \frac{u_n}{\|u_n\|} \geq N\left(\lambda_1 v_n - \frac{G_1 u_n}{\|u_n\|}\right) - t_n \frac{G_2 u_n}{\|u_n\|}, \tag{3.9}$$

with $u_n \geq \rho ||u_n||$. Then

$$v_n \geq N\big(\lambda_1 \rho - \frac{G_1(u_n)}{||u_n||}\big) - t_n \frac{G_2 u_n}{||u_n||}.$$

Set $A_n = N\left(\lambda_1\rho - \frac{G_1(u_n)}{||u_n||}\right) - t_n\frac{G_2 u_n}{||u_n||} - N(\lambda_1\rho)$. As $\lim_{n\to\infty}\frac{G_1(u_n)}{||u_n||} = \lim_{n\to\infty}\frac{G_2(u_n)}{||u_n||} = 0$, we have $\lim_{n\to\infty} A_n = 0$ and $v_n - A_n \geq N(\lambda_1\rho) \geq 0$. Since K is normal with the constant of normality $N = 1$, for $n \in N$,

$$||v_n - A_n|| \geq ||N(\lambda_1\rho))||,$$

and so

$$1 = \lim_{n\to\infty}||v_n|| = \lim_{n\to\infty}||v_n - A_n|| \geq \lambda_1||N(\rho))||,$$

then

$$1 \geq \lambda_1||N(\rho)||,$$

which contradicts 3.6. Consequently, for $R \geq R_2 = \max\{R_1, R_0\}$, $i(N_0Q_0, K_R, K) = i(NQ, P_R, P) = 0$. Thus, the proof is completed. □

4 Related lemmas

Definition 4.1. [11, 12] The Riemann-Liouville fractional integral of order $p > 0$ of $f \in L^1([a,b],\mathbb{R}^+)$, is defined by

$$I^p_{a^+}f(x) = \frac{1}{\Gamma(p)}\int_a^x (x-t)^{p-1}f(t)dt, \tag{4.1}$$

where Γ is the gamma function.

Definition 4.2. [11, 12] The Riemann-Liouville fractional derivative of order $p \geq 0$ of a function f is defined by

$$D^p_{a^+}f(x) = \frac{d^n}{dx^n}I^{n-p}_{a^+}f(x), n = [\alpha]+1 \tag{4.2}$$

where $[n]$ is the integer part of α.

Remark 4.3. If $p \in \mathbb{N}$, then $D^p_{a^+}f = \frac{\delta^p}{\delta x^p}f$, and for $p = 1$, $I^1_{a^+}f(x) = \int_a^x f(t)dt$.

Lemma 4.4. [9] Let $p > 0$, and let $u(t)$ be an integrable function in $[a,b]$.

$$I^p_{a^+}D^p_{a^+}u(x) = u(x) + c_1(x-a)^{p-1} + c_2(x-a)^{p-2}... + c_n(x-a)^{p-n}, \tag{4.3}$$

where $c_k \in \mathbb{R}$, $k \in \{1,2,...,n\}$, $n = [\alpha]+1$ and $[\alpha]$ is the integer part of α.

Definition 4.5. [16] Let $-\infty < a < b < +\infty$ and $\alpha > 0$. Also, let $\sigma(x)$ be an increasing and positive function on (a,b], having a continuous derivative $\sigma'(x)$ on (a,b). Then the left-sided fractional integral of a function u with respect to another function σ on $[a,b]$ is defined by

$$I^{\alpha,\sigma}_{a^+}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \sigma'(t)\left(\sigma(x)-\sigma(t)\right)^{\alpha-1}u(t)dt.$$

Definition 4.6. [16] Let $\alpha \in (n-1,n)$ with $n \in \mathbb{N}$, $I = [a,b]$ is the interval such that $(-\infty < a < b < +\infty)$ and u, $\sigma \in C^n(I,\mathbb{R})$ two functions such that σ is increasing and $\sigma'(t) \neq 0$, for all $t \in I$. The σ-Hilfer fractional derivative ${}^H D^{\alpha,\omega,\sigma}_{a^+}$ of u of order $n-1 < \alpha < n$ and type $0 \leq \omega \leq 1$ is defined by

$${}^H D^{\alpha,\omega,\sigma}_{a^+}u(x) = I^{\omega(n-\alpha),\sigma}_{a^+}\left(\frac{1}{\sigma'(x)}\frac{\partial}{\partial x}\right)^n I^{(1-\omega)(n-\alpha),\sigma}_{a^+}u(x).$$

Let's also recall the following important result [16]:

Theorem 4.7. If $u \in C^n(I)$, $n-1 < \alpha < n$, $0 \leq \omega \leq 1$, and $\xi = \alpha + \omega(n-\alpha)$ then

$$I^{\alpha,\sigma}_{a^+}.{}^H D^{\alpha,\omega,\sigma}_{a^+}u(x) = u(x) - \sum_{k=1}^{n}\frac{(\sigma(x)-\sigma(a))^{\xi-k}}{\Gamma(\xi-k+1)}\left(\frac{1}{\sigma'(x)}\frac{\partial}{\partial x}\right)^{n-k} I^{(1-\omega)(n-\alpha),\sigma}_{a^+}u(a).$$

Moreover, ${}^H D^{\alpha,\omega,\sigma}_{a^+}I^{\alpha,\sigma}_{a^+}u = u$.

Remark 4.8. In this paper, we assume that $\sigma(x)$ is increasing and positive on $(0,1]$ with $\sigma(0) = 0$, having a continuous derivative $\sigma'(x)$ on $(0,1)$ and $\sigma'(x) \neq 0$ for all $x \in [0,1]$. If $\alpha \in (0,1)$, then $n = 1$ and

$$I_{0^+}^{\alpha,\sigma}.{}^{H}D_{0^+}^{\alpha,\omega,\sigma}u(x) = u(x) - \frac{(\sigma(x))^{\xi-1}}{\Gamma(\xi)}(I_{0^+}^{(1-\omega)(1-\alpha),\sigma}u)(0).$$

Lemma 4.9. Let $h \in L(0,1)$. The unique continuous solution of

$$\begin{cases} {}^{H}D_{0^+}^{\alpha,\omega,\sigma}\phi(x,p(x)D_{0+}^{\beta}u(x)) + h(x) = 0, & x \in (0,1), \\ u(0) = 0, \quad u(1) = \int_0^1 g(t)u(t)dt + \sum_{n=1}^{n=+\infty}\alpha_n u(\eta_n), \end{cases} \tag{4.4}$$

is given by $u = N_0 H$, where

$$N_0 u(x) = \int_0^1 G(x,t)u(t)dt,$$

$$H(t) = \frac{1}{p(t)}\psi\Big(t, \frac{1}{\Gamma(\alpha)}\int_0^t \sigma'(t)(\sigma(t)-\sigma(s))^{\alpha-1}h(s)ds\Big),$$

and

$$G(x,t) = \frac{1}{\Gamma(\beta)}\begin{cases} x^{\beta-1}G_m(t) - (x-t)^{\beta-1} & \text{if } 0 \le t < \min\{x,\eta\}, \\ x^{\beta-1}G_\eta(t) - (x-t)^{\beta-1} & \text{if } \eta \le t \le x, \\ x^{\beta-1}G_m(t) & \text{if } x \le t < \eta, \\ x^{\beta-1}G_\eta(t) & \text{if } t \ge \max\{x,\eta\} \end{cases} \tag{4.5}$$

with $\eta = \lim_{n\to\infty}\eta_n$, and $m \in \mathbb{N}^*$ such that $\eta_{m-1} \le t \le \eta_m$, where

$$G_m(t) = \frac{\mu(t) - \sum_{n\ge m}\alpha_n(\eta_n - t)^{\beta-1}}{1-L}, \qquad G_\eta(t) = \frac{\mu(t)}{1-L}$$

and

$$\mu(t) = (1-t)^{\beta-1} - \int_t^1 (s-t)^{\beta-1}g(s)ds$$

with

$$L = \sum_{n\ge1}\alpha_n\eta_n^{\beta-1} + \int_0^1 s^{\beta-1}g(s)ds < 1.$$

Proof . Let $u \in C([0,1])$. By Theorem 4.7, equation ${}^{H}D_{0^+}^{\alpha,\omega,\sigma}\phi(x,p(x)D_{0^+}^{\beta}u(x)) + h(x) = 0$ gives

$$\phi(x,p(x)D_{0^+}^{\beta}u(x)) = -I_{0^+}^{\alpha,\sigma}h(x) + \lim_{x\to0}\frac{(\sigma(x)-\sigma(0))^{\xi-1}}{\Gamma(\xi-k+1)}I_{0^+}^{(1-\omega)(1-\alpha),\sigma}\phi(x,p(x)D_{0^+}^{\beta}u(x))$$

and from $\lim_{x\to0} I_{0^+}^{(1-\omega)(1-\alpha),\sigma}\phi(x,p(x)D_{0^+}^{\beta}u(x)) = 0$, we have that $D_{0^+}^{\beta}u(t) = -H(t)$ with

$$H(t) = \frac{1}{p(t)}\psi\Big(t, \frac{1}{\Gamma(\alpha)}\int_0^t \sigma'(s)(\sigma(t)-\sigma(s))^{\alpha-1}h(s)ds\Big).$$

And also from Lemma 4.4, we have

$$u(x) = \begin{cases} -I_{0^+}^{\beta}H(x) + d_1x^{\beta-1} + d_2x^{\beta-2} & \text{if } 1 < \beta < 2 \\ -I_{0^+}^{\beta}H(x) + d_1x + d_2' + d_3x^{-1} & \text{if } \beta = 2. \end{cases}$$

As u is continuous at 0 and $u(0) = 0$, then $d_2 = d_2' = d_3 = 0$, then

$$u(x) = -I_{0^+}^{\beta}H(x) + d_1x^{\beta-1}, \text{ for } \beta \in (1,2].$$

In addition, from equation

$$u(1) = \sum_{n\ge1}\alpha_n u(\eta_n) + \int_0^1 g(s)u(s)ds$$

we deduce that

$$\Gamma(\beta)(1-L)d_1 = -\sum_{n\geq 1}\alpha_n\int_0^{\eta_n}(\eta_n-t)^{\beta-1}H(t)dt - \int_0^1 g(t)\int_0^t (t-s)^{\beta-1}H(s)dsdt + \int_0^1(1-t)^{\beta-1}H(t)dt$$

with $L=\sum_{n\geq 1}\alpha_n\eta_n^{\beta-1}+\int_0^1 s^{\beta-1}g(s)ds$. Then

$$u(x)=\frac{1}{\Gamma(\beta)}\Big[\frac{C.x^{\beta-1}}{1-L}-\int_0^x (x-t)^{\beta-1}H(t)dt\Big]$$

where

$$C=\int_0^1(1-t)^{\beta-1}H(t)dt-\int_0^1 H(t)\int_t^1(t-s)^{\beta-1}g(s)dsdt-\sum_{n\geq 1}\alpha_n\int_0^{\eta_n}(\eta_n-t)^{\beta-1}H(t)dt.$$

Consequently, the solution of 4.4 is

$$u(x)=\int_0^1 G(x,t)H(t)dt$$

with

$$G(x,t)=\frac{1}{\Gamma(\beta)}\begin{cases} x^{\beta-1}G_m(t)-(x-t)^{\beta-1} & \text{if } 0\leq t<\min\{x,\eta\}\\ x^{\beta-1}G_\eta(t)-(x-t)^{\beta-1} & \text{if } \eta\leq t\leq x\\ x^{\beta-1}G_m(t) & \text{if } x\leq t<\eta\\ x^{\beta-1}G_\eta(t) & \text{if } t\geq \max\{x,\eta\}\end{cases} \tag{4.6}$$

with $\eta=\lim_{n\to\infty}\eta_n$, and $m\in N^*$ such that $\eta_{m-1}\leq t\leq \eta_m$, where

$$G_m(t)=\frac{\mu(t)-\sum_{n\geq m}\alpha_n(\eta_n-t)^{\beta-1}}{1-L}$$

$$G_\eta(t)=\frac{\mu(t)}{1-L}$$

and

$$\mu(t)=(1-t)^{\beta-1}-\int_t^1(s-t)^{\beta-1}g(s)ds.$$

This finishes the proof. □

Lemma 4.10. G is continuous in $[0,1]^2$, and for $x,t\in[0,1]$, we have $h_1(t)x^\beta\leq G(x,t)\leq h_2(t)x^{\beta-1}$, where

$$h_1(t)=\frac{(1-t)^{\beta-1}\int_0^t s^{\beta-1}g(s)ds}{\Gamma(\beta)(1-L)} \text{and} \quad h_2(t)=\frac{\mu(t)}{\Gamma(\beta)(1-L)}$$

with

$$L=\int_0^1 s^{\beta-1}g(s)ds+\sum_{n\geq 1}\alpha_n\eta_n^{\beta-1}.$$

Proof . It's clear that G is continuous in $[0,1]^2$, and the right hand inequality $G(x,t)\leq h_2(t)x^{\beta-1}$ is obvious. Now, we show that $G(x,t)\geq h_1(t)x^\beta$, where

$$h_1(t)=\frac{(1-t)^{\beta-1}\int_0^t s^{\beta-1}g(s)ds}{\Gamma(\beta)(1-L)}$$

with

$$L=\int_0^1 s^{\beta-1}g(s)ds+\sum_{n\geq 1}\alpha_n\eta_n^{\beta-1}.$$

Let $x, t \in [0,1]$. For $n \in N^*$, as $t \geq t\eta_n$ and $t \geq tx$, we have

$$(\eta_n - t)^{\beta-1} \leq \eta_n^{\beta-1}(1-t)^{\beta-1}, \qquad (x-t)^{\beta-1} \leq x^{\beta-1}(1-t)^{\beta-1},$$

and $t \geq ts$ gives

$$\int_t^1 (s-t)^{\beta-1} g(s)ds \leq (1-t)^{\beta-1} \int_t^1 s^{\beta-1} g(s)ds.$$

For $t, x \in [0,1]$, we have

$$G(x,t) \geq \frac{x^{\beta-1}(1-t)^{\beta-1}}{\Gamma(\beta)} \Big[\frac{1 - \int_t^1 s^{\beta-1}g(s)ds - \sum_{n\geq 1}\alpha_n\eta_n^{\beta-1}}{1-L} - 1\Big]$$

and with $x^{\beta-1} \geq x^\beta$ leads

$$G(x,t) \geq \frac{x^{\beta}(1-t)^{\beta-1}}{\Gamma(\beta)} \Big[\frac{\int_0^1 s^{\beta-1}g(s)ds - \int_t^1 s^{\beta-1}g(s)ds}{1-L}\Big]$$

then $G(x,t) \geq x^\beta h_1(t)$. □

Remark 4.11. According to Lemma 4.9, u is solution of 1.1 if and only if $u = Tu$, where $T = N_0 Q_0$ with

$$N_0 u(x) = \int_0^1 G(x,t)u(t)dt \quad \text{ and } \quad Q_0 u(t) = \frac{1}{p(t)}\psi\Big(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} q(s) f(s,u(s))ds\Big).$$

Let $E = C([0,1])$ be the Banach space equipped with the sup-norm $||u|| = sup_{x\in[0,1]}|u(x)|$, and the cone

$$K = E^+ = \{u \in E;\ u \geq 0\}.$$

Lemma 4.12. $T : E \to E$ is completely continuous and $TK \subset K$.

Proof . $T = N_0 Q_0$ is the composition of the compact operator N_0 and the continuous operator Q_0, so, the Theorem of Ascoli-Arzela garantees that T is completely continuous. Moreover, since f, p and q are positive, then $TK \subset K$. □

Remark 4.13. We have from 1.2 of the condition (A5) that

$$\psi^+(x) \leq \psi(.,x) \leq \psi^-(x) \text{ for } t \in [0,1], \tag{4.7}$$

where ψ^-, ψ^+ are the inverse functions of ϕ^-, ϕ^+ respectively , defined by

$$\psi^-(x) = \begin{cases} \psi_{p^+}(x) \text{ if } x \in [0,1] \cup (-\infty,-1] \\ \psi_{p^-}(x) \text{ if } x \in [-1,0] \cup [1,+\infty) \end{cases} \tag{4.8}$$

and

$$\psi^+(x) = \begin{cases} \psi_{p^-}(x) \text{ if } x \in [0,1] \cup (-\infty,-1] \\ \psi_{p^+}(x) \text{ if } x \in [-1,0] \cup [1,+\infty) \end{cases} \tag{4.9}$$

Then there exist $c, e > 0$ such that for all $(t,x) \in [0,1] \times \mathbb{R}^+$

$$\psi_{p^-}(x) + e \geq \psi(t,x) \geq \psi_{p^+}(x) - c. \tag{4.10}$$

5 Main results

Let $N : K \to P = K(\rho)$ be an operator defined by

$$Nu(x) = x^\beta \int_0^1 \frac{h_1(t)}{p(t)} \psi_{p^+}\Big(t, \frac{1}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1} q_0(s)\phi_{p^+}(u(s))ds\Big)dt$$

where $\rho(x) = x^\beta$, $q_0(s) = \sigma'(s).q(s)$ and set $Qu(t) = \psi_{p^+}(f(t,u(t)))$, and

$$\lambda = \phi_{p^+}\Big[\int_0^1 \frac{h_2(t)}{p(t)} \psi_{p^+}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1} q_0(s)ds\Big)dt\Big]^{-1}.$$

Theorem 5.1. Assume that there exist $r_0 > 0, r_1 > 0$ and $\gamma > \phi_{p^+}(||N(\rho)||^{-1})$, such that

$$f(t,x) < \lambda\phi_{p^+}(x), \text{ for } (t,x) \in [0,1] \times [0,r_0]. \tag{5.1}$$

and

$$f(t,x) \geq \gamma\phi_{p^+}(x), \text{ for } (t,x) \in [0,1] \times [r_1,+\infty) \tag{5.2}$$

with

$$\lim_{x\to+\infty} \frac{\sup_{t\in[0,1]}\{f(t,x)\}}{\phi_{p^-}(x)} < \infty, \tag{5.3}$$

then problem 1.1 has at least one nontrivial positive solution.

Proof . In first, we show that $i(T,K_r,K) = 1$ for some

$$r \leq \min\left\{r_0, 1, \psi_{p^+}\frac{\Gamma(\alpha+1)}{\lambda q_\infty}\right\}.$$

We have

$$\lim_{r\to0^+} \frac{1}{\Gamma(\alpha)} \int_0^t (\sigma(t)-\sigma(s))^{\alpha-1} q_0(s)\lambda\phi_{p^+}(r)ds = 0$$

uniformly in the compact $[0,1]$, and so, there exists $r \leq \min\{r_0,1\}$ such that for all $t \in [0,1]$

$$\frac{1}{\Gamma(\alpha)} \int_0^t (\sigma(t)-\sigma(s))^{\alpha-1} q_0(s)\lambda\phi_{p^+}(r)ds \leq 1.$$

From 5.1, we have $f(t,x) < \lambda\phi_{p^+}(x), (t,x) \in [0,1]\times[0,r]$. For $u \in \partial B(0,r)\cap K$,

$$\begin{aligned}
Tu(x) &= \int_0^1 \frac{G(t,x)}{p(t)}\psi\Big(t, \frac{1}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u(s))ds\Big)dt \\
&\leq \int_0^1 x^{\beta-1}\frac{h_2(t)}{p(t)}\psi\Big(t, \frac{1}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u(s))ds\Big)dt \\
&\leq \int_0^1 \frac{h_2(t)}{p(t)}\psi^-\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u(s))ds\Big)dt \\
&< \int_0^1 \frac{h_2(t)}{p(t)}\psi^-\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)\lambda\phi_{p^+}(u(s))ds\Big)dt \\
&< \int_0^1 \frac{h_2(t)}{p(t)}\psi^-\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)\lambda\phi_{p^+}(r)ds\Big)dt \\
&< \int_0^1 \frac{h_2(t)}{p(t)}\psi_{p^+}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)\lambda\phi_{p^+}(r)ds\Big)dt \\
&= r\int_0^1 \frac{h_2(t)}{p(t)}\psi_{p^+}\Big(\frac{\lambda}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)ds\Big)dt.
\end{aligned}$$

Then $||Tu|| < ||u||$. By Lemma 2.1, $i(T,K_r,K) = 1$. Now,by using Lemma 3.5, we show that there exists $R > 0$ such that $i(T,K_R,K) = 0$. In first, we have $T = N_0Q_0 \geq NQ - G_2$ and

$$\lim_{||u||\to+\infty} \frac{G_2(u)}{||u||} = 0$$

where

$$G_2u = c.x^\beta \int_0^1 \frac{h_1(t)}{p(t)}dt.$$

Then the condition 3.2 of Lemma 3.5 is satisfied. Now, we have from 5.2, for $x \geq r_1$, $\psi_{p^+}(f(t,x)) \geq \lambda_1 x$, with $\lambda_1 = \psi_{p^+}(\gamma) > ||N(\rho)||^{-1}$. Then there exists $d \in R$ such that $\psi_{p^+}(f(t,x)) \geq \lambda_1 x - d$, for $x \geq 0$. and set $G_1(u) = d$. We have for $u \in K$

$$Q(u)(t) \geq \lambda_1 u(t) - G_1(u)(t),$$

with

$$\lim_{||u||\to\infty}\frac{G_1(u)}{||u||}=0.$$

Moreover, from Remark 4.13, for $u\in K$,

$$Qu(t)=\psi_{p^+}(f(t,u(t)))\le\psi_{p^-}(f(t,u(t)))+e+c,\ \text{ for } t\in[0,1],$$

and

$$\begin{aligned}Q_0u(t)&=\psi\Big(t,\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u(s))ds\Big)\\&\le\psi_{p^-}\Big(t,\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u(s))ds\Big)+e.\end{aligned}$$

then from 5.3 we have

$$\lim_{||u||\to+\infty}\frac{||Q(u)||}{||u||}<\infty \text{ and } \lim_{||u||\to+\infty}\frac{||Q_0(u)||}{||u||}<\infty.$$

By Lemma 3.5, there exist $R>r_0$ such that $i(N_0Q_0,K_R,K)=0$. Hence, $T=N_0Q_0$ has at least one fixed point u in $K\cap(\overline{B}(0,R)\setminus B(0,r))$, which is a nontrivial positive solution for problem 1.1. □

Set

$$\lambda_2=\phi_{p^-}\Big[\int_0^1\frac{h_2(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)ds\Big)dt\Big]^{-1}$$

and

$$N_2(u)(x)=x^\beta\int_0^1\frac{h_1(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)\phi_{p^-}(u(s))ds\Big)dt.$$

Theorem 5.2. Assume that there exist $r_2>0, r_3>0$ and $\gamma>\phi_{p^-}(||N_2(\rho)||^{-1})$, such that

$$f(t,x)<\lambda_2\phi_{p^-}(x),\ \text{ for } (t,x)\in[0,1]\times[r_2,+\infty), \tag{5.4}$$

and

$$f(t,x)\ge\gamma\phi_{p^-}(x),\ \text{ for } (t,x)\in[0,1]\times[0,r_3], \tag{5.5}$$

then problem 1.1 has at least one nontrivial positive solution.

Proof . In first, by using Lemma 2.3, we show that ther exists $R\ge r_2$ such that $i(T,P_R,P)=1$. In the contrary, we assume that there exits a sequence $(u_n)_n$ in P with $\lim_{n\to\infty}||u_n||=\infty$, such that $Tu_n\ge u_n$. From 5.4, there exist $\epsilon>0$ and $b\in R$ such that

$$f(t,x)\le(\lambda_2-\epsilon)\phi_{p^-}(x)+b,\ \text{ for } (t,x)\in[0,1]\times[0,+\infty).$$

Then for $n\in N$

$$\begin{aligned}u_n\le Tu_n(x)&=\int_0^1\frac{G(t,x)}{p(t)}\psi\Big(t,\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u_n(s))ds\Big)dt\\&\le\int_0^1x^{\beta-1}\frac{h_2(t)}{p(t)}\psi\Big(t,\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u_n(s))ds\Big)dt\\&\le\int_0^1\frac{h_2(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u_n(s))ds\Big)dt+e\int_0^1\frac{h_2(t)}{p(t)}dt\\&\le\int_0^1\frac{h_2(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)[(\lambda_2-\epsilon)\phi_{p^-}(u_n(s))+b]ds\Big)dt+e\int_0^1\frac{h_2(t)}{p(t)}dt\\&\le||u_n||\psi_{p^-}(\lambda_2-\epsilon)\int_0^1\frac{h_2(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)(1+r_n)ds\Big)dt+e\int_0^1\frac{h_2(t)}{p(t)}dt\end{aligned}$$

where $r_n = \frac{b}{\phi_{p^-}(||u_n||)(\lambda_2-\epsilon)}$. Then

$$1 \quad \leq \quad \psi_{p^-}(\lambda_2-\epsilon)\int_0^1 \frac{h_2(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)(1+r_n)ds\Big)dt+\frac{e\int_0^1\frac{h_2(t)}{p(t)}dt}{||u_n||}$$

and with $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty}\frac{e\int_0^1\frac{h_2(t)}{p(t)}dt}{||u_n||}$, it follows the following contradiction $1 \leq \psi_{p^-}(\lambda_2-\epsilon)\psi_{p^-}(\lambda_2^{-1}) < 1$. Then there exists $R \geq r_2$ such that $i(T, P_R, P) = 1$. Now, we prove that $i(T, P_{r_3}, P) = 0$. Let $u \in P\cap\partial B(0, r_0)$, with

$$r_0 = \min\{1, r_3, \psi_{p^-}(\frac{\Gamma(\alpha+1)}{\gamma})\}.$$

We have $u \geq \rho||u||$, and from 5.5

$$\begin{aligned} Tu(1) &\geq \int_0^1 \frac{h_1(t)}{p(t)}\psi\Big(t, \frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)(\gamma\phi_{p^-}(u(s)))ds\Big)dt \\ &\geq \int_0^1\frac{h_1(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)(\gamma\phi_{p^-}(\rho(s)||u||)ds\Big)dt.\end{aligned}$$

Then $||Tu| \geq \psi_{p^-}(\gamma)||N_2(\rho)||||u|| > ||u||$. From Lemma 2.1, we have $i(T, P_{r_0}, P) = 0$. Consequently, $T = N_0Q_0$ has at least one fixed point u in $K\cap(\overline{B}(0,R)\setminus B(0,r_0))$, which is a nontrivial positive solution for problem 1.1. □

Example 5.3. We consider the following $(p_1(x), p_2(x), ..., p_N(x))$-Laplacian boundary value problem

$$\begin{cases}\sum_{k=1}^{k=N} HD_{0^+}^{\alpha,\omega,\sigma}\phi_{p_k(x)}(x,(x+1)D_{0^+}^{\beta}u(x)) + \frac{\sin x}{x}.h(x,u(x)) = 0, \quad x\in(0,1),\\ \lim_{x\to 0} I_{0^+}^{(1-\omega)(1-\alpha),\sigma}\phi_{p_k(x)}(x,(x+1)D_{0^+}^{\beta}u(x)) = u(0) = 0,\\ u(1) = \int_0^1 g(t)u(t)dt + \sum_{n=1}^{n=+\infty}\alpha_n u(\eta_n),\end{cases} \tag{5.6}$$

where $\phi_{p_k(t)}$ is the $p_k(t)$-Laplacian operator defined in $[0,1]\times R$ as

$$\phi_{p_k(t)}(t,x) = |x|^{p_k(t)-2}.x, \text{ for } k\in\{1,2,...,N\}, N\in\mathbb{N}^*$$

with $p_{k(t)} \in C^1([0,1],(1,+\infty))$. We consider the problem 1.1 with $f(t,x) = \frac{h(t,x)}{N}$ and $\phi(t,x) = \frac{1}{N}\sum_{k=1}^{k=N}\phi_{p_k(t)}(t,x)$, $p(x) = (x+1)$ and $q(x) = \frac{\sin x}{x}$. We assume that the conditions (A1), (A2) and (A3) are satisfied, and ϕ verifies (A4) and (A5) with

$$p^+ = \max\{p_k(t), t\in[0,1], k\in 1,2,..,N\},$$

and

$$p^- = \min\{p_k(t), t\in[0,1], k\in 1,2,..,N\}.$$

We deduce from theorems 5.1 and 5.2 that, if there exist $R_0 > 0, R_1 > 0$ and $\gamma > \gamma_0$ such that h verifies one of the following conditions;

(H1)

$$h(t,x) < N\lambda\phi_{p^+}(x), \text{ for } (t,x)\in[0,1]\times[0,R_0]. \tag{5.7}$$

$$h(t,x) \geq N\gamma\phi_{p^+}(x), \text{ for } (t,x)\in[0,1]\times[R_1,+\infty) \tag{5.8}$$

and

$$\lim_{x\to+\infty}\frac{\sup_{t\in[0,1]}\{h(t,x)\}}{\phi_{p^-}(x)} < \infty, \tag{5.9}$$

or

(H2)

$$h(t,x) < N\lambda_2\phi_{p^-}(x), \text{ for } (t,x)\in[0,1]\times[R_1,+\infty), \tag{5.10}$$

and

$$h(t,x) \geq N\gamma\phi_{p^-}(x), \text{ for } (t,x)\in[0,1]\times[0,R_0], \tag{5.11}$$

then problem 5.6 has at least one nontrivial positive solution.

References

[1] E.C. de Oliveira, J. Sousa, *Ulam-Hyers-Rassias stability for a class of fractional integro-differential equations*, Results Math. **73** (2018), no. 3, 1–16.

[2] G. Dai, *Infinitely many non-negative solutions for a Dirichlet problem involving p(x)-Laplacian*, Nonlinear Anal. TMA **71** (2009), 5840–5849.

[3] X. Fan,; *On the sub-supersolution method for p(x)-Laplacian equations*, J. Math. Anal. Appl. **330** (2007), 665–682.

[4] X. Fan, *Remarks on eigenvalue problems involving the p(x)-Laplacian*, J. Math. Anal. Appl. **352** (2009), 85–98.

[5] X. Fan, H. Wu and F. Wang, *Hartman-type results for p(t)-Laplacian systems*, Nonlinear Anal. **52** (2003), 585–594.

[6] X. Fan and Q. Zhang, *Existence of solutions for p(x)-Laplacian Dirichlet problems*, Nonlinear Anal. **52** (2003), 1843–1852.

[7] D. Guo and V. Lakshmikantaham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[8] R. Hilfer, *Applications of fractional calculus in Physics*, World Scientific, Singapore, 2000.

[9] D. Jiang and C. Yuan, *The positive properties of the green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application*, Nonlinear Anal. TMA **72** (2010), 710-719.

[10] U.N. Katugampola, *A new approach to generalized fractional derivatives*, (2011), arXiv preprint arXiv:1106.0965.

[11] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam (2006), Academic Press, San Diego, 1988.

[12] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego, 1999.

[13] T. Shen, W. Liu, and R. Zhao, *Fractional boundary value problems with p(t)-Laplacian operator*, Adv. Differ. Equ. **2016** (2016), 118.

[14] P. Ubilla, *Multiplicity results for the 1-dimensional generalized p-Laplacian*, J. Math. Anal. Appl. **190** (1995), 611–623.

[15] T. Shen and W. Liu, *Solvability of fractional p-Laplacian boundary value problems with controlled parameters*, J. Nonlinear Sci. Appl. **10** (2017), 2366–2383.

[16] J. Vanterler da C. Sousa and E. Capelas de Oliveira, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul. **60** (2018), 72–91.

[17] J. Vanterler da C. Sousa, K.D. Kucche and E. Capelas de Oliveira, *Stability of ψ-Hilfer impulsive fractional differential equations*, Appl. Math. Lett. **88** (2019), 73–80.

[18] E. Zeidler, *Nonlinear Functional Analysis and its applications*, Vol. I, Fixed point theorems, Springer-Verlag, New-York 1986.

[19] Q. Zhang, X. Liu and Z. Qiu, *Existence of solutions and multiple solutions for a class of weighted p(r)-Laplacian system*, J. Math. Anal. Appl. **355** (2009), 620–633.