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Dynamical study of addiction modelling under some control strategies

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Abstract

In this work, a paradigm for drugs and addiction is put forth and examined. It is a given that the model took the impact of governmental policies and the addict's control into account. The designer's solution's availability, uniqueness, and bounds should be first addressed. Second, we investigated each equilibrium pointe's existence and localized stability. Additionally, some of the prerequisites for the optimistic equilibrium's global stability are identified. Finally, a computational domain is used to illustrate the theoretical result.

Keywords: locally segment-dense set, vector equilibrium problem, C-sequentially sign property
2020 MSC: Primary 90C33, Secondary 26B25

1 Introduction

Drugs and alcoholic drinks behaviours have been considered as a critical problem in both health and social aspects for a long time. It is well-known that drugs and alcoholic drinks can increase the risks of having serious diseases such as cancer and cardiovascular disease [11]. The United Nations Office on Drugs and Crime (UNODC) just announced its 2019 World Drug Assessment, which estimates that 271 thousand individuals, or 5.5% of the world's population between the ages of 15 and 64, took drugs in 2016 [15].

There are various ways to stop drinking, including personality, convincing family members and friends, using drugs, or going through physiotherapy. However, despite receiving therapy, up to 70% or 80% of alcohol abusers return. As a result, the study of how to stop drinking has gained attention from academics [3]. The use of compartmental dynamic models has been expanded over the years to various additional research areas, such as characteristics of alcoholism, smoking, online viruses, rumours, or drug use [13, 14]. Resmawan et al. investigated the study of a model structure of how drug abuse spreads among educated people [12]. A model for the transmission of synthetic substances with psychiatric addicts and the frequency of generalized communication was put up by Liu et al. [10].

The impact of coverage in the media on the dynamic behaviour of the smoking model with and without spatial diffusion was investigated by A. A. Mohsen et al. [8]. A non-linear SHTR mathematical model developed by I. K. Adu et al. had been used to examine the characteristics of the drinking epidemic [1]. In South Africa, particularly in the Western Cape area, there are also Orwa and Nyabadza. To simulate the dynamics of co-using methamphetamine

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and alcohol, a mathematical model was developed [9]. A study by Grasman et al. compared self-control to addictive behaviours and cravings [6]. Ramirez and Miranda investigated the connection between teenage alcohol craving and the environmental [5].

According to the epidemiological concepts used in the model [7] provided to F. L. Matonya et al., the effectiveness to deliver is viewed as a social contact process between susceptible persons and drug users. A mathematical model that concentrates on illegal OUD and has a classification for former users was suggested by S. Cole et al. [2]. The structure of this essay is as follows. We provide the pharmacological model and demonstrate its validity in Section 2. We focused on the presence of equilibrium position in Sections 3 and 4, and these sections also provide the stability of the system for all equilibrium points. To validate our analytical findings, in Section 5 we present some quantitative simulation results. The findings are discussed in Section 6.

2 Model Formulation

In the following section, we develop a mathematical model of drug addicts who are afraid of the consequences. The following are the underlying presumptions for drug addict mathematical equations:

1. Population numbers that have either received instruction on drug misuse but have not received instruction on drug addiction may be more susceptible to mild drug addiction.
2. There cannot be moderate drug users among the community of strong drug users.
3. Neither moderate nor extreme drug addicts could be found among those who give up using drugs.

Tables 1 and 2 provide descriptions of the variables and parameters used in the model.

Table 1: Details of the variables

Variable	Details
$S_u(t)$	the susceptible population at the momentt to be drug-addicts, and do not have information about the dangers of the drugs
$S_a(t)$	the affected population first at moment t to be drug-addicts, and have information about the dangers of the drugs
$L(t)$	the mild drugs addicts' population at the moment
$H(t)$	the heavy drugs addicts' population at the moment t
$R(t)$	represents the rehabilitated drug addict's population at the moment due to treatment in rehabilitation

All characteristics are thought to be consistent and positive. Figure 1 can display a transmission that denotes the presumptions and definitions of variables in the model. The system of ordinary differential equations shown in Figure 1 is computed as follows:

$$\begin{aligned}\frac{dS_u}{dt} &= (1-b)A - \frac{\beta_1 S_u L}{1+nL} - \frac{\beta_2 S_u H}{1+nH} - \alpha S_u S_a - \mu S_u\\ \frac{dS_a}{dt} &= bA + \alpha S_u S_a - \mu S_a\\ \frac{dL}{dt} &= \frac{\beta_1 S_u L}{1+nL} + \frac{\beta_2 S_u H}{1+nH} - (\mu+\gamma+r_1+d_1)\,L\\ \frac{dH}{dt} &= \gamma L - (\mu + r_2 + d_2)\,H\\ \frac{dR}{dt} &= r_1 L + r_2 H - \mu R.\end{aligned} \tag{2.1}$$

Considering preliminary conditions $S_u(0) > 0, S_a > 0, L(0) \geq 0, H(0) \geq 0, R(0) \geq 0$.

Theorem 2.1. The following paragraphs discuss how any solution is uniformly bounded.

Table 2: Details of the variables

Parameter	Details
β_1	the frequency of interaction between mild drug users and susceptible
β_2	Is the occurrence rate between highly dependent drug users and susceptible.
A	the recruitment rate of population
d_1	The deaths among mild drug addicts
d_2	Death rates among strong drug addicts
μ	Natural human death rate
γ	The transition rate from mild drug addicts to heavy drug users addicts
n	Fear rate from the penalties
α	Drug awareness percentage
r_1	The recovery rate of the mild drug addiction
r_2	The recovery rate of the heavy drug addiction

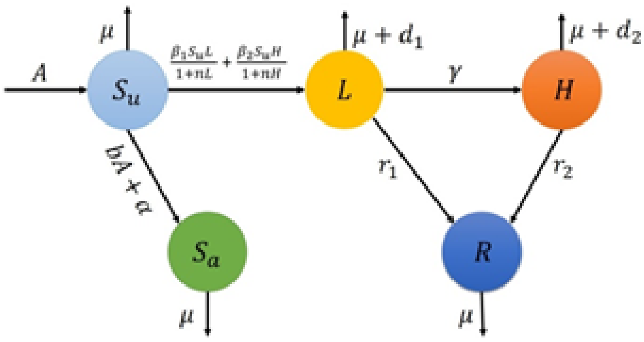

Figure 1: Model of a drug addict's transmission.

Proof . Let $(S_u(t), S_a(t), L(t), H(t), R(t))$ is the solution of the system (1) with positive initial condition $(S_u(0), S_a(0), L(0), H(0)$ which defines the function $N(t) = S_u(t) + S_a(t) + L(t) + H(t) + R(t)$ Then take the time derivative of $N(t)$ along the solution of the system (2.1); this gives

$$\begin{aligned}\frac{dN}{dt} =&(1-b)A - \frac{\beta_1 S_u L}{1+nL} - \frac{\beta_2 S_u H}{1+nH} - \alpha S_u S_a - \mu S_u + bA + \alpha S_u S_a - \mu S_a \\ &+ \frac{\beta_1 S_u L}{1+nL} + \frac{\beta_2 S_u H}{1+nH} - \mu L - \gamma L - r_1 L - d_1 L + \gamma L - \mu H - r_2 H - d_2 H \\ &+ r_1 L + r_2 H - \mu R \\ \frac{dN}{dt} =&A - \mu S_u - \mu S_a - (\mu + d_1) L - (\mu + d_2) H - \mu R \\ \frac{dN}{dt} \le& A - qN; \quad q = \min\{\mu, \mu + d_1, \mu + d_2\} = \mu \\ \frac{dN}{dt} +& qN \le A.\end{aligned}$$

Without a doubt, by resolving the aforementioned equation, we have

$$N(t) \le \frac{A}{q} + \left(N_0 - \frac{A}{q}\right) e^{-qt}.$$

Therefore , $N(t) \le \frac{A}{q}$, as $t \to \infty$ Thus, the proof is successful since these solutions are uniformly bounded. □

3 Existence of the equilibrium points

One can solve various systems instead of system (2.1) by substituting the solution values of L and H in the fifth equation of the system (2.1) and solving it individually as a system of differential equations with regards to the variable

R. It should be noted that this is because the variable R, which also represents the recovered drug, is not available in the first four equations of the system (2.1).

$$R(t)=\frac{r_1\tilde{L}+r_2\widetilde{H}}{\mu}. \tag{3.1}$$

In which $(\widetilde{L},\widetilde{H})$ stands for the system (3.2) bellow's solution values. Therefore, in place of the system, the following system will be studied (2.1)

$$\begin{aligned}
\frac{dS_u}{dt}&=(1-b)A-\frac{\beta_1 S_u L}{1+nL}-\frac{\beta_2 S_u H}{1+nH}-\alpha S_u S_a-\mu S_u\\
\frac{dS_a}{dt}&=bA+\alpha S_u S_a-\mu S_a\\
\frac{dL}{dt}&=\frac{\beta_1 S_u L}{1+nL}+\frac{\beta_2 S_u H}{1+nH}-\left(\mu+\gamma+r_1+d_1\right)L\\
\frac{dH}{dt}&=\gamma L-\left(\mu+r_2+d_2\right)H
\end{aligned} \tag{3.2}$$

The system (3.2) contains seven equilibrium points, which is obvious. The first equilibrium point (when $b=0$) that denoted by $E_1=\left(\dddot{S}_u,0,0,0\right)$, here

$$\dddot{S}_u=\frac{A}{\mu}. \tag{3.3}$$

The second equilibrium point (when $b=1$) which is denoted by $E_2=(0,S_2^*,0,0)$, here

$$S_a^*=\frac{A}{\mu}. \tag{3.4}$$

The third equilibrium point $E_3=\left(\check{S}_u,\check{S}_a,0,0\right)$, here

$$\check{S}_u=\frac{(1-b)A}{\alpha\check{S}_a+\mu}. \tag{3.5}$$

While $\check{S}_a$ is a significant real root of the quadratic equation that follows.

$$C_1S_a^2+C_2S_a+C_3=0 \tag{3.6}$$

here

$$\begin{aligned}
\check{S}_a&=\frac{-C_2+\sqrt{C_2^2-4C_1C_3}}{2C_1}\\
C_1&=-\alpha\mu\\
C_2&=\alpha A-\mu^2\\
C_3&=bA\mu.
\end{aligned}$$

The fourth equilibrium point (when $b=0,\gamma=0$) which is denoted by $E_4=\left(\tilde{S}_u,0,\tilde{L},0\right)$, where

$$\tilde{S}_u=\frac{A(1+n\tilde{L})}{\beta_1L+\mu(1+n\tilde{L})} \tag{3.7}$$

while $\tilde{L}$ is a significant real root of the quadratic equation that follows:

$$N_1L^2+N_2L+N_3=0 \tag{3.8}$$

here

$$\begin{aligned}\tilde{L} &= \frac{-N_2 + \sqrt{N_2^2 - 4N_1N_3}}{2N_1}\\ N_1 &= -\left(\mu + r_1 + d_1\right) n^2 A\\ N_2 &= \beta_1^2 + \beta_1 n - 2\left(\mu + r_1 + d_1\right) nA\\ N_3 &= \beta_1\mu - \left(\mu + r_1 + d_1\right) A.\end{aligned}$$

Clearly, E_4 exists only in R^4, interior if the following conditions are met.

$$\beta_1\mu > \left(\mu + r_1 + d_1\right) A. \tag{3.9}$$

The fifth equilibrium point (when $b = 0$) denoted by $E_5 = \left(\hat{S}_u, 0, \widehat{L}, \widehat{H}\right)$, where

$$\widehat{H} = \frac{\gamma\hat{L}}{\mu + r_2 + d_2}, \hat{S}_u = \frac{A(1 + n\hat{L})\left(\mu + r_2 + d_2 + n\hat{L}_\gamma\right)}{Q}, \tag{3.10}$$

while $\hat{L}$ the following five-order polynomial equation's positive root

$$R_1L^5 + R_2L^4 + R_3L^3 + R_4L^2 + R_5L + R_6 = 0 \tag{3.11}$$

here

$$\begin{aligned}R_1 &= -n^2\gamma\left(\mu + \gamma + r_1 + d_1\right)\left(\beta_1 n\gamma + \beta_2 n\gamma + \mu n^2\gamma\right) \quad , \quad R_2 = An^3\gamma^2\beta_2\\ R_3 &= \left(\beta n^2\gamma + \beta_2\gamma\right) An^2\gamma + An^2\beta_2\gamma\left(\mu + r_2 + d_2 + \gamma\right)\\ R_4 &= An^2\gamma\beta_1\left(\mu + r_2 + d_2\right) + An\left(\beta_1 n\gamma + \beta_2\gamma\right)\left(\mu + r_2 + d_2 + \gamma\right) + A\beta_2 n\gamma\left(\mu + r_2 + d_2\right)\\ R_5 &= An\beta_1\left(\mu + r_2 + d_2\right) + A\left(\beta_1 n\gamma + \beta_2\gamma\right)\left(\mu + r_2 + d_2\right) \quad , \quad R_6 = A\beta_1\left(\mu + r_2 + d_2\right)^2\end{aligned}$$

where

$$Q = \beta_1 L\left(\mu + r_2 + d_2 + n\gamma L\right) + \beta_2\gamma L(1 + nL) + \mu(1 + nL)\left(\mu + r_2 + d_2 + n\gamma L\right).$$

Clearly, E_5 exists only in the interior of R4 if one of the following circumstances is true.

$$\begin{cases} R_2 > 0; R_3 > 0; R_4 > 0\\ R_3 > 0; R_4 > 0; R_5 > 0\end{cases} \tag{3.12}$$

The sixth equilibrium point denoted by $E_6 = \left(\ddot{S}_u, \ddot{S}_a, \ddot{L}, 0\right)$, (where $\gamma = 0$)

$$\ddot{S}_u = \frac{\left(\mu + r_1 + d_1\right)(1 + n\ddot{L})}{\beta_1} \quad , \quad \ddot{S}_a = \frac{-bA\beta_1}{\alpha\left(\mu + r_1 + d_1\right)(1 + n\ddot{L}) + \mu\beta_1} \tag{3.13}$$

While $\ddot{L}$ the following second order polynomial equation's positive root

$$B_1L^2 + B_2L + B_3 = 0 \tag{3.14}$$

here

$$\begin{aligned}B_1 &= (1 - b)A\beta n^2 D - Dn\\ B_2 &= ((1 - b)A\beta_1 Dn + (1 - b)A\beta Dn - D - AbDn\alpha + \mu Dn)\\ B_3 &= (1 - b)A\beta_1 D + \mu\beta_1 - AbD\alpha + \mu D\end{aligned}$$

with $D = (\mu + r_1 + d_1)$. Clearly, E_6 exists uniquely in interior of R^4, assuming one of the following situations occurs

$$\left.\begin{matrix} B_1 > 0; B_3 < 0\\ B_1 < 0; B_3 > 0\end{matrix}\right\} \tag{3.15}$$

The seventh equilibrium point which is denoted by $E_7 = \left(\overline{\overline{S}}_u, \overline{\overline{S}}_a, \overline{\overline{L}}, \overline{\overline{H}}\right)$, where

$$\overline{\overline{H}} = \frac{\gamma \overline{\overline{L}}}{\mu + r_2 + d_2} \quad , \overline{\overline{S}}_a = \frac{\mu \overline{\overline{S}}_u - bA}{\alpha \overline{\overline{S}}_u} \quad , \overline{\overline{S}}_u = \frac{-m_1 \overline{\overline{L}}^2 - m_3 \overline{\overline{L}} + m_5}{m_2 \overline{\overline{L}} + m_4} \tag{3.16}$$

while $\overline{\overline{L}}$ the four possible polynomial equation's positive root

$$A_1 L^4 + A_2 L^3 + A_3 L^2 + A_4 L + A_5 = 0 \tag{3.17}$$

here

$$\begin{aligned} A_1 &= -n_2 m_1 \\ A_2 &= -n_2 m_3 - n_3 m_1 \\ A_3 &= n_1 m_4 + n_4 m_2 - n_2 m_5 - n_3 m_3 - n_5 m_1 \\ A_4 &= n_4 m_4 + n_6 m_2 - n_3 m_5 - n_5 m_3 \\ A_5 &= n_6 m_4 - n_5 m_5. \end{aligned} \tag{3.18}$$

Simple math clearly demonstrates that E_7 exists, but only if and only if one set of the conditions listed below are true

$$\begin{matrix} & A_3 < 0, A_5 > 0 \\ \text{or} & \\ & A_4 > 0, A_5 > 0 \end{matrix} \tag{3.19}$$

4 Local stability analysis

This section addresses the linearization method to examine the local stability around each equilibrium point before establishing the system's persistence criteria. System (3.2)'s general Jacobian matrix at the location (S_u, S_a, L, H), can be written

$$J = \begin{bmatrix} -\frac{\beta_1 L}{1+nL} - \frac{\beta_2 H}{1+nH} - \alpha S_a - \mu & -\alpha S_u & -\frac{\beta_1 S_u}{(1+nL)^2} & -\frac{\beta_2 S_u}{(1+nH)^2} \\ \alpha S_a & \alpha S_u - \mu & 0 & 0 \\ \frac{\beta_1 L}{1+nL} + \frac{\beta_2 H}{1+nH} & 0 & \frac{\beta_1 S_u}{(1+nL)^2} - (\mu + \gamma + r_1 + d_1) & \frac{\beta_2 S_u}{(1+nH)^2} \\ 0 & 0 & \gamma & -(\mu + r_2 + d_2) \end{bmatrix}.$$

So that at the equilibrium position, the Jacobian matrix $E_1 = \left(\dddot{S}_u, 0, 0, 0\right)$ is determined as

$$J(E_1) = (\ddot{a}_{ij})_{4\times 4}; \quad i, j = 1, 2, 3, 4 \tag{4.1}$$

here

$$\begin{aligned} &\ddot{a}_{11} = -\mu \quad , \quad \ddot{a}_{12} = -\alpha \frac{A}{\mu}, \ \ddot{a}_{13} = -\beta_1 \dddot{S}_u, \quad , \ddot{a}_{14} = -\beta_2 \dddot{S}_u, \quad \ddot{a}_{22} = \alpha \frac{A}{\mu} - \mu \\ &\ddot{a}_{33} = \beta_1 \dddot{S}_u - (\mu + \gamma + r_1 + d_1), \quad \ddot{a}_{34} = \beta_2 \dddot{S}_u, \quad \ddot{a}_{43} = \gamma, \quad \ddot{a}_{44} = -(\mu + r_2 + d_2) \\ &\ddot{a}_{21} = a_{23} = \ddot{a}_{24} = \ddot{a}_{31} = \ddot{a}_{32} = \ddot{a}_{41} = \ddot{a}_{42} = 0. \end{aligned}$$

Because of this, the characteristic equation of the Jacobian matrix of system (3.2) at the E_1 is

$$(\ddot{a}_{22} - \lambda)\left[\lambda^3 + \dddot{A}_1 \lambda^2 + \dddot{A}_2 \lambda + \dddot{A}_3\right] = 0 \tag{4.2}$$

where

$$\begin{aligned} \dddot{A}_1 &= -(\ddot{a}_{11} + \ddot{a}_{22} + \ddot{a}_{33}) \\ \dddot{A}_2 &= \ddot{a}_{11}\ddot{a}_{22} - \ddot{a}_{12}\ddot{a}_{21} + \ddot{a}_{11}\ddot{a}_{33} - \ddot{a}_{13}\ddot{a}_{31} + \ddot{a}_{22}\ddot{a}_{33} - \ddot{a}_{23}\ddot{a}_{32} \\ \ddot{A}_3 &= -\ddot{a}_{11}\ddot{a}_{22}\ddot{a}_{33} - \ddot{a}_{12}\ddot{a}_{23}\ddot{a}_{31} - \ddot{a}_{13}\ddot{a}_{21}\ddot{a}_{32} + \ddot{a}_{13}\ddot{a}_{22}\ddot{a}_{31} + \ddot{a}_{11}\ddot{a}_{23}\ddot{a}_{32} + \ddot{a}_{12}\ddot{a}_{21}\ddot{a}_{33}. \end{aligned}$$

Whichever way $(\dddot{a}_{22} - \lambda) = 0$, that also provides eigenvalues in the X-direction by $\dddot{\lambda}_x = \dddot{a}_{22}$ or

$$\left[\lambda^3 + \dddot{A}_1\lambda^2 + \dddot{A}_2\lambda + \dddot{A}_3\right] = 0. \tag{4.3}$$

Therefore, all of the eigenvalues are determined by the Routh-Hawirtiz Criterion of E_1 include real roots that are detrimental simply if and when if $\ddot{A}_i (i = 1, 3) > 0$ and $\Delta = \dddot{A}_1 \dddot{A}_2 - \dddot{A}_3 > 0$. So, $E_1 = \left(\dddot{S}_u, 0, 0, 0\right)$ if and only if the essential requirements are met is a locally asymptotically stable equilibrium.

$$\dddot{S}_u < \min\left\{\frac{(\mu + \gamma + r_1 + d_1)(\mu + r_2 + d_2)}{\gamma\beta_2 + \beta_1(\mu + r_2 + d_2)}, \frac{(\mu + \gamma + r_1 + d_1)}{\beta_1}\right\}. \tag{4.4}$$

The equilibrium point's evaluation of the Jacobain matrix $E_2 = (0, S_a^*, 0, 0)$ can be written as

$$J(E_2) = \begin{bmatrix} -\alpha\frac{A}{\mu} - \mu & 0 & 0 & 0 \\ -\alpha\frac{A}{\mu} & -\mu & 0 & 0 \\ 0 & 0 & -(\mu + \gamma + r_1 + d_1) & 0 \\ 0 & 0 & \gamma & -(\mu + r_2 + d_2) \end{bmatrix} \tag{4.5}$$

$$\lambda_1^* = -\alpha\frac{A}{\mu} - \mu, \quad \lambda_2^* = -\mu, \lambda_3^* = -(\mu + \gamma + r_1 + d_1), \lambda_4^* = -(\mu + r_2 + d_2). \tag{4.6}$$

So, $E_2 = (0, S_a^*, 0, 0)$ is an equilibrium that is locally asymptotically stable? The Jacobian matrix at $E_3 = \left(\check{S}_u, \check{S}_a, 0, 0\right)$, can always be expressed in writing as:

$$J(E_3) = (\breve{a}_{ij})_{4\times 4}; \quad i, j = 1, 2, 3, 4 \tag{4.7}$$

where

$$\begin{aligned} &\breve{a}_{11} = \alpha\check{S}_a - \mu, \quad \breve{a}_{12} = -\alpha\check{S}_u, \breve{a}_{13} = -\beta_1\check{S}_u, \breve{a}_{14} = -\beta_2\check{S}_u, \breve{a}_{21} = -\alpha\check{S}_a \\ &\breve{a}_{22} = \alpha\check{S}_u - \mu \quad , \quad \breve{a}_{33} = \beta_1\check{S}_u - (\mu + \gamma + r_1 + d_1), \quad \breve{a}_{34} = \beta_2\check{S}_u, \breve{a}_{43} = \gamma \\ &\breve{a}_{44} = -(\mu + r_2 + d_2) \quad , \breve{a}_{23} = \breve{a}_{24} = \breve{a}_{31} = \breve{a}_{32} = \breve{a}_{41} = \breve{a}_{42} = 0. \end{aligned}$$

The characteristic equation follows of $J(E_3)$ can be determined as follows

$$\left[\lambda^2 + \breve{A}_1\lambda + \breve{A}_2\right]\left[\lambda^2 + \breve{B}_1\lambda + \breve{B}_2\right] = 0 \tag{4.8}$$

$$\begin{aligned} \breve{A}_1 &= \breve{a}_{11} + \breve{a}_{22} \\ \breve{A}_2 &= \breve{a}_{11}\breve{a}_{22} - \breve{a}_{12}\breve{a}_{21} \\ \breve{B}_1 &= \breve{a}_{33} + \breve{a}_{44} \\ \breve{B}_2 &= \breve{a}_{33}\breve{a}_{44} - \breve{a}_{34}\breve{a}_{43}. \end{aligned}$$

Consequently, the eigenvalues are written as

$$\begin{aligned} \check{\lambda}_{1,2} &= -\frac{\breve{A}_1}{2} \mp \frac{1}{2}\sqrt{\breve{A}_1^2 - 4\breve{A}_2} \\ \check{\lambda}_{3,4} &= -\frac{\breve{B}_1}{2} \mp \frac{1}{2}\sqrt{\breve{B}_1^2 - 4\breve{B}_2}. \end{aligned} \tag{4.9}$$

Given that the following requirements are met, the E_3 is locally asymptotically stable and all of the eigenvalues have negative real portions.

$$\left(\check{S}_u + \check{S}_a\right) < \min\left\{\frac{2\mu}{\alpha}, \frac{2\alpha^2\check{S}_u\check{S}_a + \mu^2}{\mu\alpha}\right\} \tag{4.10}$$

$$\check{S}_u < \min\left\{\frac{(\mu+\gamma+r_1+d_1)+(\mu+r_2+d_2)}{\beta_1}, \frac{(\mu+\gamma+r_1+d_1)(\mu+r_2+d_2)}{\beta_1(\mu+r_2+d_2)+\gamma\beta_2}\right\}. \tag{4.11}$$

Evaluation of the Jacobian matrix at the equilibrium position $E_4 = \left(\tilde{S}_u, 0, \tilde{L}, 0\right)$ is given by

$$J(E_4) = \begin{bmatrix} -\frac{B\tilde{L}}{1+n\tilde{L}} - \mu & -\alpha\tilde{S}_u & -\frac{\beta_1\tilde{S}_u}{(1+n\tilde{L})^2} & -\beta_1\tilde{S}_u \\ 0 & \alpha\tilde{S}_u - \mu & 0 & 0 \\ \frac{\beta_1\tilde{L}}{1+n\tilde{L}} & 0 & \frac{\beta_1\tilde{S}_u}{(1+n\tilde{L})^2} - (\mu+\gamma+r+d_1) & \beta_2\tilde{S}_u \\ 0 & 0 & 0 & -(\mu+r_2+d_2) \end{bmatrix} = [\tilde{a}_{ij}]. \tag{4.12}$$

The equation for the characteristic of $J(E_4)$ is given by

$$(\tilde{a}_{44}-\lambda)(\tilde{a}_{22}-\lambda)\left(\lambda^2+\tilde{A}_1\lambda+\tilde{A}_2\right) = 0 \tag{4.13}$$

here

$$\tilde{A}_1 = \tilde{a}_{11} + \tilde{a}_{33}$$
$$\tilde{A}_2 = \tilde{a}_{11}\tilde{a}_{33} - \tilde{a}_{13}\tilde{a}_{31}.$$

Consequently, the eigenvalues are written as:

$$\left.\begin{array}{l} \tilde{\lambda}_2 = \alpha\tilde{S}_u - \mu \\ \tilde{\lambda}_4 = -(\mu+r_2+d_2) \\ \tilde{\lambda}_{1,3} = -\frac{\tilde{A}_1}{2} \mp \frac{1}{2}\sqrt{\tilde{A}_1^2 - 4\tilde{A}_2} \end{array}\right\}. \tag{4.14}$$

Given that the following requirements are met, the E_4 is locally asymptotically stable but all of the eigenvalues have negative real portions.

$$S_u < \frac{\mu}{\alpha} \tag{4.15}$$

$$\frac{\beta_1\tilde{s}_u}{(1+\tilde{L})^2} < \min\left\{\frac{\beta_1\tilde{L}}{1+n\tilde{L}} + \mu + (\mu+\gamma+r_1+d_1), \left(\frac{\beta_1\tilde{L}}{\mu(1+n\tilde{L})}(\mu+\gamma+r_1+d_1) + \frac{\beta_1^2\tilde{S}_u^2}{\mu(1+n\tilde{L})^3} - \frac{\beta_1^2\tilde{s}_u\tilde{L}}{\mu(1+n\tilde{L})^2}\right) + (\mu+\gamma+r_1+d_1)\right\}. \tag{4.16}$$

The Jacobian matrix of system (3.2) around $E_5 = (\hat{S}_u, 0, \hat{L}, \hat{H})$ is determined as

$$J(E_5) = (\hat{a}_{ij})_{4\times 4}; \quad i,j = 1,2,3,4 \tag{4.17}$$

$$\hat{a}_{11} = -\frac{\beta_1\hat{L}}{1+n\hat{L}} - \frac{\beta_2\widehat{H}}{1+n\hat{H}} - \mu, \quad \hat{a}_{12} = -\alpha\hat{S}_u, \quad \hat{a}_{13} = -\frac{\beta_1\hat{s}_u}{(1+n\hat{L})^2}, \quad \hat{a}_{14} = -\frac{\beta_2\hat{s}_u}{(1+n\hat{H})^2}$$
$$\hat{a}_{22} = \alpha\hat{S}_u - \mu \quad, \hat{a}_{31} = \frac{\beta_1\hat{L}}{1+n\hat{L}} + \frac{\beta_2\hat{H}}{1+n\hat{H}}, \quad \hat{a}_{33} = \frac{\beta_1\hat{s}_u}{(1+n\hat{L})^2} - (\mu+\gamma+r+d_1), \quad \hat{a}_{34} = \frac{\beta_2\hat{S}_u}{(1+n\hat{H})^2}$$
$$\hat{a}_{34} = \gamma, \hat{a}_{44} = -(\mu+r_2+d_2) \quad, \hat{a}_{21} = \hat{a}_{23} = \hat{a}_{24} = \hat{a}_{32} = \hat{a}_{41} = \hat{a}_{42} = 0.$$

Consequently, the Jacobian matrix of the system's characteristic equation is (3.2) at the E_5 is given by

$$(\hat{a}_{22}-\lambda)\left[\lambda^3 + \hat{A}_1\lambda^2 + \hat{A}_2\lambda + \hat{A}_3\right] = 0 \tag{4.18}$$

here

$$\hat{A}_1 = -\left(\hat{a}_{11} + \hat{a}_{22} + \hat{a}_{33}\right)$$
$$\hat{A}_2 = \hat{a}_{11}\hat{a}_{22} - \hat{a}_{12}\hat{a}_{21} + \hat{a}_{11}\hat{a}_{33} - \hat{a}_{13}\hat{a}_{31} + \hat{a}_{22}\hat{a}_{33} - \hat{a}_{23}\hat{a}_{32}$$
$$\hat{A}_3 = -\hat{a}_{11}\hat{a}_{22}\hat{a}_{33} - \hat{a}_{12}\hat{a}_{23}\hat{a}_{31} - \hat{a}_{13}\hat{a}_{21}\hat{a}_{32} + \hat{a}_{13}\hat{a}_{22}\hat{a}_{31} + \hat{a}_{11}\hat{a}_{23}\hat{a}_{32} + \hat{a}_{12}\hat{a}_{21}\hat{a}_{33}.$$

Whichever way $(\hat{a}_{22} - \lambda) = 0$, it provides the X-direction eigenvalues by $\widehat{\lambda}_x = \hat{a}_{22}$ or $\lambda^3 + \hat{A}_1\lambda^2 + \hat{A}_2\lambda + \hat{A}_3 = 0$. Now, if and only if all of the eigenvalues of E_5 have roots with negative real portions, then the Routh-Hawirtiz Criterion is satisfied if $\hat{A}_i(i = 1,3) > 0$ and $\Delta = \hat{A}_1\hat{A}_2 - \hat{A}_3 > 0$. So, $E_5 = \left(\hat{S}_u, 0, \hat{L}, \widehat{H}\right)$ is a locally asymptotically stable equilibrium if

$$\hat{S}_u < \min\left\{\frac{-\beta_2\gamma(1+n\hat{H})^2\left[-\beta_1\hat{L}(1+n\hat{H}) - \beta_2\hat{H}(1+n\hat{L}) - \mu(1+n\hat{L})(1+n\hat{H})\right]}{\beta_1\hat{L}(1+n\hat{H}) + \beta_2\hat{H}(1+n\hat{L})},\right.$$
$$\left.\frac{\left(\mu+\gamma+r_1+d_1\right)(1+n\hat{L})^2}{\beta_1}\right\}. \tag{4.19}$$

The system's variational matrix (3.2) at $E_6 = \left(\ddot{S}_u, \ddot{S}_a, \ddot{L}, 0\right)$ be expressed as

$$J\left(E_6\right) = \left(\ddot{a}_{ij}\right)_{4\times4}; \quad i,j = 1,2,3,4 \tag{4.20}$$

where

$$\ddot{a}_{11} = -\frac{\beta_1\ddot{L}}{1+n\ddot{L}} - \alpha\ddot{S}_a - \mu, \quad \ddot{a}_{12} = -\alpha\ddot{S}_u, \ddot{a}_{13} = -\frac{\beta_1\ddot{S}_u}{(1+n\tilde{L})^2}, \quad \ddot{a}_{14} = -\beta_2\ddot{S}_u, \ddot{a}_{21} = -\alpha\ddot{S}_a$$
$$\ddot{a}_{22} = \alpha\ddot{S}_u - \mu, \ddot{a}_{31} = \frac{\beta_1\ddot{L}}{1+n\ddot{L}}, \quad \ddot{a}_{33} = \frac{\beta_1\ddot{S}_u}{(1+n\ddot{})^2} - \left(\mu+\gamma+r_1+d_1\right), \ddot{a}_{34} = \beta_2\ddot{S}_u$$
$$\ddot{a}_{44} = -\left(\mu + r_2 + d_2\right), \quad \ddot{a}_{23} = \ddot{a}_{24} = \ddot{a}_{32} = \ddot{a}_{41} = \ddot{a}_{42} = 0.$$

Then eigenvalues are given by

$$\left(\ddot{a}_{44} - \lambda\right)\left[\lambda^3 + \ddot{A}_1\lambda^2 + \ddot{A}_2\lambda + \ddot{A}_3\right] = 0 \tag{4.21}$$

here

$$\ddot{A}_1 = -\left(\ddot{a}_{11} + \ddot{a}_{22} + \ddot{a}_{33}\right)$$
$$\ddot{A}_2 = \ddot{a}_{11}\ddot{a}_{22} - \ddot{a}_{12}\ddot{a}_{21} + \ddot{a}_{11}\ddot{a}_{33} - \ddot{a}_{13}\ddot{a}_{31} + \ddot{a}_{22}\ddot{a}_{33} - \ddot{a}_{23}\ddot{a}_{32}$$
$$\ddot{A}_3 = -\ddot{a}_{11}\ddot{a}_{22}\ddot{a}_{33} - \ddot{a}_{12}\ddot{a}_{23}\ddot{a}_{31} - \ddot{a}_{13}\ddot{a}_{21}\ddot{a}_{32} + \ddot{a}_{13}\ddot{a}_{22}\ddot{a}_{31} + \ddot{a}_{11}\ddot{a}_{23} + \ddot{a}_{12}\ddot{a}_{21}\ddot{a}_{33}.$$

Whichever way $(\ddot{a}_{44} - \lambda) = 0$, it provides the X-direction eigenvalues by $\ddot{\lambda}_x = \ddot{a}_{44}$ or

$$\left[\lambda^3 + \ddot{A}_1\lambda^2 + \ddot{A}_2\lambda + \ddot{A}_3\right] = 0.$$

Now, if and only if all of the eigenvalues of E_6 have roots with negative real portions, then the Routh-Hawirtiz Criterion is satisfied if $\ddot{A}_i(i = 1,3) > 0$ and $\Delta = \ddot{A}_1\ddot{A}_2 - \ddot{A}_3 > 0$. So, $E_6 = \left(\ddot{S}_u, \ddot{S}_a, \ddot{L}, 0\right)$ is a locally asymptotically stable equilibrium if

$$\mu > \alpha\ddot{S}_u \tag{4.22}$$
$$\left(\mu + \gamma + r + d_1\right) > \frac{\beta_1\ddot{s}_u}{(1+n\ddot{L})^2} \tag{4.23}$$
$$\ddot{a}_{11}\ddot{a}_{22}\ddot{a}_{33} + \ddot{a}_{13}\ddot{a}_{22}\ddot{a}_{31} > \ddot{a}_{12}\ddot{a}_{21}\ddot{a}_{33}. \tag{4.24}$$

The system's variational matrix (3) at $E_7 = \left(\bar{\bar{S}}_u, \bar{\bar{S}}_a, \bar{\bar{L}}, \bar{\bar{H}}\right)$ is given by

$$J(E_7) = (\bar{\bar{a}}_{ij})_{4\times 4}; \quad i,j = 1,2,3,4 \tag{4.25}$$

here,

$$\bar{\bar{a}}_{11} = -\frac{\beta_1 \bar{\bar{L}}}{1+n\bar{\bar{L}}} - \frac{\beta_2 \bar{\bar{H}}}{1+n\bar{\bar{H}}} - \alpha \bar{\bar{S}}_a - \mu, \bar{\bar{a}}_{12} = -\alpha \bar{\bar{S}}_u, \bar{\bar{a}}_{13} = -\frac{\beta_1 \bar{\bar{S}}_u}{(1+n\bar{\bar{S}})^2} \quad, \bar{\bar{a}}_{14} = -\frac{\beta_2 \bar{\bar{S}}_u}{(1+n\bar{\bar{H}})^2}$$

$$\bar{\bar{a}}_{21} = \alpha \bar{\bar{S}}_a, \bar{\bar{a}}_{22} = -\alpha \bar{\bar{S}}_u - \mu, \bar{\bar{a}}_{31} = \frac{\beta_1 \bar{\bar{L}}}{1+n\bar{\bar{L}}} + \frac{\beta_2 \bar{\bar{H}}}{1+n\bar{\bar{H}}}, \quad \bar{\bar{a}}_{33} = \frac{\beta_1 \bar{\bar{S}}_u}{(1+n\bar{\bar{L}})^2} - (\mu+\gamma+r+d_1)$$

$$\bar{\bar{a}}_{34} = \frac{\beta_2 \bar{\bar{S}}_u}{(1+n\bar{\bar{H}})^2}, \bar{\bar{a}}_{43} = \gamma, \bar{\bar{a}}_{44} = -(\mu+r_2+d_2), \bar{\bar{a}}_{23} = \bar{\bar{a}}_{24} = \bar{\bar{a}}_{32} = \bar{\bar{a}}_{41} = \bar{\bar{a}}_{42} = 0.$$

Applying the Gersgorin theorem's condition now [4]

$$|d_{ii}| > \sum_{\substack{i=1\\ i\neq j}}^{4} |d_{ij}|. \tag{4.26}$$

Consequently, the sub reign Ω contains all of the eigenvalues of the Jacobian matrix at (E_7).

$$\Omega = \cup \left\{ u^* \in C : |u^* - d_{ii}| < \sum_{\substack{i=1\\ i\neq j}}^{4} |d_{ij}| \right\}.$$

As a result, the disc centered at d_{ii} contains all of the eigenvalues of $J(E_7)$. Thus, all of the eigenvalues would exist in the left half plane as well as the E_7 is locally asymptotically stable if and only if the following necessary criterion is fulfilled if the diagonal elements are negative and requirement (57) holds.

$$\bar{\bar{S}}_u < \min\left\{ \frac{(1+n\bar{\bar{L}})^2 (\mu+\gamma+r_1+d_1)}{2\beta_1}, \frac{(1+n\bar{\bar{H}})^2 (\mu+r_2+d_2)}{2\beta_1} \right\} \tag{4.27}$$

5 Global stability analysis:

The equilibrium points of system (3.2), which are provided as mentioned in the previous theorems, have a basin of attraction that we may identify in this part by discussing the global stability conditions.

Theorem 5.1. The equilibrium point E_1 is assumed to be locally asymptotically stable in the range $\Re^4_+$. If the following criteria are met holds, it is a globally asymptotically stable.

$$\dddot{S}_u < \min\{(S_u - 1), \beta_1 A + \mu(\mu + r_1 + d_1), \beta_2 A + \mu(\mu + r_2 + d_2)\}. \tag{5.1}$$

Proof . Take into account the subsequent positive definite function.

$$V_1(S_u, S_a, L, H) = \frac{1}{2}\left(S_u - \dddot{S}_u\right)^2 + S_a + L + H.$$

It is easy to see that $V_1 = (S_u, S_a, L, H) \in C^1\left(R^4_+, R\right)$ in addition $V_1\left(\dddot{S}_u, 0, 0, 0\right) = 0$ while $V_1(S_u, S_a, L, H) > 0$, for all $(S_u, S_a, L, H) \in R^4_+$ and $(S_u, S_a, L, H) \neq \left(\dddot{S}_u, 0, 0, 0\right)$.

$$\frac{dV_1}{dt} = \left(S_u - \dddot{S}_u\right)\frac{dS_u}{dt} + \frac{dS_a}{dt} + \frac{dL}{dt} + \frac{dH}{dt}.$$

Additionally, we obtain that by taking the derivative and simplifying the resulting terms:

$$\frac{dV_1}{dt} = \left(S_u - \dddot{S}_u\right)\left[A - \frac{\beta_1 S_u L}{1+nL} - \frac{\beta_2 S_u H}{1+nH} - \alpha S_u S_a - \mu S_u\right] + [\alpha S_u S_a - \mu S_a]$$
$$+ \left[\frac{\beta_1 S_u L}{1+nL} + \frac{\beta_2 S_u H}{1+nH} - (\mu+\gamma+r_1+d_1)\,L\right] + [\gamma L - (\mu + r_2 + d_2)\,H].$$

By using the above condition (49), we obtain that

$$\frac{dV_1}{dt} \leq -\,\mu\left(S_u - \dddot{S}_u\right)^2 - \alpha\left[S_u - \left(\dddot{S}_u + 1\right)\right] S_u S_a - \mu S_a$$
$$-\left[(\mu + +r_1 + d_1) - \beta_1\left(\dddot{S}_u + 1\right)\frac{A}{\mu}\right] L - \left[(\mu + +r_2 + d_2) - \beta_2\left(\dddot{S}_u + 1\right)\frac{A}{\mu}\right] H.$$

As a result of the aforementioned circumstance $\frac{dV_1}{dt} \leq 0$ is unambiguously negative and hence the proof is finished since E_1 is a globally asymptotically stable function V_1 is the Lyapunov function in relation to in the area where the specified criterion is met. The proof is finished since E_1 is a globally asymptotically stable function. □

Theorem 5.2. Suppose that E_2 be locally asymptotically stable. Then it's a globally asymptotically stable assuming the following sufficient conditions hold

$$S_a < 1 + S_a^*. \tag{5.2}$$

Proof . Take into consideration the following real valued positive definite function.

$$V_2\,(S_u, S_a, L, H) = S_u + \frac{1}{2}\,(S_a - S_a^*)^2 + L + H.$$

It is easy to see that $V_2 = (S_u, S_a, L, H) \in C^1\left(R_+^4, R\right)$ in addition $V_2\,(0, S_a^*, 0, 0) = 0$ while $V_2\,(S_u, S_a, L, H) > 0$, for all $(S_u, S_a, L, H) \in R_+^4$ and $(S_u, S_a, L, H) \neq (0, S_a^*, 0, 0)$

$$\frac{dV_2}{dt} = \frac{dS_u}{dt} + \left(S_a - \dot{S}_a\right)\frac{dS_a}{dt} + \frac{dL}{dt} + \frac{dH}{dt}.$$

Then the derivative of this function with respect to time can be written as

$$\frac{dV_2}{dt} = \left[-\frac{\beta_1 S_u L}{1+nL} - \frac{\beta_2 S_u H}{1+nH} - \alpha S_u S_a - \mu S_u\right] + (S_a - S_a^*)\,[A + \alpha S_u S_a - \mu S_a] +$$
$$\left[\frac{\beta_1 S_u L}{1+nL} + \frac{\beta_2 S_u H}{1+nH} - (\mu+\gamma+r_1+d_1)\,L\right] + [\gamma L - (\mu + r_2 + d_2)\,H]\,.$$

By using the above condition (50), we obtain that

$$\frac{dV_2}{dt} \leq -\,\mu\,(S_a - S_a^*)^2 - \alpha\,[1 - (S_a - S_a^*)]\,S_u S_a - \mu S_u$$
$$-\,(\mu + +r_1 + d_1)\,L - [(\mu + +r_2 + d_2)]\,H$$

As a result of the aforementioned circumstance $\frac{dV_2}{dt} \leq 0$ is unambiguously negative and hence V_2 is Lyapunov function with respect to E_2 in the area where the specified criterion is met. Thus E_2 is a globally asymptotically stable and the evidence is conclusive □

Theorem 5.3. Assume that the equilibrium point E_3 is asymptotically stable locally. If the following sufficient condition applies, it is then globally asymptotically stable.

$$S_u < \min\left\{\check{S}_u + 1, \frac{\mu}{\alpha}\right\} \tag{5.3}$$

$$P_{12}^2 < 4P_{11}P_{22}. \tag{5.4}$$

Symbol locations $p_{ij}, i, j = 1, 2$ are provided as evidence.

Proof . Think about the subsequent function

$$V_3\left(S_u, S_a, L, H\right)=\frac{1}{2}\left(S_u-\check{S}_u\right)^2+\frac{1}{2}\left(S_a-\check{S}_a\right)^2+L+H.$$

It is easy to see that $V_3=\left(S_u, S_a, L, H\right) \in C^1\left(R_{+}^4, R\right)$ in addition $V_3\left(\check{S}_u, \check{S}_a, 0,0\right)=0$ while $V_3\left(S_u, S_a, L, H\right)>0$, for all $\left(S_u, S_a, L, H\right) \in R_{+}^4$ and $\left(S_u, S_a, L, H\right) \neq\left(\check{S}_u, \check{S}_a, 0,0\right)$.

$$\frac{d V_3}{d t}=\left(S_u-\check{S}_u\right) \frac{d S_u}{d t}+\left(S_a-\check{S}_a\right) \frac{d S_a}{d t}+\frac{d L}{d t}+\frac{d H}{d t}.$$

Furthermore, by the derivative and simplifying the

$$\begin{aligned}\frac{d V_3}{d t}= & \left(S_u-\check{S}_u\right)\left[(1-b) A-\frac{\beta_1 S_u L}{1+n L}-\frac{\beta_2 S_u H}{1+n H}-\alpha S_u S_a-\mu S_u\right]+\left(S_a-\check{S}_a\right)\left[b A+\alpha S_u S_a-\mu S_a\right] \\ & +\left[\frac{\beta_1 S_u L}{1+n L}+\frac{\beta_2 S_u H}{1+n H}-\left(\mu+\gamma+r_1+d_1\right) L\right]+\left[\gamma L-\left(\mu+r_2+d_2\right) H\right].\end{aligned}$$

By using the above condition (51), we obtain that

$$\begin{aligned}\frac{d V_3}{d t}= & -\left[P_{11}\left(S_u-\check{S}_u\right)^2+P_{12}\left(S_u-\check{S}_u\right)\left(S_a-\check{S}_a\right)+P_{22}\left(S_a-\check{S}_a\right)^2\right]-\frac{\beta_1 S_u L}{1+n L}\left[\left(S_u-\check{S}_u\right)-1\right] \\ & -\frac{\beta_2 S_u H}{1+n H}\left[\left(S_u-\check{S}_u\right)-1\right]-\left(\mu+r_1+d_1\right) L-\left(\mu+r_2+d_2\right) H\end{aligned}$$

where

$$\begin{aligned}& P_{11}=\alpha \check{S}_a+\mu \\ & P_{12}=\alpha\left(S_u+\check{S}_a\right) \\ & P_{22}=\mu-\alpha S_u .\end{aligned}$$

Therefore, V_3 is a Lyapunov function with respect to E_3 in the region that fulfills the stated constraints since $\frac{d V_3}{d t} \leq 0$ is negative definite according to the criterion previously. Thus, the evidence that E_3 is a globally asymptotically stable is complete. □

Theorem 5.4. Let the equilibrium point E_4 be locally asymptotically stable. If the following necessary condition applies, it is then globally asymptotically stable.

$$\left(\mu+r_1+d_1\right)>\frac{\beta_1 s_u}{K_1} \tag{5.5}$$

$$S_u>\max \left\{\left(\tilde{S}_u+1\right),\left(\tilde{S}_u+L-\tilde{L}\right),(\tilde{L}+n \tilde{L} L)\right\} \tag{5.6}$$

$$\tilde{L}>L \tag{5.7}$$

$$R_{12}^2<4 R_{11} R_{22}. \tag{5.8}$$

Symbol locations $R_{ij}, i, j=1,2$ are provided as evidence.

Proof . Think about the subsequent function

$$V_4\left(S_u, S_a, L, H\right)=\frac{1}{2}\left(S_u-\tilde{S}_u\right)^2+S_a+\frac{1}{2}(L-\widetilde{L})^2+H.$$

It is easy to see that $V_4=\left(S_u, S_a, L, H\right) \in C^1\left(R_{+}^4, R\right)$ in addition $V_4\left(\tilde{S}_u, 0, \tilde{L}, 0\right)=0$ while $V_4\left(S_u, S_a, L, H\right)>0$, for all $\left(S_u, S_a, L, H\right) \in R_{+}^4$ and $\left(S_u, S_a, L, H\right) \neq\left(\tilde{S}_u, 0, \tilde{L}, 0\right)$.

$$\frac{d V_4}{d t}=\left(S_u-\tilde{S}_u\right) \frac{d S_u}{d t}+\frac{d S_a}{d t}+(L-\tilde{L}) \frac{d L}{d t}+\frac{d H}{d t}.$$

Furthermore, by the derivative and simplifying the

$$\frac{dE_4}{dt} = \left(S_u - \tilde{S}_u\right)\left[A - \frac{\beta_1 S_u L}{1+nL} - \frac{\beta_2 S_u H}{1+nH} - \alpha S_u S_a - \mu S_u\right] + \left[\alpha S_u S_a - \mu S_a\right] +$$
$$(L - \widetilde{L})\left[\frac{\beta_1 S_u L}{1+nL} + \frac{\beta_2 S_u H}{1+nH} - (\mu + r_1 + d_1)\, L\right] + \left[\gamma L - (\mu + r_2 + d_2)\, H\right].$$

By using the above condition (52a)-(52b), we obtain that

$$\frac{dV_4}{dt} = -\left[R_{11}\left(S_u - \tilde{S}_u\right)^2 + R_{12}\left(S_u - \tilde{S}_u\right)\left(S_a - \tilde{S}_a\right) + R_{22}\left(S_a - \tilde{S}_a\right)^2\right] - \frac{\beta_2 S_u H}{1+nH}$$
$$\left[\left(S_u - \check{S}_u\right) - (L - \tilde{L})\right] - \alpha S_u S_a\left[\left(S_u - \tilde{S}_u\right) - 1\right] - (\mu + r_2 + d_2)\, H - \mu S_a$$

where

$$R_{11} = \frac{\beta_1 L}{K_1} + \frac{\beta_1 nLL}{K_1} + \mu$$
$$R_{12} = \frac{\beta_1 s_u}{K_1} - \frac{\beta_1 L}{K_1} - \frac{\beta_1 nLL}{K_1}$$
$$R_{22} = (\mu + r_1 + d_1) - \frac{\beta_1 s_u}{K_1}$$
with $K_1 = (1+nL)(1+n\tilde{L})$.

Therefore, V_4 is a Lyapunov function with respect to E_4 in the region that fulfills the stated constraints since $\frac{dV_4}{dt} \le 0$ is negative definite according to the criterion previously. Thus, the evidence that E_4 is a globally asymptotically stable is complete. □

Theorem 5.5. Let the equilibrium point E_5 be locally asymptotically stable. Then it's a globally asymptotically stable provided that the following sufficient condition holds

$$\begin{aligned} d_{12}^2 &< d_{11}d_{22} \\ d_{13}^2 &< d_{11}d_{33} \\ d_{13}^2 &< d_{11}d_{33} \end{aligned} \tag{5.9}$$

$$(\mu + \gamma + r_1 + d_1) > \frac{\beta_1 s_u}{K_1} \tag{5.10}$$

$$S_u > \hat{S}_u + 1. \tag{5.11}$$

Symbol locations $d_{ij}, i, j = 1, 2, 3$ are provided as evidence.

Proof . Think about the subsequent function

$$V_5\left(S_u, S_a, L, H\right) = \frac{1}{2}\left(S_u - \hat{S}_u\right)^2 + S_a + \frac{1}{2}(L - \widehat{L})^2 + \frac{1}{2}(H - \widehat{H})^2.$$

It is easy to see that $V_5 = (S_u, S_a, L, H) \in C^1\left(R_+^4, R\right)$ in addition $V_5\left(\hat{S}_u, 0, \widehat{L}, \widehat{H}\right) = 0$ while $V_5\left(S_u, S_a, L, H\right) > 0$ for all $(S_u, S_a, L, H) \in R_+^4$ and $(S_u, S_a, L, H) \neq \left(\hat{S}_u, 0, \hat{L}, \widehat{H}\right)$.

$$\frac{dV_5}{dt} = \left(S_u - \hat{S}_u\right)\frac{dS_u}{dt} + \frac{dS_a}{dt} + (L - \widehat{L})\frac{dL}{dt} + (H - \widehat{H})\frac{dH}{dt}.$$

Furthermore, by the derivative and simplifying the

$$\frac{dV_5}{dt} = \left(S_u - \hat{S}_u\right)\left[A - \frac{\beta_1 S_u L}{1+nL} - \frac{\beta_2 S_u H}{1+nH} - \alpha S_u S_a - \mu S_u\right] + \left[\alpha S_u S_a - \mu S_a\right] + (L - \widehat{L})$$
$$\left[\frac{\beta_1 S_u L}{1+nL} + \frac{\beta_2 S_u H}{1+nH} - (\mu + \gamma + r_1 + d_1)\, L\right] + (H - \widehat{H})\left[\gamma L - (\mu + r_2 + d_2)\, H\right].$$

By using the above condition(51a)-(51c), we obtain that

$$\frac{dV_5}{dt} \leq -\left[\sqrt{\frac{d_{11}}{2}}\left(S_u-\hat{S}_u\right)+\sqrt{\frac{d_{22}}{2}}(L-\hat{L})\right]^2-\left[\sqrt{\frac{d_{11}}{2}}\left(S_u-\hat{S}_u\right)+\sqrt{\frac{d_{33}}{2}}(H-\widehat{H})\right]^2$$
$$-\left[\sqrt{\frac{d_{22}}{2}}(L-\widehat{L})+\sqrt{\frac{d_{33}}{2}}(H-\widehat{H})\right]^2-\alpha S_u S_a\left[\left(S_u-\hat{S}_u\right)-1\right]$$

where

$$d_{11}=\frac{\beta_1\hat{L}}{K_1}+\frac{\beta_1 nL\hat{L}}{K_1}+\frac{\beta_2\hat{H}}{K_2}+\frac{\beta_2 nH\hat{H}}{K_2}+\mu$$
$$d_{22}=(\mu+\gamma+r_1+d_1)-\frac{\beta_1 s_u}{K_1}$$
$$d_{33}=(\mu+r_2+d_2).$$

Therefore, V_5 is a Lyapunov function with respect to E_5 in the region that fulfills the stated constraints since $\frac{dV_5}{dt}\leq 0$ is negative definite according to the criterion previously. Thus, the evidence that E_5 is a globally asymptotically stable is complete. □

Theorem 5.6. Let the equilibrium point E_6 be locally asymptotically stable. Then it's a globally asymptotically stable provided that the following sufficient condition holds

$$C_{12}^2<2C_{11}C_{22} \tag{5.12}$$
$$C_{13}^2<2C_{11}C_{33}$$
$$\frac{\beta_2\ddot{\mathrm{S}}_\mathrm{u}\mathrm{S}_\mathrm{u}}{1+\mathrm{nH}}+\frac{\beta_2\ \mathrm{S}_\mathrm{u}\mathrm{L}}{1+\mathrm{nH}}<(\mu+\mathrm{r}_2+\mathrm{d}_2)\,. \tag{5.13}$$

Symbol locations $c_{ij}, i,j=1,2,3$ are provided as evidence.

Proof . Think about the subsequent function

$$V_6\left(S_u,S_a,L,H\right)=\frac{1}{2}\left(S_u-\ddot{S}_u\right)^2+\frac{1}{2}\left(S_a-\ddot{S}_a\right)^2+\frac{1}{2}(L-\ddot{L})^2+H.$$

It is easy to see that $V_6=(S_u,S_a,L,H)\in C^1\left(R_+^4,R\right)$ in addition $V_6\left(\ddot{S}_u,\ddot{S}_a,\ddot{L},0\right)=0$ while $V_6\left(S_u,S_a,L,H\right)>0$, for all $(S_u,S_a,L,H)\in R_+^4$ and $(S_u,S_a,L,H)\neq\left(\ddot{S}_u,\ddot{S}_a,\ddot{L},0\right)$.

$$\frac{dV_5}{dt}=\left(S_u-\ddot{S}_u\right)\frac{dS_u}{dt}+\left(S_a-\ddot{S}_a\right)\frac{dS_a}{dt}+(L-\ddot{L})\frac{dL}{dt}+\frac{dH}{dt}.$$

Furthermore, by the derivative and simplifying the

$$\frac{\mathrm{dV}_6}{\mathrm{dt}}=\left(\mathrm{S_u}-\ddot{\mathrm{S}}_\mathrm{u}\right)\left[(1-\mathrm{b})\mathrm{A}-\frac{\beta_1\ \mathrm{S_uL}}{1+\mathrm{nL}}-\frac{\beta_2\ \mathrm{S_uH}}{1+\mathrm{nH}}-\alpha\mathrm{S_uS_a}-\mu\mathrm{S_u}\right]+\left(\mathrm{S_a}-\ddot{\mathrm{S}}_\mathrm{a}\right)[\mathrm{bA}+\alpha\mathrm{S_uS_a}$$
$$-\mu\mathrm{S_a}]+(\mathrm{L}-\ddot{\mathrm{L}})\left[\frac{\beta_1\ \mathrm{S_uL}}{1+\mathrm{nL}}+\frac{\beta_2\ \mathrm{S_uH}}{1+\mathrm{nH}}-(\mu+\gamma+\mathrm{r}_1+\mathrm{d}_1)\,\mathrm{L}\right]-(\mu+\mathrm{r}_2+\mathrm{d}_2)\,H.$$

By using the above condition, we obtain that

$$\frac{dV_6}{dt}\leq-\left[\sqrt{\frac{C_{11}}{2}}\left(S_u-\ddot{S}_u\right)+\sqrt{C_{22}}\left(S_a-\ddot{S}_a\right)\right]^2-\left[\sqrt{\frac{C_{11}}{2}}\left(S_u-\ddot{S}_u\right)+\sqrt{C_{33}}(L-\ddot{L})\right]^2$$
$$-\left[(\mu+r_2+d_2)-\left(\frac{\beta_2\ddot{\mathrm{S}}_\mathrm{u}\mathrm{S}_\mathrm{u}}{1+\mathrm{nH}}+\frac{\beta_2\ \mathrm{S_uL}}{1+\mathrm{nH}}\right)\right]H$$

where

$$C_{11}=\frac{\beta_1 L}{K_1}+\frac{\beta_1 nLZ}{K_1}+\alpha\ddot{S}_a+\mu, C_{13}=\frac{\beta_1 S_u}{K_1}-\frac{\beta_1 L}{K_1}-\frac{\beta_1 nLZ}{K_1}$$
$$C_{22}=\mu-\alpha S_u, C_{12}=\alpha S_u-\alpha\ddot{S}_a, C_{33}=(\mu+r_1+d_1)-\frac{\beta_1 S_u}{K_1}.$$

Therefore, V_6 is a Lyapunov function with respect to E_6 in the region that fulfills the stated constraints since $\frac{dV_6}{dt}\leq 0$ is negative definite according to the criterion previously. Thus, the evidence that E_6 is a globally asymptotically stable is complete. □

Theorem 5.7. Let the equilibrium point E_7 be locally asymptotically stable. Then it's a globally asymptotically stable provided that the following sufficient condition holds

$$\begin{aligned}
&f_{12}^2<\frac{4}{3}f_{11}f_{22}\\
&f_{13}^2<\frac{4}{6}f_{11}f_{33}\\
&f_{14}^2<\frac{4}{6}f_{11}f_{44}\\
&f_{34}^2<f_{33}f_{44}
\end{aligned} \tag{5.14}$$

$$S_u<\min\left\{(\mu+\gamma+r_1+d_1)\frac{\bar{\bar{K}}_1}{\beta_1},\frac{\mu}{\alpha}\right\}. \tag{5.15}$$

Symbol locations $f_{ij}, i,j=1,2,3,4$ are provided as evidence.

Proof . Think about the subsequent function

$$V_7\left(S_u,S_a,L,H\right)=\frac{1}{2}\left(S_u-\bar{\bar{S}}_u\right)^2+\frac{1}{2}\left(S_a-\bar{\bar{S}}_a\right)^2+\frac{1}{2}(L-\bar{\bar{L}})^2+\frac{1}{2}(H-\bar{\bar{H}})^2.$$

It is easy to see that $V_7=(S_u,S_a,L,H)\in C^1\left(R_+^4,R\right)$ in addition $V_7\left(\bar{S}_u,\bar{S}_a,\bar{\bar{L}},\bar{\bar{H}}\right)=0$, while $V_7\left(S_u,S_a,L,H\right)>0$, for all $(S_u,S_a,L,H)\in R_+^4$ and $(S_u,S_a,L,H)\neq\left(\bar{S}_u,\bar{\bar{S}}_a,\bar{\bar{L}},\bar{\bar{H}}\right)$.

$$\frac{dV_6}{dt}=\left(S_u-\overline{\bar{S}}_u\right)\frac{dS_u}{dt}+\left(S_a-\overline{\bar{S}}_a\right)\frac{dS_a}{dt}+(L-\overline{\bar{L}})\frac{dL}{dt}+(H-\overline{\bar{H}})\frac{dH}{dt}.$$

Furthermore, by the derivative and simplifying the

$$\begin{aligned}
\frac{dV_6}{dt}=&\left(S_u-\overline{\bar{S}}_u\right)\left[(1-b)A-\frac{\beta_1 S_u L}{1+nL}-\frac{\beta_2 S_u H}{1+nH}-\alpha S_u S_a-\mu S_u\right]+\left(S_a-\overline{\bar{S}}_a\right)\left[bA+\alpha S_u S_a-\mu S_a\right]\\
&+(L-\overline{\bar{L}})\left[\frac{\beta_1 S_u L}{1+nL}+\frac{\beta_2 S_u H}{1+nH}-(\mu+\gamma+r_1+d_1)L\right]+(H-\overline{\bar{H}})\left[\gamma L-(\mu+r_2+d_2)\right]H.
\end{aligned}$$

By using the above conditions, we obtain that

$$\begin{aligned}
\frac{dV_7}{dt}\leq&-\left[\sqrt{\frac{f_{11}}{3}}\left(S_u-\bar{S}_u\right)+\sqrt{f_{22}}\left(S_a-\bar{S}_a\right)\right]^2-\left[\sqrt{\frac{f_{11}}{3}}\left(S_u-\bar{S}_u\right)+\sqrt{\frac{f_{33}}{2}}(L-\overline{\bar{L}})\right]^2\\
&-\left[\sqrt{\frac{f_{11}}{3}}\left(S_u-\bar{S}_u\right)+\sqrt{\frac{f_{44}}{2}}(H-\overline{\bar{H}})\right]^2-\left[\sqrt{\frac{f_{33}}{2}}(L-\overline{\bar{L}})+\sqrt{\frac{f_{44}}{2}}(H-\overline{\bar{H}})\right]^2.
\end{aligned}$$

Therefore, V_7 is a Lyapunov function with respect to E_7 in the region that fulfills the stated constraints since $\frac{dV_7}{dt}\leq 0$ is negative definite according to the criterion previously. Thus, the evidence that E_7 is a globally asymptotically stable is complete. □

6 Numerical Simulation

In this section, numerical simulation is carried out in order to visualize the aforementioned numerical solutions and comprehend the impact of changing the parameters on the overall dynamics of the system (2.1). As a result, system (2.1) is numerically solved for a variety of initial conditions and parameter sets. As can be shown in the following figures, the system (2.1) exhibits a globally asymptotically stable equilibrium point for the following units of fictitious parameters:

$$A = 500, b = 0.002, \beta_1 = 0.002, \beta_2 = 0.003 \tag{6.1}$$

$$n = 0.004, \alpha = 0.003, \mu = 0.1, \gamma = 0.03 \tag{6.2}$$

$$r_1 = 0.004, d_1 = 0.005, r_2 = 0.003, d_2 = 0.02 \tag{6.3}$$

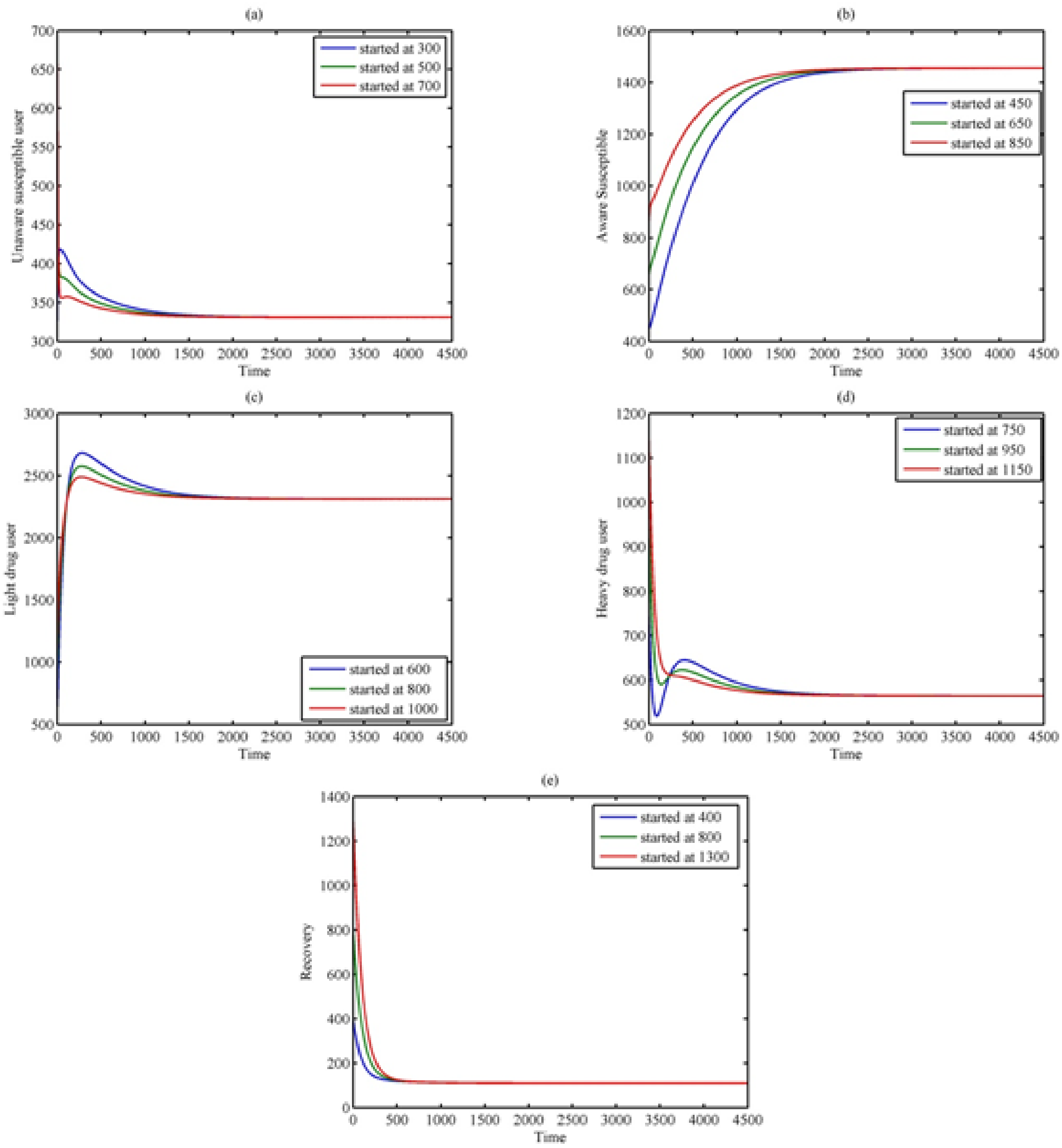

Figure 2: A timeline of eventsof system (2.1)'s trajectory for equation (54). a) Unaware susceptible user trajectory b) Aware susceptible trajectory c) Light drug user trajectory d) Heavy drug user trajectory e) Recovery trajectory

Due to convergence from three distinct initial data, following figure clearly illustrates the presence of a globally asymptotically stable seventh equilibrium point $E_7 = (331.04, 1455.02, 2312.19, 563.94, 109.40)$ for system (2.1). But for the information provided by Eq. (54) with $\theta = 0$ and $\mu_1 = 0.2$, Solutions to system (2.1)'s resolution asymptotically to the first equilibrium point $E_1 = (5000, 0, 0, 0, 0)$ as representative examples in the following, Figure(3)

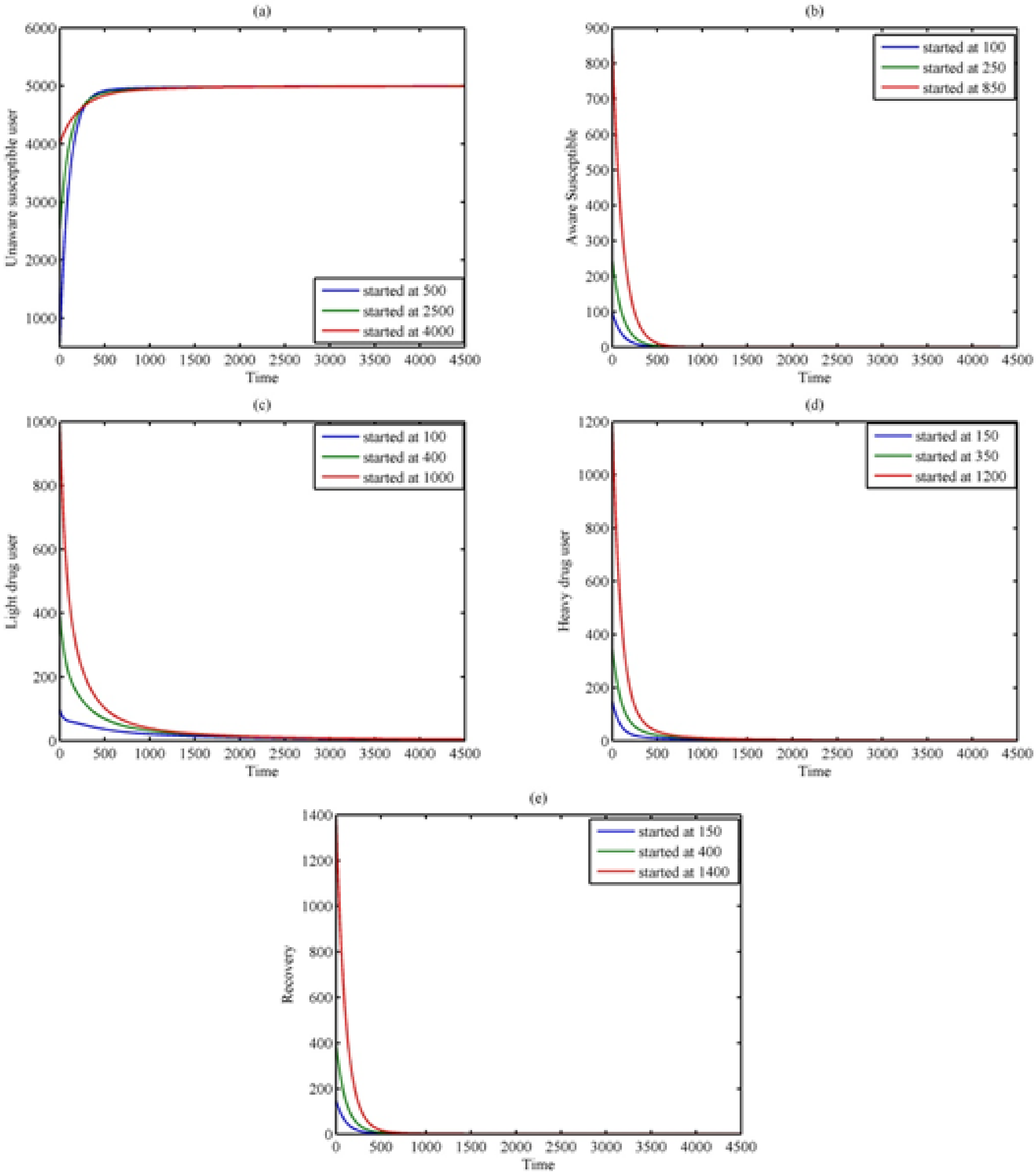

Figure 3: A timeline of events of the trajectory of system (2.1) for the Eq.(54). (a) Trajectories of Unaware susceptible user (b) Trajectories of aware susceptible (c) Trajectories of light drug user, (d) Trajectories of heavy drug user (e) Trajectories of recovery.

Obviously, Figure 3. However, for the data given in Eq.(54) with $b = 0$ and $\beta_1 = 0.00002, \beta_2 = 0.00003, \alpha = 0.000003$ the system's solution (2.1) approaches asymptotically to the second equilibrium point $E_2 = (0, 5000, 0, 0, 0)$ as representative examples in the following, Figure (4)

Additionally, the settings for the parameters listed in Eq.(54) with $b = 1$ the system's solution (2.1) approaches asymptotically $E_3 = (332.619, 4667.38, 0, 0, 0)$ as representative examples in the following, Figure (5)

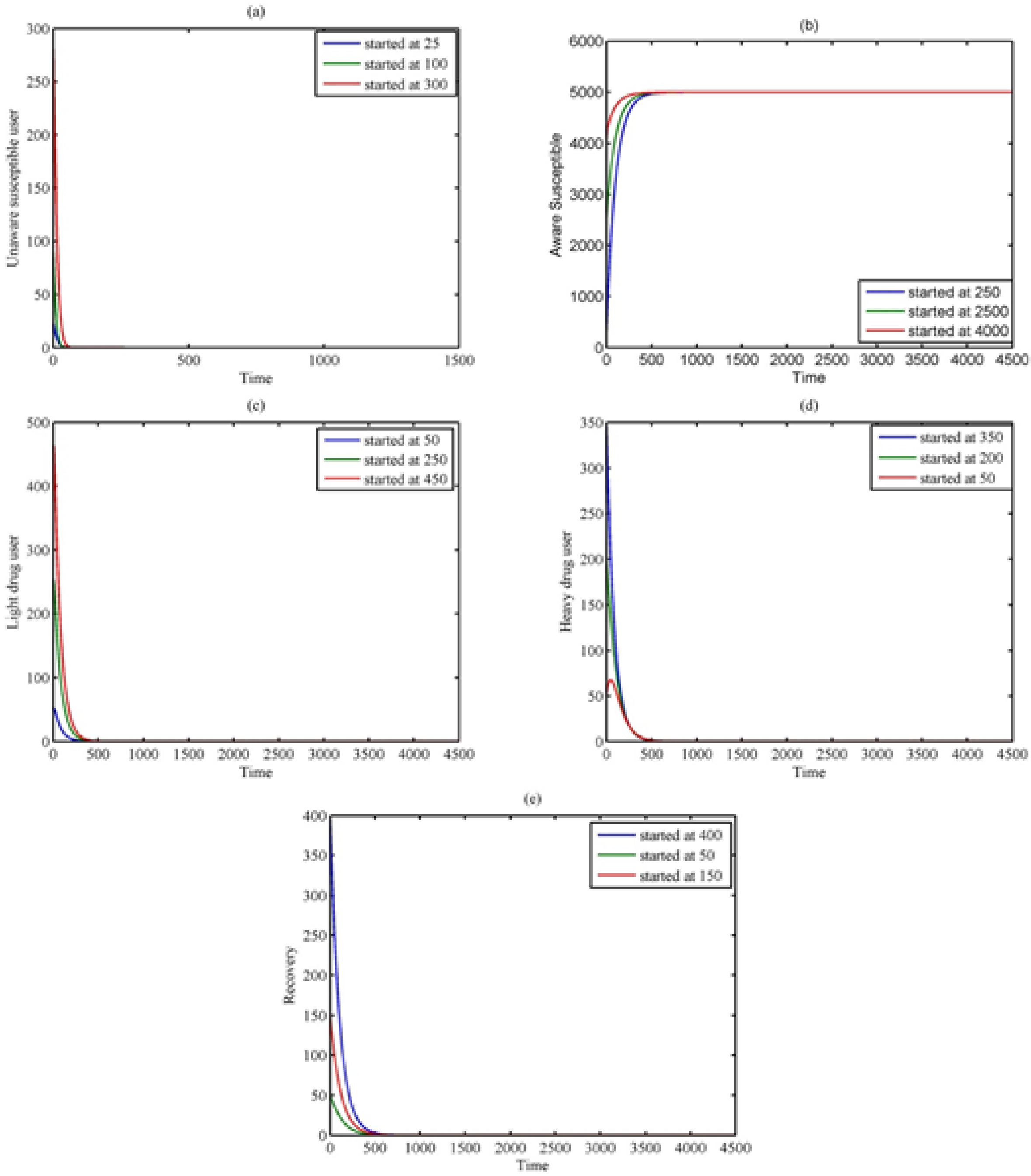

Figure 4: A timeline of events of the trajectory of system (2.1) for the Eq.(54). (a) Trajectories of Unaware susceptible user (b) Trajectories of aware susceptible (c) Trajectories of light drug user, (d) Trajectories of heavy drug user (e) Trajectories of recovery.

Additionally, the settings for the parameters listed in Eq. (54) with different $\beta_1 = 0.0002, \beta_2 = 0.0003$ the system's solution (2.1) approach asymptotically to the $E_4 = (878.75, 0, 3780.96, 0, 151.239)$ as representative examples in the following, Figure (6)

We select the values $b = 0, \gamma = 0, \alpha = 0.00003$ leaving other parameters constant as shown in Eq.(54), we obtain the system (2.1) trajectories still makes its way toward the point of equilibrium to $E_5 = (440.124, 0, 328049, 800.119, 155223)$, as representative examples in the following Figure (7)

Additionally, the settings for the parameters listed in Eq.(54) with $b = 0, \alpha = 0.00003$ the system's (2.1) solution approaches asymptotically to $E_6 = (332.317, 3278.6, 1274.39, 0, 50.9755)$ as representative examples in the following

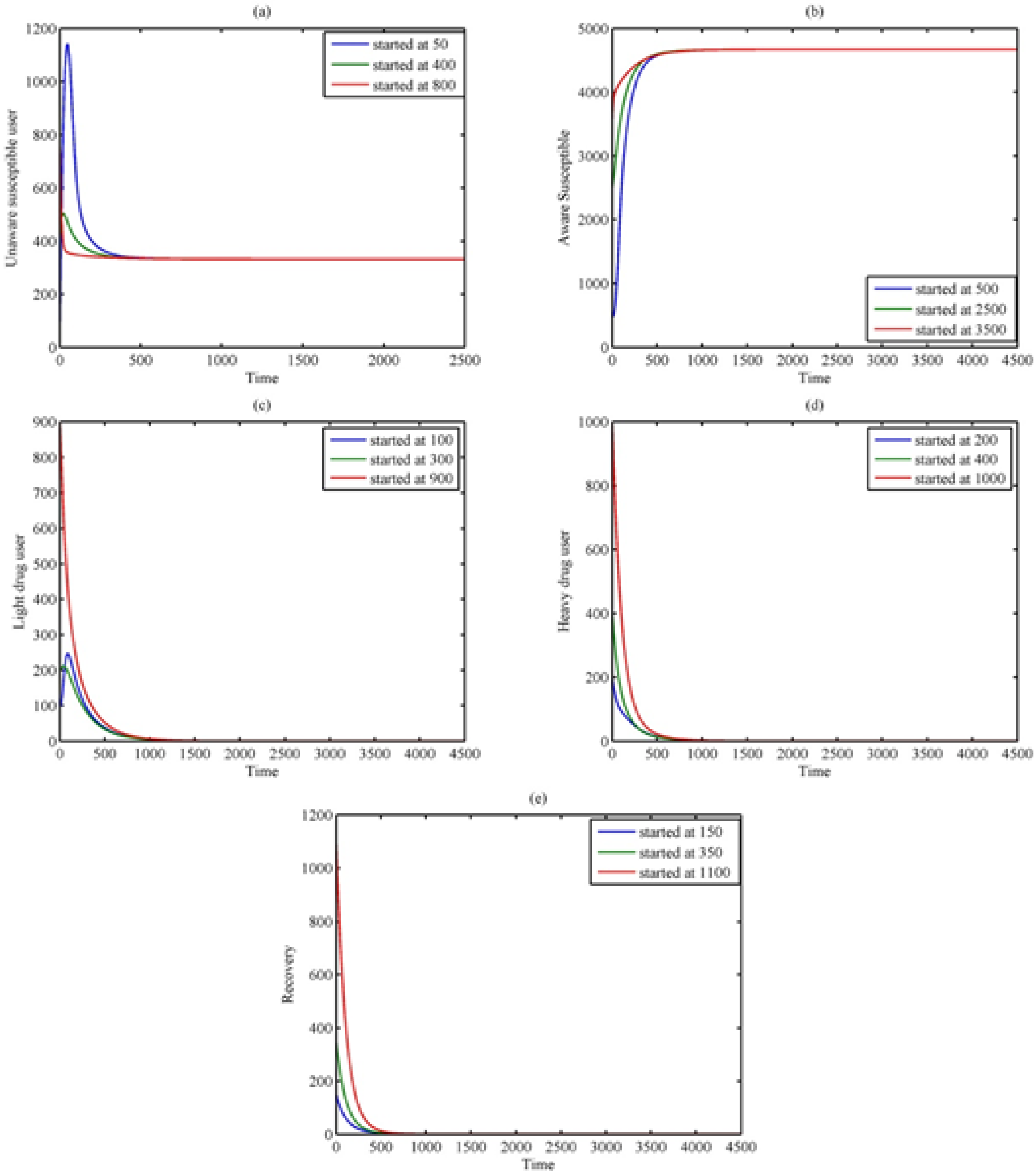

Figure 5: A timeline of events of the trajectory of system (2.1) for the Eq.(54). (a) Trajectories of Unaware susceptible user (b) Trajectories of aware susceptible (c) Trajectories of light drug user, (d) Trajectories of heavy drug user (e) Trajectories of recovery.

,Figure (8)

It should be noted that the system is numerically computed for the data provided in Eq. (54), with one parameter being changed every time, in attempt to explore the impact of the parameter values of system (2.1) on the dynamical behavior of system (2.1).

The system's solution(2.1) clearly approaches the seventh equilibrium point asymptotically, as shown by Figure (8). Let's choose a mild drug interaction rate addicts and susceptible from asymptomatic $\beta_1 = 0.002$ and 0.02 respectively, keeping other parameter fixed as given in equation (54), get the trajectories of system (2.1) still approaches to the seventh equilibrium point but the number of an aware susceptible and aware susceptible decreases while the light

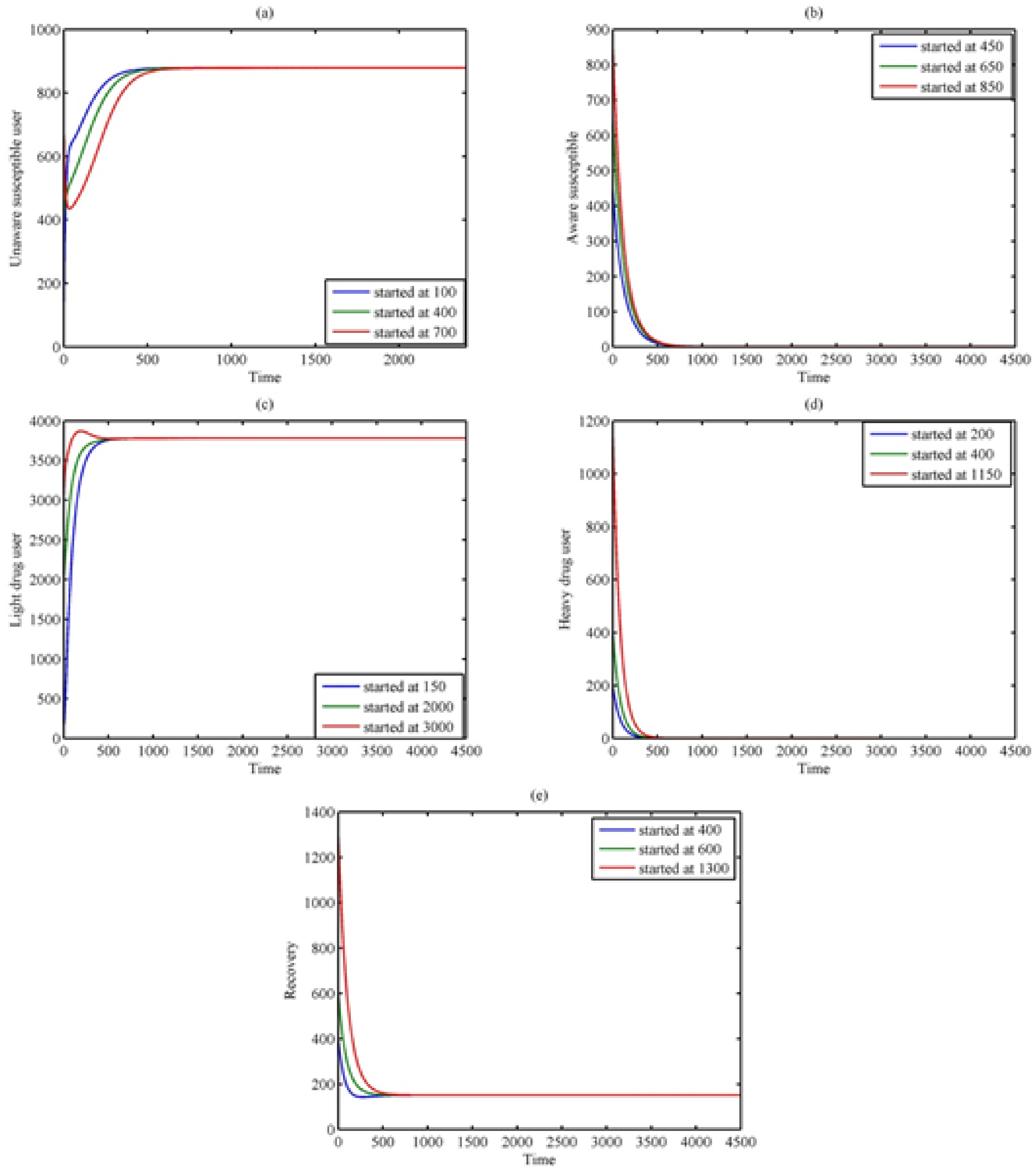

Figure 6: A timeline of events of the trajectory of system (2.1) for the Eq.(54). (a) Trajectories of Unaware susceptible user (b) Trajectories of aware susceptible (c) Trajectories of light drug user, (d) Trajectories of heavy drug user (e) Trajectories of recovery.

drugs user increases as shown in Figure(9).

The system's solution(2.1) is clearly shown to asymptotically approach the seventh equilibrium point in Figure (9). Let's choose the rate of contact between strong drugs. addicts and susceptible from asymptomatic $\beta_2 = 0.003$ and 0.03 The trajectory of system (2.1) continuously approaching the seventh equilibrium point while maintaining the other parameters fixed as stated in equation (54), however the proportion of unaware (susceptible) and conscious susceptible individuals falls and the proportion of mild drug users increases as depicted in Figure (10).

Figure (10), which makes it evident how the system's solution (2.1) works asymptotically to the seventh equilibrium pointe except if $\alpha = 0.003$ the system's solution (2.1) approaches asymptotically stable to the second equilibrium pointe

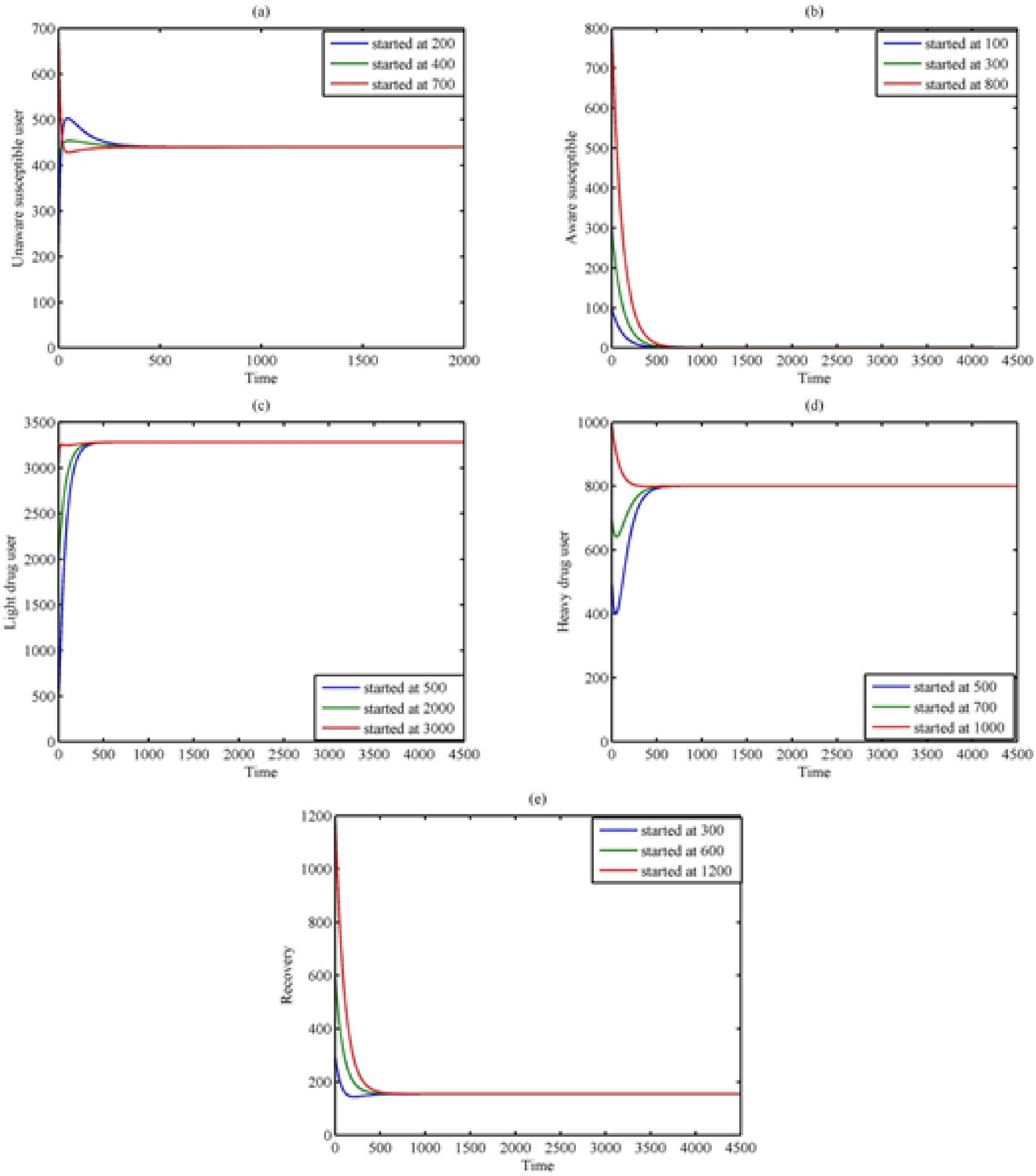

Figure 7: A timeline of events of the trajectory of system (2.1) for the Eq.(54). (a) Trajectories of Unaware susceptible user (b) Trajectories of aware susceptible (c) Trajectories of light drug user, (d) Trajectories of heavy drug user (e) Trajectories of recovery.

as shown in Figure(11).

Figure (11), which makes it evident how the system's solution (1) works asymptotically to the seventh equilibrium point except if $n = 0.4$ the system's solution (1) approaches asymptotically stable to the third equilibrium point as shown in Figure (12).

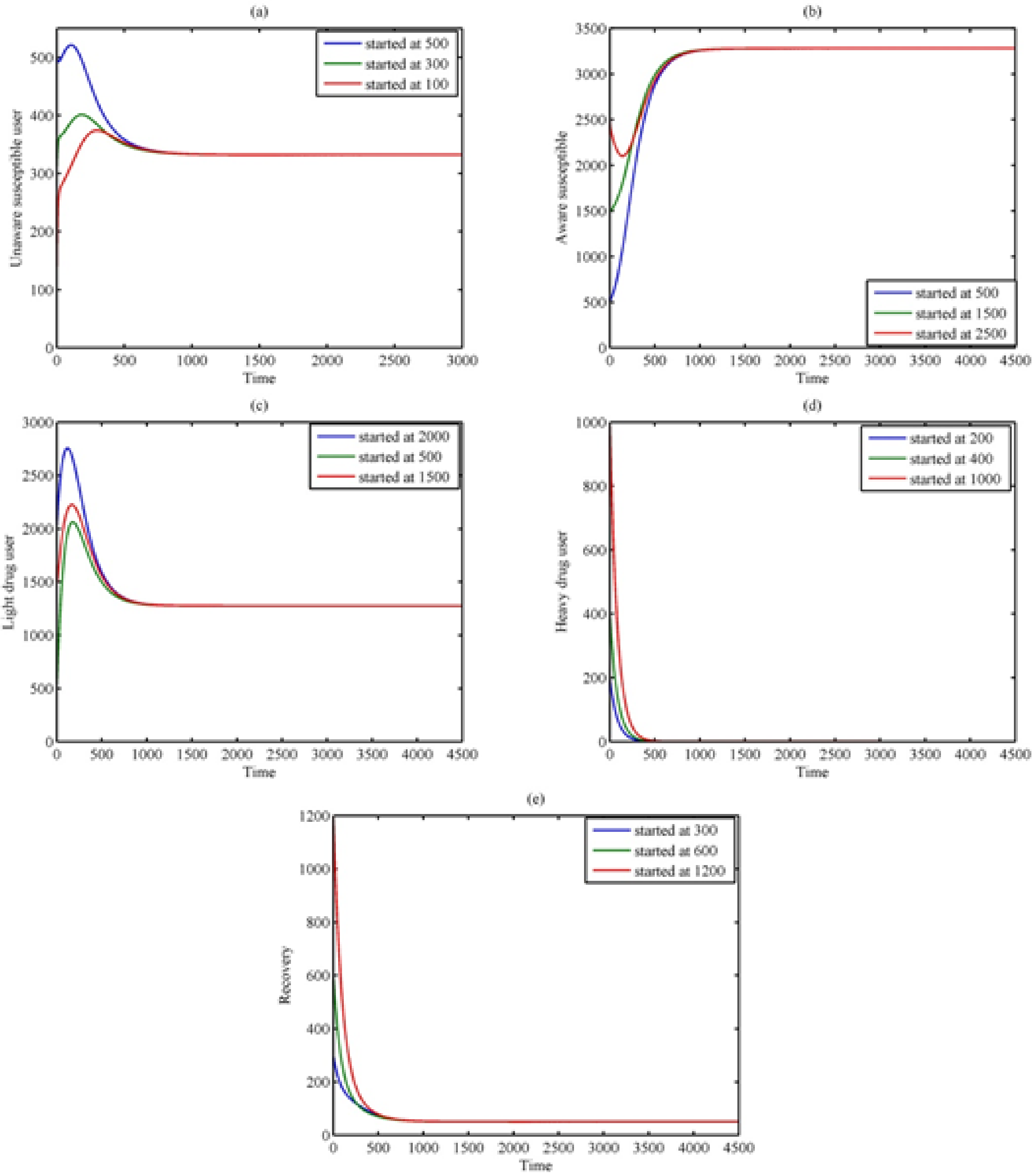

Figure 8: A timeline of events of the trajectory of system (2.1) for the Eq. (54). (a) Trajectories of Unaware susceptible user (b) Trajectories of aware susceptible (c) Trajectories of light drug user, (d) Trajectories of heavy drug user (e) Trajectories of recovery.

7 Conclusions and Discussion

In this essay, we created and examined an eco-epidemiological model that explained drug and alcohol addiction as well as the effects of the legal system and social stratification on addicts. Five non-linear autonomous ordinary differential equations were included in model to capture the dynamics of five distinct species: ignorant Susceptible (S_u), aware Susceptible (S_a), light drugs (L), strong drugs (H), and recovery rate (R). System (2.1)'s roundness has been debated.

All potential equilibrium points' existence conditions are discovered. These points had their local and global stability evaluations done. The control set of factors that affect the dynamics of the system is finally specified by

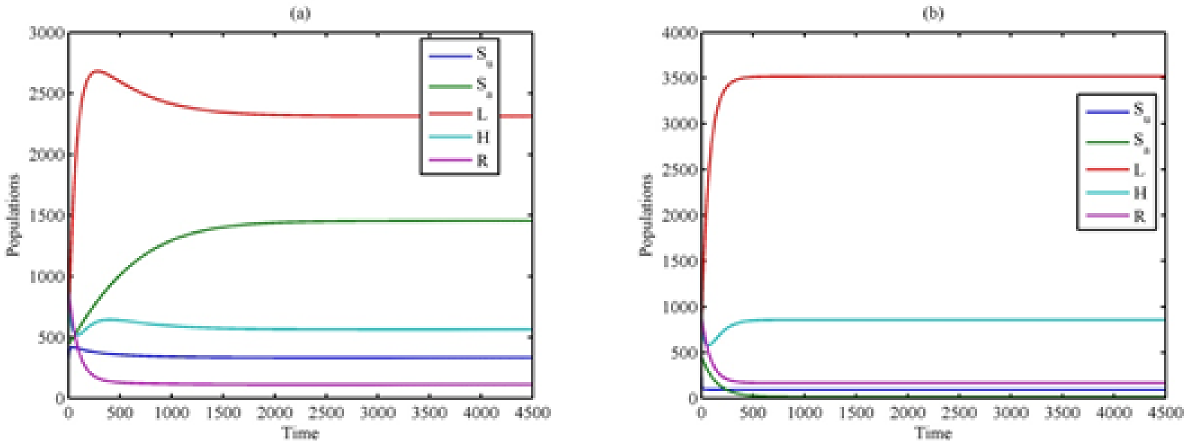

Figure 9: A timeline of events of the trajectory of system (2.1). The system approaches asymptotically to the seventh equilibrium point when (a) $\beta_1 = 0.002$. (b) $\beta_1 = 0.02$.

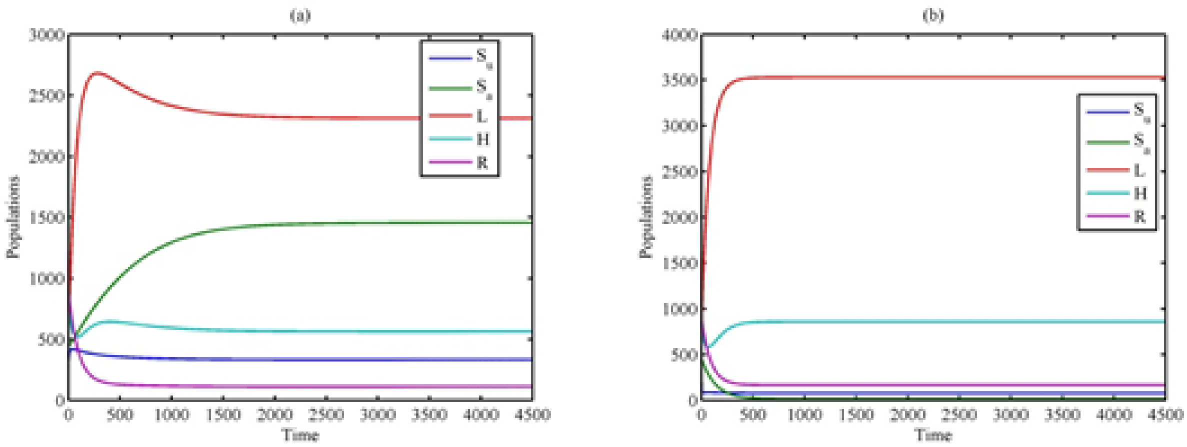

Figure 10: A timeline of events of the trajectory of system (2.1). The system approaches asymptotically to the seventh equilibrium point when (a) $\beta_2 = 0.003$. (b) $\beta_2 = 0.03$

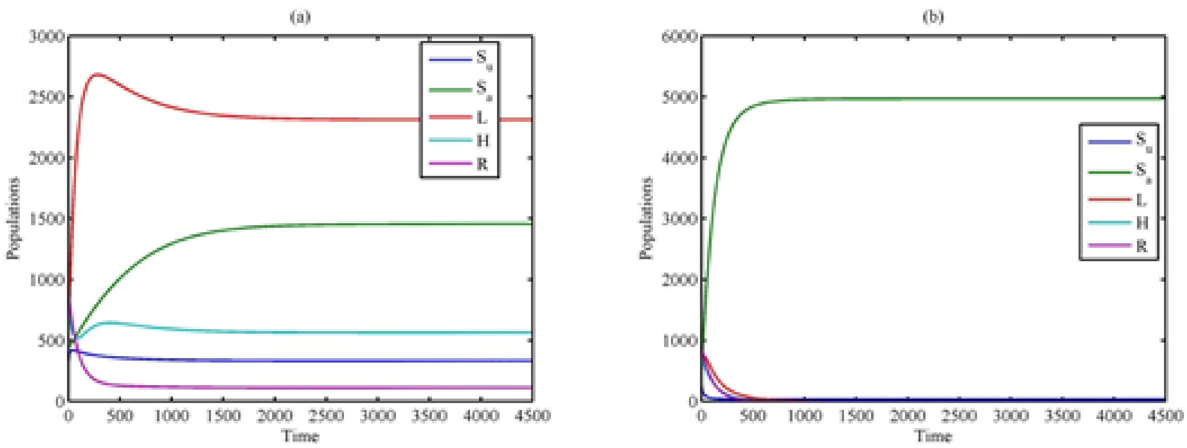

Figure 11: A timeline of events of the trajectory of system (1) for the data (54) different in value of α. The system approaches asymptotically to the seventh equilibrium point when (a) $\alpha = 0.0003$. (b) The system approaches asymptotically to the second equilibrium point when $\alpha = 0.003$

numerical simulation, which also serves to validate the conclusions drawn from our analytical work. As a result,

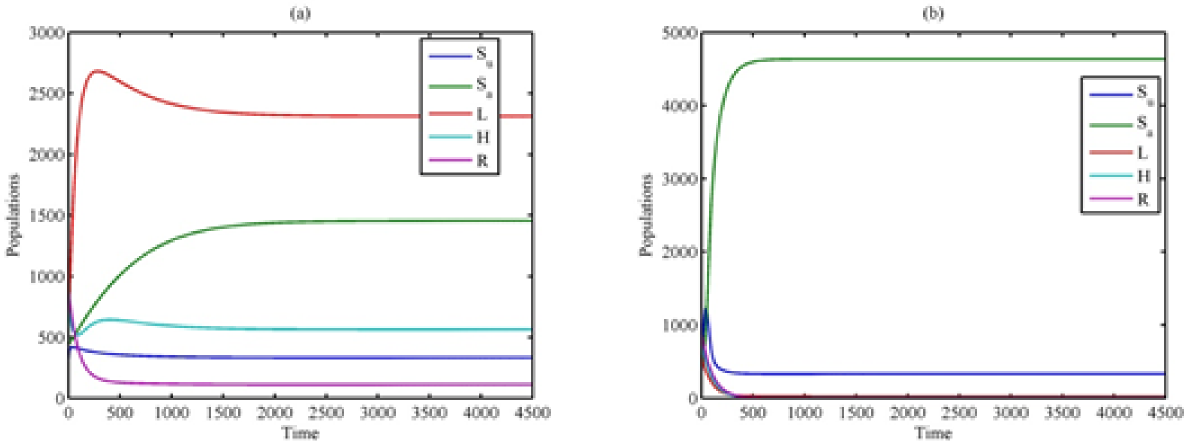

Figure 12: A timeline of events of the system's solution (1), for the data given by Eq. (54) with different values of n. (a) Globally asymptotically stable seventh equilibrium point for $n = 0.004$ (b) globally asymptotically stable for third equilibrium point for $n = 0.4$

starting with the hypothetical set of data provided by Eq. (54), system (2.1) has been solved numerically for various sets of initial points and various sets of parameters, and the following observations are obtained.

1- System (2.1) lacks periodic dynamics; instead, one of its equilibrium points is approached asymptotically by the system's solution (2.1).

2- The system (2.1) approached the global stable seventh equilibrium point E_7 asymptotically for the hypothetical parameter values provided by Eq. (54).

3- The seventh equilibrium point $E7$ becomes unstable when the value rises while maintaining the other parameters as in eq. (54), and the trajectory of the system (1) moves toward the second equilibrium point E_2.

4- The system's solution (1) approaches asymptotically to the third equilibrium point E_3 as the value of n increases while maintaining the other parameters as in equation (54).

5- The dynamical behaviour of the system (1) is unaffected qualitatively by changing the parameter values β_1 and β_2, and the system continues to move toward the seventh equilibrium point E_7.

With the aforementioned in mind, each of these results is dependent on the fictitious set of parameter values provided by Eq. (54), and various results may be reached for various sets of data.

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