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Half inverse problems for the singular Sturm-Liouville operator

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Abstract

In this paper, we consider the inverse spectral problem for the impulsive Sturm-Liouville differential pencils on $[\,0,\pi]$ with the Robin boundary conditions and the jump conditions at the point $\frac{\pi}{2}$. We prove that two potentials functions on the whole interval and the parameters in the boundary and jump conditions can be determined from a set of eigenvalues for two cases: (i) The potentials are given on $\left(0,\frac{\pi}{4}\left(\,\alpha+\beta\right)\,\right)$. (ii) The potentials is given on $\left(\,\frac{\pi}{4}\left(\,\alpha+\beta\right)\,,\frac{\pi}{2}\left(\,\alpha+\beta\right)\,\right)$, where $0<\alpha<\beta<1$ and $\alpha+\beta>1$, respectively.

Keywords: Inverse spectral problems, Sturm-Liouville Operator, spectrum, uniqueness
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1 Introduction

Consider the following impulsive Sturm-Liouville problem:

$$\ell y := -y^{''} + q(x)y\left(x\right) = \lambda\rho\left(x\right)y\left(x\right), \qquad x\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right), \tag{1.1}$$

with the boundary conditions

$$U\left(y\right) := y^{'}\left(0\right) - hy(0) = 0, \tag{1.2}$$

$$V\left(y\right) := y^{'}\left(\pi\right) + Hy(\pi) = 0, \tag{1.3}$$

and the jump conditions

$$\begin{aligned} y\left(\frac{\pi}{2}+0\right) &= ay\left(\frac{\pi}{2}-0\right), \\ y^{'}\left(\frac{\pi}{2}+0\right) &= a^{-1}y^{'}\left(\frac{\pi}{2}-0\right) + \gamma y\left(\frac{\pi}{2}-0\right), \end{aligned} \tag{1.4}$$

where λ is the spectral parameter, $q(x)$ is a real-valued function in $L_2\left(0,\pi\right)$,

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$$\rho(x)=\begin{cases}\alpha^2, & 0<x<\dfrac{\pi}{2},\\ \beta^2, & \dfrac{\pi}{2}<x<\pi,\end{cases}$$

$\alpha,\beta,h,H,a,\gamma$ are real, and $a>0, |a-1|^2+\gamma^2\neq 0, 0<\alpha<\beta<1, \alpha+\beta>1$.

Inverse spectral problems consist in recovering the coefficients of an operator from their spectral characteristics. The first results on inverse problems theory of classical Sturm-Liouville operator where given by Ambarzumyan and Borg ([3, 12]). Inverse Sturm-Liouville problems which appear in mathematical physics, mechanics, electronics, geophysics an other branches of natural sciences have been studied for about ninety years ([21, 22, 25, 26, 36] and the references there in).

The half inverse Sturm-Liouville problem which is one of the important subjects of the inverse spectral theory has been studied firstly by Hochstadt and Lieberman in 1978 [27]. They proved that spectrum of the problem

$$-y^{''}+q(x)y=\lambda y, \quad x\in(0,1),$$

$$y^{'}(0)-hy(0)=0=y^{'}(1)+Hy(1),$$

and potential $q(x)$ on the $\left(\frac{1}{2},1\right)$ uniquely determine the potential $q(x)$ on the whole interval $[0,1]$ almost everywhere. Since then, this result has been generalized to various versions. In 1984, Hald [6] proved similar results in the case when there exist a impulse conditions inside the interval. He also gave some applications of this kinds of problem to geophysics. Recently, some new uniqueness results in inverse spectral analysis with partial information on the potential for some classes of differential equations have been given (see for example [8, 23, 28]) . These kinds of results are known as Hochstadt and Lieberman type theorems. In particular, in the work [36] studied the inverse spectral problem for the impulsive Sturm-Liouville problem on $(0,\pi)$ with the Robin boundary conditions and the jump conditions at the point $\frac{\pi}{2}$. They proved that the potential $q(x)$ on the whole interval and the paremeters in the boundary conditions and jump conditions can be determined from a set of eigenvalues for two cases:

i) The potential $q(x)$ is given on $\left(0,\frac{1+\alpha}{4}\pi\right)$,

ii) The potential $q(x)$ is given on $\left(\frac{1+\alpha}{4}\pi,\pi\right)$, where $0<\alpha<1$,

and also shown that the potential and all the parameters can be uniquely recovered by one spectrum and some information on the eigenfunctions at some interior point. Similary problem studied in [9]. In particular, they discuss Gesztesy-Simon theorem and show that if the potential function $q(x)$ is prescribed on the interval $\left[\frac{\pi(1-\alpha)}{2},\pi\right]$ for some $\alpha\in(0,1)$, then parts of a finite number of spectra suffice to determine $q(x)$ on $[0,\pi]$.

2 Preliminaries

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of the equation (1.1),satisfying the initial conditions $\varphi(0,\lambda)=1, \varphi'(0,\lambda)=h, \psi(\pi,\lambda)=1, \psi'(\pi,\lambda)=-H$ and the jump condition (1.4). Denote

$$\sigma(x)=\int_0^x\sqrt{\rho(t)}dt,\ \lambda=k^2,\ \tau=\operatorname{Im}k.$$

Theorem 2.1. The solution $y(x,\lambda)$ of impulsive equation (1.1) with initial conditions $y(0,k)=1, y'(0,k)=ik$ and jump conditions (1.4) can be expressed by the formula

$$y(x,\lambda)=r^+e^{ik\sigma(x)}+r^-e^{ik(\alpha\pi-\sigma(x))}+\int_{-\sigma(x)}^{\sigma(x)}R(x,t)e^{ikt}dt, \tag{2.1}$$

where $r^{\pm} = \frac{1}{2}\left(a \pm \frac{\alpha}{a\beta}\right)$ and the kernel $R(x,t)$ satisfies the inequality

$$\int_{-\sigma(x)}^{\sigma(x)} |R\left(x,t\right)| \, dt \leq c\left\{e^{t(x)} - 1\right\},$$

with $t(x) = \int_0^x (x-s)\,|q(s)|\,ds.$

The kernel $R(x,t)$ of the integral representation (2.1) has both partial derivatives $D_x R(x,t), D_t R(x,t)$ belonging to the space $L_1\left(\sigma\left(x\right), -\sigma\left(x\right)\right)$ for every $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and the following relations hold:

i) $\frac{d}{dx} R(x, \sigma\left(x\right)) = \frac{r^{+}}{4\sqrt{\rho\left(x\right)}} q(x),$

ii) $\frac{d}{dx}\left\{R(x,t)\left|_{t=\alpha\pi-\sigma(x)-0}^{t=\alpha\pi-\sigma(x)+0}\right.\right\} = \frac{r^{-}}{4\sqrt{\rho\left(x\right)}} q(x),$

iii) $R(x, -\sigma\left(x\right)) = 0.$

Moreover, if $q(x)$ is differentable on $(0,\pi)$, than $R\left(x,t\right)$ satisfies the partial differential equation

$$\rho\left(x\right) R_{tt}\left(x,t\right) - R_{xx}\left(x,t\right) + q(x) R(x,t) = 0, (0 \leq x \leq \pi, |t| \leq \sigma\left(x\right)).$$

Proof . Similar to the proof of [20], so we omit the proof.

It is easy to verify from the integral representation (2.1) above that the solution $\varphi\left(x,\lambda\right)$ following asimptotic relation is valid as $|k| \to \infty$. For $\frac{\pi}{2} < x < \pi$,

$$\varphi(x,\lambda) = r^{+}\cos k\sigma(x) + r^{-}\cos k(\alpha\pi - \sigma\left(x\right)) + \frac{h}{k\alpha}\left\{r^{+}\sin k\sigma(x) + r^{-}\sin k(\alpha\pi - \sigma\left(x\right))\right\} + \\ + O\left(k^{-2}\exp\left(|\tau|\,\sigma\left(x\right)\right)\right), \tag{2.2}$$

$$\varphi^{'}\left(x,\lambda\right) = -kr^{+}\beta\sin k\sigma(x) + kr^{-}\beta\sin k(\alpha\pi - \sigma\left(x\right)) + \frac{h}{\alpha}\beta\left\{r^{+}\cos k\sigma(x) - r^{-}\cos k(\alpha\pi - \sigma\left(x\right))\right\} + \\ + O\left(k^{-1}\exp\left(|\tau|\,\sigma\left(x\right)\right)\right). \tag{2.3}$$

Similarly, for the solution $\psi\left(x,\lambda\right)$ following asimptotic relation hold as $|k| \to \infty$. For $0 < x < \frac{\pi}{2}$,

$$\psi(x,\lambda) = R^{+}\cos k(\sigma(\pi) - \sigma\left(x\right)) + R^{-}\cos k(\beta\pi - \left(\sigma\left(\pi\right) - \sigma\left(x\right)\right) + \\ + \frac{1}{k}\left\{\left(\frac{H}{\beta}R^{+} + \frac{\gamma}{\alpha}\right)\sin k\left(\sigma(\pi) - \sigma(x)\right) + \left(\frac{H}{\beta}R^{-} + \frac{\gamma}{\alpha}\right)\sin k\left(\beta\pi - \left(\sigma(\pi) - \sigma(x)\right)\right)\right\} + \\ + O\left(k^{-2}\exp\left(|\tau|\left(\sigma\left(\pi\right) - \sigma\left(x\right)\right)\right)\right), \tag{2.4}$$

$$\psi^{'}\left(x,\lambda\right) = k\alpha R^{+}\sin k(\sigma(\pi) - \sigma\left(x\right)) - k\alpha R^{-}\sin k(\beta\pi - \left(\sigma\left(\pi\right) - \sigma\left(x\right)\right)) - \\ - \left\{\left(\frac{H}{\beta}R^{+} + \frac{\gamma}{\alpha}\right)\alpha k\cos k\left(\sigma(\pi) - \sigma(x)\right) + \left(\frac{H}{\beta}R^{-} + \frac{\gamma}{\alpha}\right)\alpha k\cos k\left(\beta\pi - \left(\sigma(\pi) - \sigma(x)\right)\right)\right\} + \\ + O\left(k^{-1}\exp\left(|\tau|\left(\sigma\left(\pi\right) - \sigma\left(x\right)\right)\right)\right), \tag{2.5}$$

where $R^{\pm} = \frac{1}{2}\left(\frac{1}{a} \pm \frac{\beta a}{\alpha}\right)$. $\square$

Definition 2.2. It is easy to verify that if $y(x)$ and $z(x)$ satisfy equation (1.1) and impulsive condition (1.4), then $\langle y, z\rangle$ is independent of x, and

$$\langle y, z\rangle \left|_{x=\frac{\pi}{2}-0} = \langle y, z\rangle \right|_{x=\frac{\pi}{2}+0} .$$

Denote

$$\Delta(\lambda) = \langle \varphi, \psi\rangle = V(\varphi) = -U(\psi). \tag{2.6}$$

The function $\Delta(\lambda)$ is called the characteristic function of L, which is entire in λ, and it has an at most countable set of zeros $\{\lambda_n\}_{n\geq 0}$.

$$\Delta(\lambda) = \Delta_0(\lambda) + \int_0^{\sigma(\pi)} \widetilde{R}(\pi, t) \cos ktdt, \tag{2.7}$$

where $\Delta_0(\lambda) = \varphi_0^{'}(x, \lambda) + H\varphi_0(x, \lambda)$,

$$\begin{aligned} \varphi_0(x, \lambda) = r^+ \cos k\sigma(\pi) + r^- \cos k\left(\alpha\pi - \sigma\left(\pi\right)\right) + \\ + \frac{h}{k\alpha}\left\{r^+ \sin k\sigma\left(\pi\right) + r^- \sin k\left(\alpha\pi - \sigma\left(\pi\right)\right)\right\}, \end{aligned}$$

$\langle \varphi\left(x, \lambda\right), \psi\left(x, \lambda\right)\rangle := \varphi\left(x, \lambda\right)\psi^{'}\left(x, \lambda\right) - \varphi^{'}\left(x, \lambda\right)\psi\left(x, \lambda\right)$.

Lemma 2.3. The following statements hold:

i) The zeros $\{\lambda_n\}_{n\geq 0}$ of the characteristic function $\Delta\left(\lambda\right)$ coincide with the eigenvalues of the boundary value problem L.

ii) The functions $\varphi\left(x, \lambda_n\right)$ and $\psi\left(x, \lambda_n\right)$ are corresponding eigenfunctions and exists a sequence $\{\beta_n\}$, $\beta_n \neq 0$, $n = 0, 1, 2, ...$, such that

$$\psi(x, \lambda_n) = \beta_n \varphi\left(x, \lambda_n\right). \tag{2.8}$$

Next, we denote by $L_2\left((0, \pi); \rho\left(x\right)\right)$ a space which has the inner product

$$(\varphi, \psi) = \int_0^{\pi} \rho\left(x\right)\varphi(x, \lambda)\psi(x, \lambda)dx.$$

Lemma 2.4. The eigenvalues $\{k_n\}_{n\geq 0}$ of the problem L are real and simple. The eigenfunctions corresponding to the different eigenvalues are orthogonal in the weighted space $L_2\left((0, \pi); \rho\left(x\right)\right)$ and for sufficiently large values of n, the eigenvalue k_n has the following behavior

$$k_n = k_n^0 + \frac{d_n}{k_n^0} + \frac{k_n}{k_n^0}, \tag{2.9}$$

where, λ_n^0 are zeros of $\Delta_0\left(\lambda\right) = \varphi_0'(\pi, \lambda) + H\varphi_0\left(\pi, \lambda\right)$, d_n is bounded and $k_n \in \ell_2$,

$$k_n^0 = \frac{n\pi}{\sigma\left(\pi\right)} + \theta_n, \quad \sup_n |\theta_n| < +\infty.$$

Proof of lemmas similarly to the proof of [20], so we omit the proof. Let α_n $(n \geq 0)$ be the normalized constants, which are defined as

$$\alpha_n := \int_0^{\pi} \rho\left(x\right)\varphi^2\left(x, \lambda_n\right)dx \quad \text{ for all } n \geq 0.$$

Lemma 2.5. The following relation holds:
$$\overset{\bullet}{\Delta}(k_n) = -2\alpha_n\beta_n k_n \tag{2.10}$$

where $\overset{\bullet}{\Delta}(k_n) = \left(\frac{d}{d\lambda}\Delta(\lambda)\right)_{k=k_n}, \beta_n = -[\varphi(\pi, k_n)]^{-1}$.

Proof . The functions $\varphi(x,k)$ and $\psi(x,k_n)$ are solutions of the equation (1.1), then

$$-\varphi''(x,k) + q(x)\varphi(x,k) = k^2\rho(x)\varphi(x,k),$$

and
$$-\psi''(x,k_n) + q(x)\psi(x,k_n) = k_n^2\rho(x)\psi(x,k_n).$$

Multiplying the two equations by $\psi(x,k_n)$ and $\varphi(x,k)$, respectively, and subtracting the second equation from the first equation, it follows that

$$\frac{d}{dx}\left[\varphi(x,k)\psi'(x,k_n) - \varphi'(x,k)\psi(x,k_n)\right] = (k^2 - k_n^2)\rho(x)\varphi(x,k)\psi(x,k_n).$$

Integrating the above equality from 0 to π and considering the discontinouty point, we can obtain that

$$(k^2 - k_n^2)\int_0^\pi \rho(x)\varphi(x,k)\psi(x,k_n)\,dx = -\Delta(\lambda).$$

Dividing the two sides by $k^2 - k_n^2$ and letting $k \to k_n$ yields

$$-\overset{\bullet}{\Delta}(k_n) = \int_0^\pi \rho(x)\varphi(x,k_n)\psi(x,k_n)\,dx.$$

Combining (2.8) with the definition of α_n, we arrive at (2.10). In particular, it follows from (2.10) that all eigenvalues k_n are simple.

Let be $\delta > 0$ and fixed. Define $G_\delta := \left\{k \in \mathbb{C} : \left|k - k_n^0\right| \geq \delta, n = 1,2,...\right\}$. The following inequality can be deduced using the asymptotic formula for $\Delta(\lambda)$,

$$\Delta_0(k) \geq C|k|\exp(|\tau|\sigma(\pi)), \quad k \in G_\delta, \tag{2.11}$$

for some pozitive constant C. □

3 Main Results

Now we state the main result of this work. It is assumed in what follows that if a certain symbol s denotes an object related to L, then the corresponding symbol $\widetilde{s}$ with tilde denote the analogous object related to $\tilde{L}$.

Lemma 3.1. If $k_n = \widetilde{k}_n$, $n = 0,1,2,...$ then $\sigma(\pi) = \widetilde{\sigma}(\pi)$,that is the sequence $\{\lambda_n\}_{n\geq 0}$ uniquely determines $\sigma(\pi)$.

Proof of Lemma is easily obtained from the asymptotic expression of λ_n.

Lemma 3.2. The specification of the spectrum $\{k_n\}_{n\geq 0}$ uniquely determines the characteristic function $\Delta(k)$ by the formula
$$\Delta(k) = \sigma(\pi)(k_0^2 - k^2)\prod_{n=1}^{\infty}\left[\frac{k_n^2 - k^2}{(k_n^0)^2}\right], \tag{3.1}$$

and the estimation
$$|\Delta(k)| \geq C_\delta|k|\exp(|\tau|\sigma(\pi)), \tag{3.2}$$

holds for some $C_\delta > 0$.

Proof . From (2.7) we have that $\Delta(k)$ is an entire function of order one, and consequenly by Hadamard's theorem [17]

$$\Delta(k) = C\prod_{n=0}^{\infty}\left[1-\frac{k^2}{k_n^2}\right], \tag{3.3}$$

with some constant C. Because

$$\Delta_0(k) = -k^2\sigma(\pi)\prod_{n=1}^{\infty}\left[1-\frac{k^2}{(k_n^0)^2}\right], \tag{3.4}$$

then

$$\frac{\Delta(k)}{\Delta_0(k)} = \frac{C}{\sigma(\pi)}\left(\frac{k^2-k_0^2}{k^2k_0^2}\right)\prod_{n=1}^{\infty}\frac{(k_n^0)^2}{k_n^2}\left[1+\frac{k_n^2-(k_n^0)^2}{(k_n^0)^2-k^2}\right]. \tag{3.5}$$

Taking representain of $\Delta(k)$ and (2.9) into our account, we calculate

$$\lim_{k\to+i\infty}\frac{\Delta(k)}{\Delta_0(k)} = 1,\ \lim_{k\to+i\infty}\prod_{n=1}^{\infty}\left[1+\frac{k_n^2-(k_n^0)^2}{(k_n^0)^2-k^2}\right] = 1,$$

which imply

$$C = \sigma(\pi)\left(k_0^2-k^2\right)\prod_{n=1}^{\infty}\frac{k_n^2}{(k_n^0)^2}.$$

Substituting this into (3.4), we have

$$\Delta(\lambda) = \sigma(\pi)\left(k_0^2-k^2\right)\prod_{n=1}^{\infty}\left[\frac{k_n^2-k^2}{(k_n^0)^2}\right]. \tag{3.6}$$

Now from (3.4) and (3.6), we can write

$$\frac{\Delta_0(k)}{\Delta(k)} = \left(\frac{k^2}{k^2-k_0^2}\right)\prod_{n=1}^{\infty}\left[1+\frac{(k_n^0)^2-k_n^2}{k_n^2-k^2}\right]. \tag{3.7}$$

Because, $\lambda_n = \lambda_n^0 + O\left(\frac{1}{n}\right)$, $n\to\infty$ and

$$\left|\frac{(k_n^0)^2-k_n^2}{k_n^2-k^2}\right| \le \frac{C_\delta}{n^2},$$

for all $k\in G_\delta = \left\{k : \left|k-k_n^0\right| \ge \delta, \delta > 0\right\}$, where $C_\delta > 0$, we have $\left|\frac{\Delta_0(k)}{\Delta(k)}\right| \le M_\delta$ for some constant $M_\delta > 0$. Then using (3.2) we obtain the estimation (3.3). □

Lemma 3.3. If $k_n = \widetilde{k}_n$, $n = 0,1,2,\ldots$ then $a = \widetilde{a}$, $\rho(x) = \widetilde{\rho}(x)$, $h = \widetilde{h}$ and $H = \widetilde{H}$.

Proof . Since, $k_n = \widetilde{k}_n$, $n = 0,1,2,\ldots$, Lemma 2.3 requires $\sigma(\pi) = \widetilde{\sigma}(\pi)$ or $\alpha+\beta = \widetilde{\alpha}+\widetilde{\beta}$. $\Delta(k), \widetilde{\Delta}(k)$ are entire functions of order one by Hadamard factorization theorem, for $\lambda\in\mathbb{C}$

$$\Delta(k) \equiv C\widetilde{\Delta}(k). \tag{3.8}$$

Then from Lemma 2.4 and $\sigma(\pi) = \widetilde{\sigma}(\pi)$ we obtain $C = 1$.

On the other hand, (3.8) can be written as

$$\Delta_0(k) - C\widetilde{\Delta}_0(k) = \left[\widetilde{\Delta}(k) - \widetilde{\Delta}_0(k)\right] - \left[\Delta(k) - \Delta_0(k)\right]. \tag{3.9}$$

Hence

$$\begin{aligned}
\left[\widetilde{\Delta}(k) - \widetilde{\Delta}_0(k)\right] - \left[\Delta(k) - \Delta_0(k)\right] = {} & -r^+ k\beta \sin k\sigma(\pi) + r^- k\beta \sin k(\alpha\pi - \sigma(\pi)) \\
& + h\frac{\beta}{\alpha}\left[r^+ \cos k\sigma(\pi) - r^- \cos k(\alpha\pi - \sigma(\pi))\right] \\
& + H\left\{r^+ \cos k\sigma(\pi) + r^- \cos k(\alpha\pi - \sigma(\pi))\right. \\
& \left. + \frac{h}{k\alpha}\left[r^+ \sin k\sigma(\pi) + r^- \sin k(\alpha\pi - \sigma(\pi))\right]\right\} \\
& - \left\{\widetilde{r}^+ k\beta \sin k\sigma(\pi) + \widetilde{r}^- k\beta \sin k(\alpha\pi - \sigma(\pi))\right. \\
& \left. + \widetilde{h}\frac{\beta}{\alpha}\left[\widetilde{r}^+ \cos k\sigma(\pi) - \widetilde{r}^- \cos k(\alpha\pi - \sigma(\pi))\right]\right\} \\
& - \widetilde{H}\left\{\widetilde{r}^+ \cos k\sigma(\pi) + \widetilde{r}^- \cos k(\alpha\pi - \sigma(\pi))\right. \\
& \left. + \frac{\widetilde{h}}{k\alpha}\left[\widetilde{r}^+ \sin k\sigma(\pi) + \widetilde{r}^- \sin k(\alpha\pi - \sigma(\pi))\right]\right\},
\end{aligned} \tag{3.10}$$

if we multiply both sides of (3.10) with $\sin k\sigma(\pi)$ and integrate with respect to k in (ε, T) (ε is sufficiently small pozitive number) for any pozitive real number T, then we get

$$\begin{aligned}
& \int_{\varepsilon}^{T} \left(\left[\widetilde{\Delta}(k) - \widetilde{\Delta}_0(k)\right] - \left[\Delta(k) - \Delta_0(k)\right]\right) \sin k\sigma dk = \\
& \int_{\varepsilon}^{T} \left\{-r^+ k\beta \sin k\sigma(\pi) + r^- k\beta \sin k(\alpha\pi - \sigma(\pi)) + h\frac{\beta}{\alpha}\left[r^+ \cos k\sigma(\pi) - r^- \cos k(\alpha\pi - \sigma(\pi))\right]\right. \\
& + H\left[r^+ \cos k\sigma(\pi) - r^- \cos k(\alpha\pi - \sigma(\pi)) + \frac{h}{k\alpha}\left(r^+ \sin k\sigma(\pi) + r^- \sin k(\alpha\pi - \sigma(\pi))\right)\right] \\
& - \left[\widetilde{r}^+ k\beta \sin k\sigma(\pi) + \widetilde{r}^- k\beta \sin k(\alpha\pi - \sigma(\pi)) + \widetilde{h}\frac{\beta}{\alpha}\left(\widetilde{r}^+ \cos k\sigma(\pi) - \widetilde{r}^- \cos k(\alpha\pi - \sigma(\pi))\right)\right] \\
& \left. - \widetilde{H}\left[\widetilde{r}^+ \cos k\sigma(\pi) + \widetilde{r}^- \cos k(\alpha\pi - \sigma(\pi)) + \frac{\widetilde{h}}{k\alpha}\left(\widetilde{r}^+ \sin k\sigma(\pi) + \widetilde{r}^- \sin k(\alpha\pi - \sigma(\pi))\right)\right]\right\} \sin k\sigma dk.
\end{aligned}$$

Since

$$\Delta(k) - \Delta_0(k) = O\left(k^{-2}\exp(|\tau|\sigma(\pi))\right), \widetilde{\Delta}(k) - \widetilde{\Delta}_0(k) = O\left(k^{-2}\exp(|\tau|\sigma(\pi))\right),$$

for all k in (ε, T)

$$\frac{\beta}{4}\widetilde{r}^+ - \frac{\beta}{4}r^+ = O(\frac{1}{T^2}).$$

By letting T tend to infinity we see that

$$r^+ = \widetilde{r}^+. \tag{3.11}$$

Similarly, if we multiply both sides of (3.10) with $\sin k(\alpha\pi - \sigma(\pi))$ and integrate again with respect to k in (ε, T), and by letting T tend to infinity, then we get

$$r^- = \widetilde{r}^-. \tag{3.12}$$

Taking $a > 0$ into account, (3.11) and (3.12) implies that $a = \widetilde{a}, \alpha = \widetilde{\alpha}, \beta = \widetilde{\beta}$.

Considering that Lemma 3.2, and $a = \widetilde{a}$, if both sides of the last expression are multiplied by the $\cos k\sigma(\pi)$ and integrate with respect to k in (ε, T), then we get

$$h\frac{\beta}{\alpha}r^+ + Hr^+ = \widetilde{h}\frac{\beta}{\alpha}r^+ + \widetilde{H}r^+. \tag{3.13}$$

Similarly, if we multiply both sides of the last expression are with $\cos k\left(\alpha\pi-\sigma(\pi)\right)$ and integrate again with respect to k in (ε,T) , and by letting T tend to infinity, then we get

$$h\frac{\beta}{\alpha}r^{-}-Hr^{-}=\widetilde{h}\frac{\beta}{\alpha}r^{-}-\widetilde{H}r^{-}. \tag{3.14}$$

Finaly, from (3.13) and (3.14) implies that $h=\widetilde{h}$ and $H=\widetilde{H}$. □

Theorem 3.4. If $\lambda_n=\widetilde{\lambda}_n$ for all $n\geq 0, q\left(x\right)=\widetilde{q}\left(x\right)$ on $\left(0,\frac{\alpha+\beta}{4}\pi\right)$, then $q\left(x\right)=\widetilde{q}\left(x\right)$ almost every where on $\left(\frac{\alpha+\beta}{4}\pi,\frac{\alpha+\beta}{2}\pi\right)$, $h=\widetilde{h}$, $H=\widetilde{H}$, $\rho\left(x\right)=\widetilde{\rho}\left(x\right)$ and $a=\widetilde{a}$.

Proof . Let the boundary value problems L and $\widetilde{L}$ satisfy the conditions of Theorem2.1, then by virtue of Lemma2.4 and Lemma2.5 $h=\widetilde{h}, H=\widetilde{H}$, $\rho\left(x\right)=\widetilde{\rho}\left(x\right)$ and $a=\widetilde{a}$. For brevity, denote $c_1=\frac{\alpha+\beta}{4}\pi$ and $c_2=\frac{\alpha+\beta}{2}\pi$. Let $\psi\left(x,\lambda\right), \widetilde{\psi}\left(x,\lambda\right)$ be the solutions of the equations

$$-\psi^{''}\left(x,\lambda\right)+q(x)\psi\left(x,\lambda\right)=\lambda\rho\left(x\right)\psi\left(x,\lambda\right), \tag{3.15}$$

$$-\widetilde{\psi}^{''}\left(x,\lambda\right)+\widetilde{q}(x)\widetilde{\psi}\left(x,\lambda\right)=\lambda\widetilde{\rho}\left(x\right)\widetilde{\psi}\left(x,\lambda\right), \tag{3.16}$$

with the initial valued conditions, respectively

$$\psi\left(\pi,\lambda\right)=1, \psi^{'}\left(\pi,\lambda\right)=-H \tag{3.17}$$

$$\widetilde{\psi}\left(\pi,\lambda\right)=1, \widetilde{\psi}^{'}\left(\pi,\lambda\right)=-\widetilde{H} \tag{3.18}$$

and the impulsive conditions (1.4). After multiplying (3.15) by $\widetilde{\psi}\left(x,\lambda\right)$ and (3.16) by $\psi\left(x,\lambda\right)$, we subtract these equations from each other. Then by integrating on $[c_1,\pi]$ with respect to x, using the initial conditions (3.17), (3.18) and impulsive conditions (1.4), we have

$$\int_{c_1}^{\pi}(q(x)-\widetilde{q}(x))\psi(x,\lambda)\widetilde{\psi}(x,\lambda)dx=\psi^{'}(c_1,\lambda)\widetilde{\psi}(c_1,\lambda)-\widetilde{\psi}^{'}(c_1,\lambda)\psi(c_1,\lambda), \tag{3.19}$$

from the hypothesis $q(x)=\widetilde{q}(x)$ on $(0,c_1)$.

Denote $Q(x)=q(x)-\widetilde{q}(x)$ and

$$F_0\left(\lambda\right)=\int_{c_1}^{\pi}Q\left(x\right)\psi(x,\lambda)\widetilde{\psi}(x,\lambda)dx. \tag{3.20}$$

It follows from (2.4) and (3.19) that $F_0\left(\lambda\right)$ is an entire function of exponential type, and there are same pozitive constants c_1 and c_2 such that

$$\left|F_0\left(\lambda\right)\right|\leq C\left|k\right|\exp\left(\left|\tau\right|\sigma\left(\pi\right)\right), \quad \text{for all } \lambda\in\mathbb{C}. \tag{3.21}$$

It is clear from the properties of $\psi(x,\lambda)$, $\psi^{'}(x,\lambda)$ and the boundary conditions (1.2)

$$F_0\left(k_n^2\right)=0, \quad n=0,1,2,..., \tag{3.22}$$

for each eigenvalue $\lambda_n=k_n^2$.

Define

$$F\left(\lambda\right):=\frac{F_0\left(\lambda\right)}{\Delta\left(\lambda\right)},$$

which is an entire function from the above arguments and it follows from Lemma 3.2 and (3.21) that

$$F\left(k^2\right) = O\left(1\right),$$

for sufficiently large $|k|$, $k \in G_\delta$. Using Liouville' s theorem [15], we obtain for all $\lambda = k^2$ that

$$F\left(k^2\right) = C,$$

where C is a constant.

Let as show that the $C = 0$. Now, we can rewrite the equation $F_0\left(\lambda\right) = C\Delta\left(\lambda\right)$ as

$$\begin{aligned}\int_{c_1}^{\pi} Q\left(x\right)\psi(x,k)\widetilde{\psi}(x,k)dx = C\left[-kr^{+}\beta\sin k\sigma\left(\pi\right) + kr^{-}\beta\sin k\left(\alpha\pi - \sigma\left(\pi\right)\right)\right.\\ \left.+Hr^{+}\cos k\sigma\left(\pi\right) + Hr^{-}\cos k\left(\alpha\pi - \sigma\left(\pi\right)\right)\right]\\ +O\left(\exp\left(\left|\tau\right|\sigma\left(\pi\right)\right)\right).\end{aligned}$$

By use of Riemann-Lebesque Lemma [15], we see that the limit of the left-hand side of the above equality exists as $x \to \infty, k \in \mathbb{R}$. therefore, we get that $C = 0$. So, we have

$$F_0\left(\lambda\right) = 0, \quad \text{for all } \lambda \in \mathbb{C}.$$

Then, from the equality (3.19) we obtain

$$\psi^{'}\left(c_1,\lambda\right)\widetilde{\psi}(c_1,\lambda) - \widetilde{\psi}^{'}\left(c_1,\lambda\right)\psi\left(c_1,\lambda\right) = 0,$$

for all $\lambda \in \mathbb{C}$. Hence,

$$\frac{\psi\left(c_1,\lambda\right)}{\psi'\left(c_1,\lambda\right)} = \frac{\widetilde{\psi}(c_1,\lambda)}{\widetilde{\psi}'\left(c_1,\lambda\right)}. \tag{3.23}$$

Note that $M\left(\lambda\right) = -\dfrac{\psi\left(c_1,\lambda\right)}{\psi'\left(c_1,\lambda\right)}$ is the Weyl function, defined in [35], of the boundary value problem for equation (1.1) on the interval (c_1,π) with the boundary conditions $y^{'}\left(c_1\right) = 0, V\left(y\right) = 0$ and the impulsive conditions (1.4). It has been shown in [35] that the Weyl function uniquely specifies the function $q\left(x\right)$ on $\left(c_1,\pi\right)$, consequently on $\left(c_1,c_2\right)$. Theorem is proved. □

Theorem 3.5. If $\lambda_n = \widetilde{\lambda}_n$ for all $n \geq 0$, $q(x) = \widetilde{q}\left(x\right)$ on $\left(\dfrac{\alpha+\beta}{4}\pi, \dfrac{\alpha+\beta}{2}\pi\right)$, then $q(x) = \widetilde{q}\left(x\right)$ almost every where on $\left(0, \dfrac{\alpha+\beta}{4}\pi\right)$ and $\left(\dfrac{\alpha+\beta}{2}\pi, \pi\right)$, $h = \widetilde{h}, H = \widetilde{H}, a = \widetilde{a}$ and $\rho\left(x\right) = \widetilde{\rho}\left(x\right)$.

Proof . By Lemma 2.5 and the conditions of Theorem 3.4, we have $h = \widetilde{h}, H = \widetilde{H}, a = \widetilde{a}$, $\rho\left(x\right) = \widetilde{\rho}\left(x\right)$ and $q(x) = \widetilde{q}\left(x\right)$ on $\left(c_1,c_2\right)$. Let $\varphi(x,\lambda), \widetilde{\varphi}\left(x,\lambda\right)$ be the solutions of the equations

$$-\varphi^{''}\left(x,\lambda\right) + q(x)\varphi\left(x,\lambda\right) = \lambda\rho\left(x\right)\varphi(x,\lambda), \tag{3.24}$$

$$-\widetilde{\varphi}^{''}\left(x,\lambda\right) + \widetilde{q}(x)\widetilde{\varphi}\left(x,\lambda\right) = \lambda\widetilde{\rho}\left(x\right)\widetilde{\varphi}(x,\lambda), \tag{3.25}$$

with the initial valued conditions, respectively

$$\varphi\left(0,\lambda\right) = 1, \varphi^{'}\left(0,\lambda\right) = h, \tag{3.26}$$

$$\widetilde{\varphi}\left(0,\lambda\right) = 1, \widetilde{\varphi}^{'}\left(0,\lambda\right) = \widetilde{h}, \tag{3.27}$$

and the impulsive conditions (1.4). Multiplying (3.24) by $\widetilde{\varphi}(x,\lambda)$ and (3.25) by $\varphi(x,\lambda)$, we subtract these equations from each other. Then by integrating on $[0,c_2]$ with respect to x, using the initial conditions (3.26), (3.27) and impulsive conditions (1.4), we have

$$H(\lambda)=\int_0^{c_1} Q(x)\,\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)dx=\varphi^{'}(c_1,\lambda)\widetilde{\varphi}(c_1,\lambda)-\widetilde{\varphi}^{'}(c_1,\lambda)\varphi(c_1,\lambda), \tag{3.28}$$

from the hypothesis $q(x)=\widetilde{q}(x)$ on (c_1,c_2). Similarly to proof of Theorem 3.4, we have that $H(\lambda)=0$, for all $\lambda\in\mathbb{C}$. Then, from the equality (3.28) we obtain

$$\varphi^{'}(c_1,\lambda)\widetilde{\varphi}(c_1,\lambda)-\widetilde{\varphi}^{'}(c_1,\lambda)\varphi(c_1,\lambda)=0$$

for all $\lambda\in\mathbb{C}$, so

$$\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)}=\frac{\widetilde{\varphi}(c_1,\lambda)}{\widetilde{\varphi}'(c_1,\lambda)}.$$

The function $M(\lambda)=\dfrac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)}$ is the Weyl function of the impulsive boundary value problem for equation (1.1) on $(0,c_1)$ with boundary conditions $U(y)=0, y^{'}(c_1)=0$ and without jump conditions (1.4) [1]. By [1], the Weyl function uniquely specifes $q(x)$ on $(0,c_1)$. Next, now using Theorem 3.4 we obtain $q(x)=\widetilde{q}(x)$ on (c_2,π). Theorem 3.5 is proved. □

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