Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 287–298
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.23737.2592

Haar wavelet based numerical method for solving proportional delay variant of Dirichlet boundary value problems

Basharat Hussain[a], Ahmad Afroz[a], Abdullah Abdullah[b,*]

[a] *Department of Mathematics, MANUU, Hyderabad, India*

[b] *Department of Mathematics, Zakir Hussain Delhi College, University of Delhi, India*

(Communicated by Haydar Akca)

Abstract

In this paper, We studied an application of the Haar wavelet basis in solving a particular class of delay differential equations. We have extended the Haar wavelet series(HWS) method to develop a numerical technique to solve linear and nonlinear Dirichlet boundary value problems of proportional delay nature. Some problems are presented to test the efficiency of the proposed technique, where a remarkable agreement between approximate and analytic solutions is obtained. The numerical simulation indicates that error drops with the increase in the level of resolution. Also, it is observed that the rate of convergence tends to be 2.

Keywords: Delay differential equation, Dirichlet boundary Condition, Haar wavelet
2020 MSC: 34B05, 34B15, 65L03, 65Q20, 65T60

1 Introduction

Differential equations play a decisive role in the mathematical modeling of plentiful real-life phenomena. They have been proven as an efficient tool to accurately capture the behavior of the models where the system's current state determines the system's future behavior. Such models are called deterministic models. Several authors have investigated different aspects and applications of ordinary and fractional differential equations [13, 18, 28, 38, 39, 40, 41, 42]. In many real-time models, consistency plays an important role and can be improved by incorporating delay terms in its modeling. When delay terms are included in the differential model, a new class of differential equations is known as delay differential equations or functional differential equations. A delay differential equation is an evolutionary system in which the rate of change of a time-dependent process is defined by the current state of the system and a specific past state. The theory of delay differential equations has drawn the interest of several mathematicians and physicists. These equations are frequently used in the simulation of many real-life events, and they have proven to be more accurate in the simulation of natural phenomena. These equations are used in a wide range of disciplines, including industry, biological processes, chemical kinetics, electronics and transportation systems, ship navigational control, infectious diseases, and population dynamics [6, 16, 20, 23, 43].

*Corresponding author
Email addresses: basharathussain_rs@manuu.edu.in (Basharat Hussain), afroz.ahmad@manuu.edu.in (Ahmad Afroz), abd.zhc.du@gmail.com (Abdullah Abdullah)

Received: June 2021 *Accepted:* July 2022

The proportional delay variant of the delay differential equation is generally referred to as the Pantograph equation. The name Pantograph comes from Ockendon and Tayler's first work for collecting current by the Pantograph head of an electric locomotive. Such systems appear in a wide range of applications such as adaptive control, number theory, electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structure, quantum mechanics, cell growth, engineering, economics, and so on [10, 20, 21, 43, 44].

It is difficult to obtain the exact solution for the majority of the delay differential models. As a result, the solutions to such equations have developed great interest among researchers and they have used various numerical approaches to estimate the solutions of these equations. Agarwal and Chow extended the finite-difference method to offer the approximate solution to two-point boundary-value problems with deviating argument [4]. Li and Liu presented a novel numerical approach for solving multi-pantograph delay equations based on the Runge-Kutta scheme [27]. Aibinu et al. applied Sumudu transformation to solve general delay differential equations [5]. Shakeri et al. used the homotopy perturbation method (HPM) to solve certain delay differential equations [35]. To solve generalized pantograph equations, Saadatmandi et al. used a Lagrange multipliers-based variational iteration scheme (VIM) [32]. VIM and HPM yield correct results, but the computational cost is significant due to the use of symbolic integrations. Shakeri and Dehghan have investigated the numerical solution of delay differential equations using a domain decomposition method [36]. Sedaghat et al. [33] adopted a Chebyshev polynomial-based numerical method to the approximate solution of delay differential equations. The aim of the method [33] is to utilize the operational matrix and its derivative to simplify the problem into a series of algebraic equations from which the solution can be derived. Authors in [11, 19, 45, 46] have developed some numerical techniques based on collocation method in the recent past. These techniques are quite effective for investigating proportional delay differential equations.

The authors were inspired by the aforementioned findings to expand the Haar wavelet series approach to investigate the approximate numerical solutions of Dirichlet boundary value problems of proportional delay nature. The method is extremely useful for solving boundary value problems because it considers the boundary condition automatically into account. In this proposed study, we consider the following proportional delay variant of the Dirichlet boundary value problem.

$$\begin{aligned} \rho''(t) = \Omega(t, \rho(t), \rho(\Lambda t), \rho'(t), \rho'(\Lambda t)), \quad t \in [0,1], \\ \text{with } \rho(0) = \zeta_1,\ \rho(1) = \zeta_2, \end{aligned} \tag{1.1}$$

where $\Lambda \in (0,1)$ and ζ_1, ζ_2 are arbitrary constants.

Boundary value problems(BVPs) are now prevalent in all applied sciences. The growing popularity of this branch of differential equations has prompted numerous researchers to investigate various physical models utilizing mathematical tools and computer simulation software. Many problems in science and technology such as the study of the mechanical behavior of the nanomaterial in nanomechanics [22], turbulence modeling [25], modeling of chemical reactors [24], a study of molecular structure in chemical engineering [9], heat transfer model and study of deflection in cables can be formulated mathematically in BVPs for second-order differential equations. Another application of BVPs appears in mathematical modeling design to reflect the real mechanical properties of smart material [30]. The smart material also called intellectual material of the 21st century is a material with shape memory effects. Such materials are widely used in medicine, engineering, aircraft building, construction, etc.

The present paper is organized as follows, for clarity's sake, we provide a few fundamental facts and definitions associated with Haar wavelets and their integrals in section 2. In section 3, we provide a numerical method for approximate solution of equation (1.1). An outline of the method in the form of a flow chart is demonstrated in section 4. In section 5, we solve some numerical problems to examine the computational efficiency of the numerical method. Section 6 is reserved for discussion on error analysis and rate of convergence. Finally, section 7 concludes the work done.

2 Fundamental definitions and Notations

In this section, we summarize some essential definitions and mathematical preliminaries.

Over the past two decades, Haar wavelets gained interest as a fundamental tool in different numerical techniques for solving a wide variety of differential equations. Haar wavelets are first-order Daubechies wavelets that are not only have mathematical expressions but also form an orthonormal basis for $L^2(\mathbb{R})$.

Haar wavelets belong to the family of box functions, these wavelets acquired only three values, that is, $0, 1, -1$. Due to the presence of jump discontinuity, it is impossible to utilize these wavelets directly for solving differential equations. To avoid this shortcoming two approaches are mentioned in [14, 15]. Chen-Hsiao expanded the highest order derivatives involved in the differential equation into the series of Haar basis [15]. Castro et al. regularize the square wave using interpolating spline [14]. The Haar wavelet series method utilized Chen-Hsiao's idea and approximated the highest order derivative with the truncated series of the Haar basis. This method exhibits several advantages, such as it does not involve the computing of symbolic integrals, also conversion of delay differential equation into the ordinary differential equation is not required. This method handles boundary value problems very efficiently since it permits the automatic inclusion of boundary conditions. The method can be implemented easily in software packages (such as MATLAB).

Over the years, this wavelet has been exploited widely to tackle several sciences and engineering problems. A substantial amount of research work has been done using the Haar wavelet. It is an important tool for producing solutions of differential and integral equations, Image quality assessment, detecting false data injection attacks (cyber security), and several other applications, as described in [1, 2, 3, 7, 17, 26, 31, 34, 37] and references therein.

To construct the Haar wavelet system $\{\mathfrak{h}_i(\Theta)\}_{i=1}^{\infty}$ on $[\Gamma_1, \Gamma_2]$ we define two basic functions:

The Haar scaling function (father wavelet):

$$\mathfrak{h}_1(\Theta) = \mathcal{X}_{[\Gamma_1,\Gamma_2)}(\Theta), \tag{2.1}$$

and the mother wavelet:

$$\mathfrak{h}_2(\Theta) = \mathcal{X}_{[\Gamma_1,(\Gamma_1+\Gamma_2)/2)}(\Theta) - \mathcal{X}_{[(\Gamma_1+\Gamma_2)/2,\Gamma_2)}(\Theta), \tag{2.2}$$

where $\mathcal{X}_{[a,b]}(\Theta)$ is characteristics function.

For generating the Haar wavelet series, let j be dilation and k as translation parameter. Then, i-th Haar wavelet is defined as follows:

$$\mathfrak{h}_i(\Theta) = \begin{cases} 1 & \text{for } \Theta \in [\vartheta_1(i), \vartheta_2(i)) \\ -1 & \text{for } \Theta \in [\vartheta_2(i), \vartheta_3(i)) \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

where $\vartheta_1(i) = \Gamma_1 + (\Gamma_2 - \Gamma_1)k/2^j$, $\vartheta_2(i) = \Gamma_1 + (\Gamma_2 - \Gamma_1)(k + 0.5)/2^j$, $\vartheta_3(i) = \Gamma_1 + (\Gamma_2 - \Gamma_1)(k + 1)/2^j$. The index $i = 2^j + k + 1$, $j = 0, 1, \ldots, J$, where J is the maximum level of wavelet and $k = 0, 1, \ldots, 2^j - 1$.

To apply the Haar wavelet method following integrals are required,

$$\mathcal{P}_i(\Theta) = \int_{\Gamma_1}^{\Theta} \mathfrak{h}_i(\Theta) d\Theta, \mathcal{Q}_i(\Theta) = \int_{\Gamma_1}^{\Theta} \mathcal{P}_i(\Theta) d\Theta.$$

Analytic integrations of 2.3 yields,

$$\mathcal{P}_i(\Theta) = \begin{cases} \Theta - \vartheta_1(i), & \text{for } \Theta \in [\vartheta_1(i), \vartheta_2(i)) \\ \vartheta_3(i) - \Theta, & \text{for } \Theta \in [\vartheta_2(i), \vartheta_3(i)) \\ 0, & \text{otherwise,} \end{cases} \tag{2.4}$$

and

$$\mathcal{Q}_i(\Theta) = \begin{cases} \frac{1}{2}(\Theta - \vartheta_1(i))^2, & \text{for } \Theta \in [\vartheta_1(i), \vartheta_2(i)) \\ \frac{(\Gamma_2 - \Gamma_1)^2}{2^{2j+2}} - \frac{1}{2}(\vartheta_3(i) - \Theta)^2, & \text{for } \Theta \in [\vartheta_2(i), \vartheta_3(i)) \\ \frac{(\Gamma_2 - \Gamma_1)^2}{2^{2j+2}}, & \text{for } \Theta \in [\vartheta_3(i), 1] \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

3 Construction of Haar wavelet series method

Haar wavelets series method for the numerical solution of (1.1) constructed as follows:

$$\begin{aligned} \rho''(t) = \Omega(t, \rho(t), \rho(\Lambda t), \rho'(t), \rho'(\Lambda t)), \quad t \in [0,1], \\ \text{with } \rho(0) = \zeta_1,\ \rho(1) = \zeta_2, \end{aligned} \tag{3.1}$$

where $\Lambda \in (0,1)$ and ζ_1, ζ_2 are arbitrary constants. To apply the Haar wavelet series method, firstly, we expand the $\rho''(t)$ in terms of truncated Haar series as:

$$\rho''(t) = \sum_{i=0}^{2^{J+1}} a_i \mathfrak{h}_i(t). \tag{3.2}$$

Integrate Eq. (3.2), from 0 to t, we get

$$\rho'(t) = \sum_{i=0}^{2^{J+1}} a_i \mathcal{P}_i(t) + \rho'(0). \tag{3.3}$$

Further integration yields

$$\rho(t) = \sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(t) + \rho'(0)t + \rho(0). \tag{3.4}$$

Quantity $\rho'(0)$ in Eq. (3.3) and (3.4) are yet to determined, for that integrate Eq. (3.3) from t to 1, we have

$$-\rho(t) = \sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(1) - \sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(t) + \rho'(0)(1-t) - \rho(1). \tag{3.5}$$

Now, from Eq. (3.4) and (3.5), we get $\rho'(0) = -\sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(1) - \rho(0) + \rho(1)$. Utilizing $\rho'(0)$, $\rho(0) = \zeta_1$ and $\rho(1) = \zeta_2$ in Eq. (3.4), we obtain the following

$$\rho(t) = \sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(t) + \Big(-\sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(1) - \zeta_1 + \zeta_2\Big)t + \zeta_1, \tag{3.6}$$

and

$$\rho(\Lambda t) = \sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(\Lambda t) + \Big(-\sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(1) - \zeta_1 + \zeta_2\Big)(\Lambda t) + \zeta_1. \tag{3.7}$$

Similarly, putting $\rho'(0)$, $\rho(0) = \zeta_1$ and $\rho(1) = \zeta_2$ in Eq. (3.3), we get

$$\rho'(t) = \sum_{i=0}^{2^{J+1}} a_i \mathcal{P}_i(t) - \sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(1) - \zeta_1 + \zeta_2, \tag{3.8}$$

and

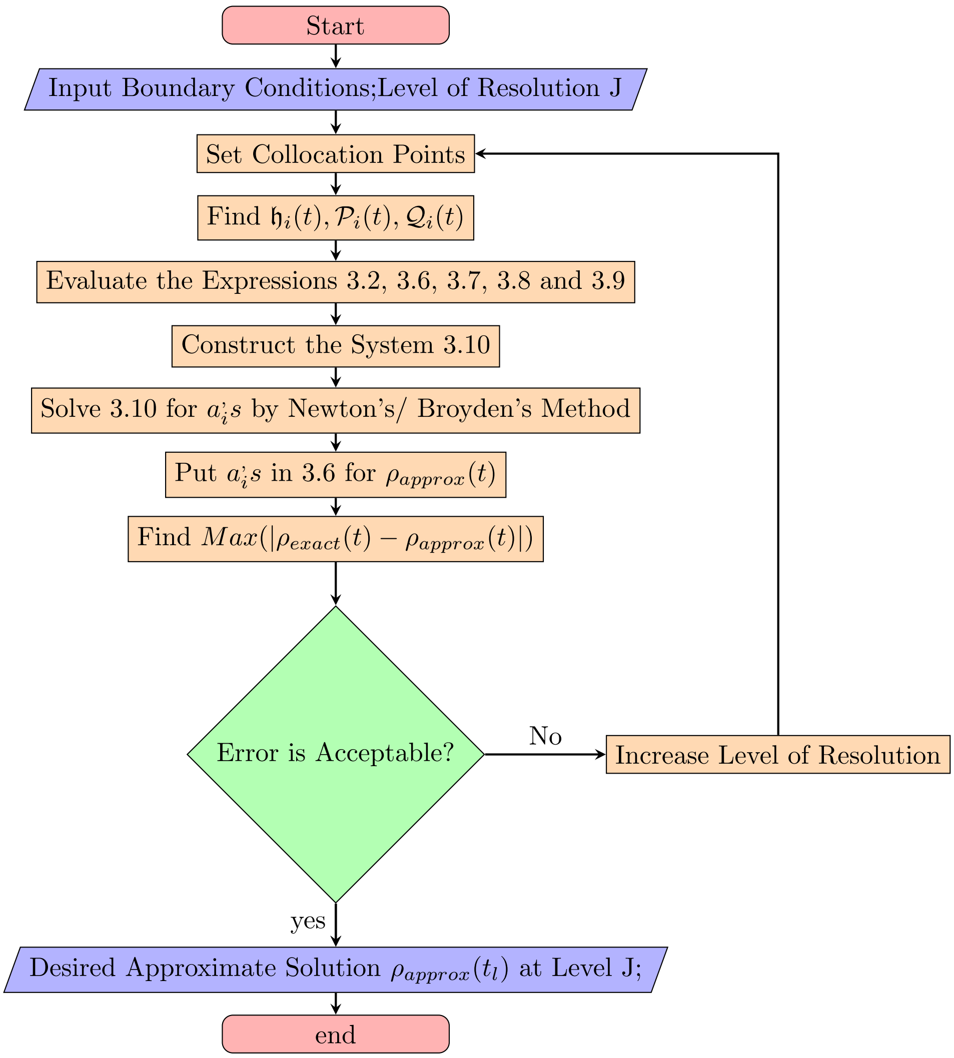
$$\rho'(\Lambda t) = \sum_{i=0}^{2^{J+1}} a_i \mathcal{P}_i(\Lambda t) - \sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(1) - \zeta_1 + \zeta_2. \tag{3.9}$$

Substituting Eq. (3.2) and Eqs. (3.6) to (3.9), in Eq. (3.1), we have

$$
\begin{aligned}
\sum_{i=0}^{2^{J+1}} a_i \mathfrak{h}_i(t) = \Omega\Bigg(t, \Big[\sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(t) + \Big(-\sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(1) - \zeta_1 + \zeta_2\Big)t + \zeta_1\Big], \\
\Big[\sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(\Lambda t) + \Big(-\sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(1) - \zeta_1 + \zeta_2\Big)(\Lambda t) + \zeta_1\Big], \\
\Big[\sum_{i=0}^{2^{J+1}} a_i \mathcal{P}_i(t) - \sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(1) - \zeta_1 + \zeta_2\Big], \Big[\sum_{i=0}^{2^{J+1}} a_i \mathcal{P}_i(\Lambda t) - \sum_{i=0}^{2^{J+1}} a_i \mathcal{Q}_i(1) - \zeta_1 + \zeta_2\Big]\Bigg).
\end{aligned} \tag{3.10}
$$

Discretization of the above Eq. using collocation procedure leads to the algebraic system for $a_i^{,}s$. After determining $a_i^{,}s$ using Newton's iterative method or any other suitable method, we get the approximate solution from Eq. (3.6).

4 Flow chart



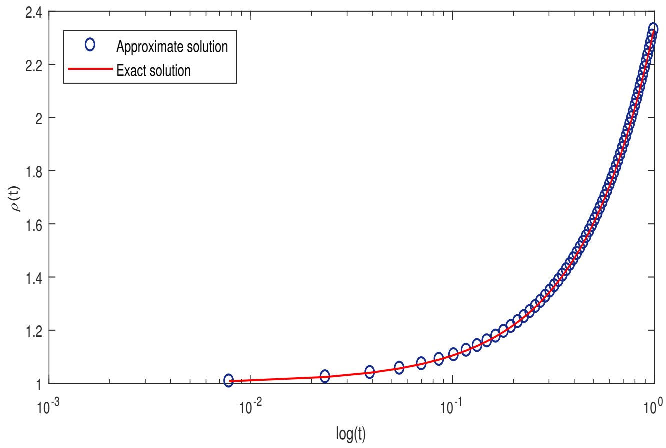


Figure 1: Plot for Problem 1 at J=5

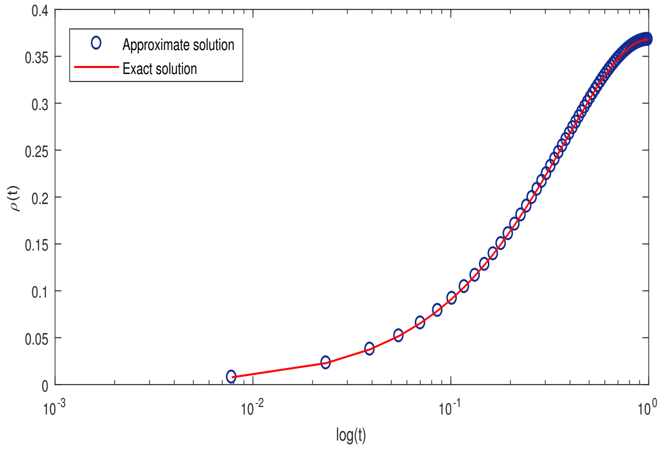


Figure 2: Plot for Problem 2 at J=5

5 Illustrative examples:

Problem 1. Consider the boundary value problem:

$$\begin{gathered}\rho''(t) - 1 - 2(1 + t^2/8)\cos(t/2) + 2\cos(t/2)\rho(t/2) = 0,\ t \in [0,1] \\ \text{with Dirichlet boundary condition } \rho(0) = 1,\ \rho(1) = \frac{3}{2} + \sin(1).\end{gathered} \tag{5.1}$$

The exact solution of this problem is $\rho(t) = \frac{t^2}{2} + \sin(t) + 1$.

Problem 2. Consider the boundary value problem:

$$\begin{gathered}\rho''(t) + 2e^{-t} - \rho(t)/2 - e^{-t/2}\rho(t/2) = 0, \quad t \in [0,1] \\ \text{with Dirichlet boundary condition } \ \rho(0) = 0,\ \rho(1) = e^{-1}.\end{gathered} \tag{5.2}$$

The exact solution of this problem is $\rho(t) = te^{-t}$.

Problem 3. Let us assume the boundary value problem:

$$\begin{gathered}\rho''(t) - 4e^{-t/2}\sin(t/2)\rho(t/2) = 0, \quad t \in [0,1], \\ \text{along with Dirichlet boundary condition } \ \rho(0) = 1,\ \rho(1) = e^{-1}\cos(1).\end{gathered} \tag{5.3}$$

The exact solution of this problem is $\rho(t) = e^{-t}\cos(t)$.

Problem 4. Assume the nonlinear boundary value problem as:

$$\begin{gathered}\rho''(t) - ([\rho(t)]^2 + [\rho(t)]^3)\rho(t/2) = 0, \quad t \in [0,1], \\ \text{with Dirichlet boundary condition } \ \rho(0) = 1,\ \rho(1) = 1/2.\end{gathered} \tag{5.4}$$

The exact solution of the above nonlinear boundary value problem is $\rho(t) = \frac{1}{t+1}$.

Problem 5. Let us consider the following boundary value problem:

$$\begin{gathered}\rho''(t) - \rho'(t)\rho(t/2) + 8t^2\rho(t/2) + \ominus(t) = 0, \quad t \in [0,1], \\ \text{subject to Dirichlet boundary condition } \ \rho(0) = 1,\ \rho(1) = 3,\end{gathered} \tag{5.5}$$

and the $\ominus(t)$ is chosen such that the exact solution of the above nonlinear boundary value problem is $\rho(t) = 1 + t + t^3$.

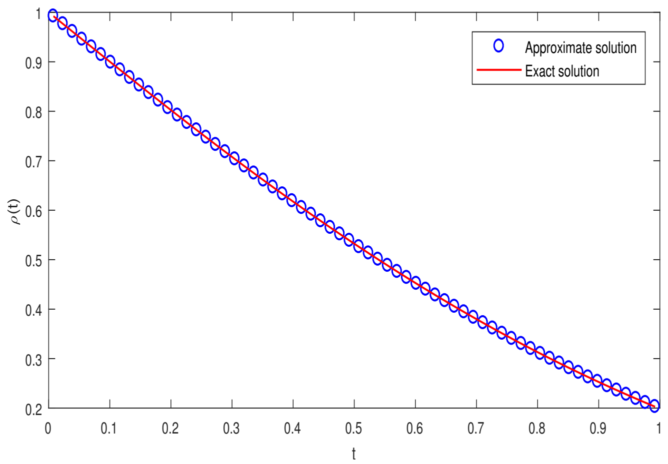


Figure 3: Solution curves for Problem 3 at J=5

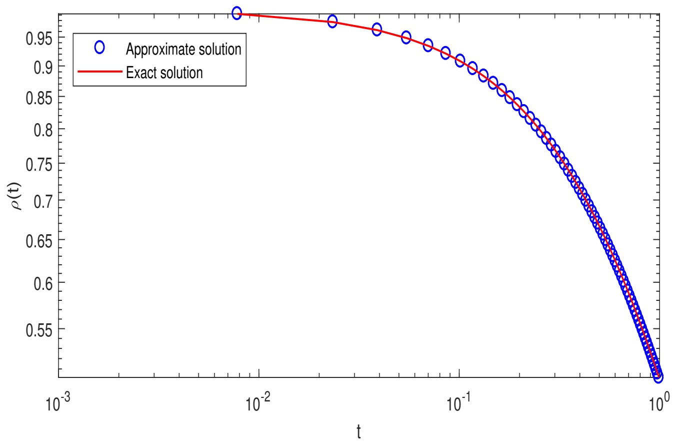


Figure 4: loglog Plot for Problem 4 at J=5

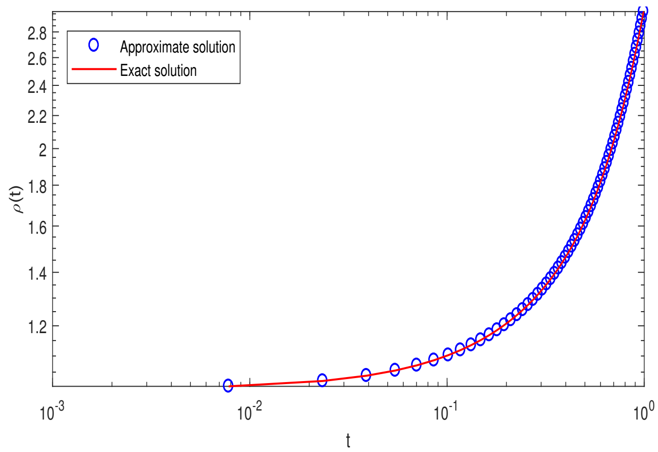


Figure 5: loglog Plot for Problem 5 at J=5

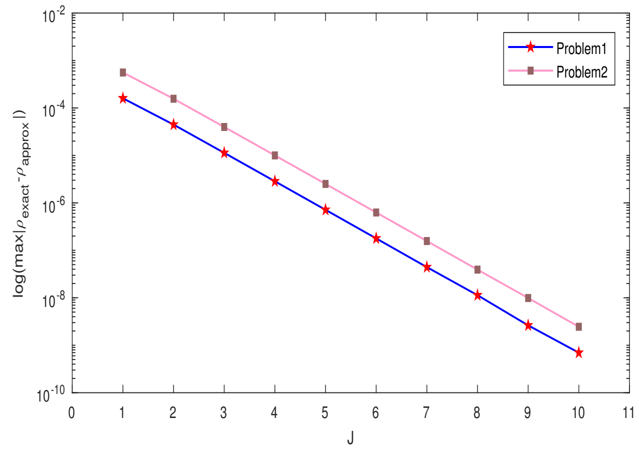


Figure 6: Maximum absolute errors of Problems 1-2 for different choices of resolution

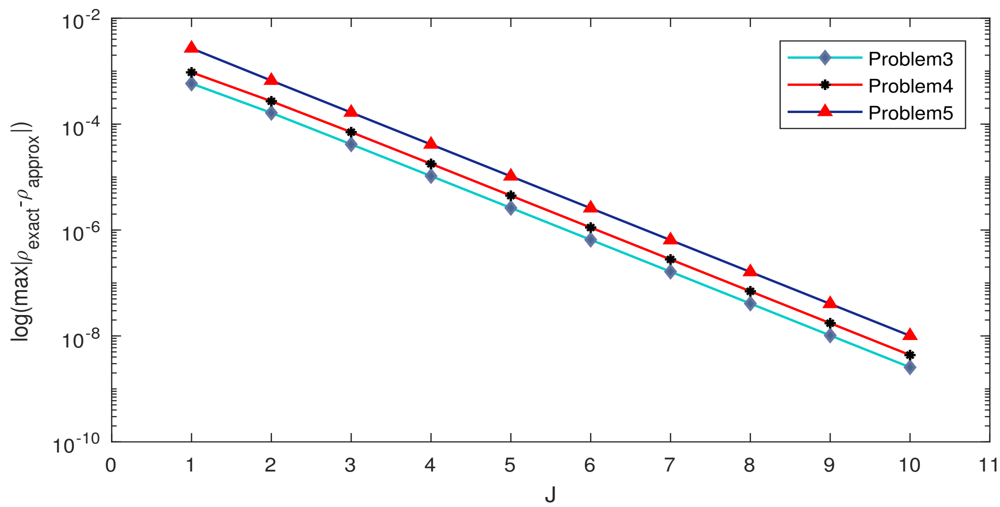


Figure 7: Maximum absolute errors of Problem 3-5 for different choices of resolution

Table 1: Maximum Absolute Error for Problems 1 to 5.

J	**Problem 1**	**Problem 2**	**Problem 3**	**Problem 4**	**Problem 5**
1	$1.5997e-04$	$5.5835e-04$	$5.8345e-04$	$9.4871e-04$	$2.6907e-03$
2	$4.4574e-05$	$1.5644e-04$	$1.6306e-04$	$2.7026e-04$	$6.6283e-04$
3	$1.1324e-05$	$3.9757e-05$	$4.1522e-05$	$7.0508e-05$	$1.6553e-04$
4	$2.8500e-06$	$1.0014e-05$	$1.0454e-05$	$1.7787e-05$	$4.1311e-05$
5	$7.1382e-07$	$2.5082e-06$	$2.6172e-06$	$4.4582e-06$	$1.0331e-05$
6	$1.7831e-07$	$6.2735e-07$	$6.5449e-07$	$1.1152e-06$	$2.5828e-06$
7	$4.4275e-08$	$1.5692e-07$	$1.6362e-07$	$2.7884e-07$	$6.4569e-07$
8	$1.1342e-08$	$3.9226e-08$	$4.0890e-08$	$6.9712e-08$	$1.6142e-07$
9	$2.6235e-09$	$9.8905e-09$	$1.0218e-08$	$1.7427e-08$	$4.0355e-08$
10	$6.9708e-10$	$2.4508e-09$	$2.5589e-09$	$4.3559e-09$	$1.0088e-08$

Table 2: Rate of convergence R_c for Problems 1 to 5.

J	**Problem 1**	**Problem 2**	**Problem 3**	**Problem 4**	**Problem 5**
1	−−−−−−	−−−−−−	−−−−−−	−−−−	−−−−−
2	1.8435	1.8356	1.8392	1.8116	2.0213
3	1.9768	1.9763	1.9735	1.9385	2.0015
4	1.9903	1.9892	1.9898	1.9870	2.0025
5	1.9973	1.9973	1.9980	1.9963	1.9995
6	2.0012	1.9993	1.9996	1.9992	2.0000
7	2.0098	1.9992	2.0000	1.9998	2.0000
8	1.9648	2.0001	2.0005	2.0000	2.0000
9	2.1121	1.9877	2.0006	2.0001	2.0000
10	1.9121	2.0128	1.9975	2.0003	2.0001

Table 3: Comparison of Errors.

Problem	**HWS**	**Bica[12]**
1	$1.7831e-07$ at $J=6$	$6.9097e-07$ at $h=\frac{\pi}{400}$
1	$6.9708e-10$ at $J=10$	$6.9100e-09$ at $h=\frac{\pi}{4000}$
2	$9.8905e-09$ at $J=9$	$1.2770e-08$ at $h=\frac{1}{1000}$

Table 4: Comparison between approximate and analytic solution Problem 1 at J=6.

t	ρ_{exact}	ρ_{approx}	$\lvert\rho_{exact}-\rho_{approx}\rvert$
0.1	1.1108	1.1108	$4.9295e-08$
0.2	1.2270	1.2270	$9.3465e-08$
0.3	1.3415	1.3415	$1.2867e-07$
0.4	1.4725	1.4725	$1.5786e-07$
0.5	1.6098	1.6098	$1.7498e-07$
0.6	1.7524	1.7524	$1.7727e-07$
0.7	1.8995	1.8995	$1.6214e-07$
0.8	2.0385	2.0385	$1.3054e-07$
0.9	2.1919	2.1919	$7.5335e-08$

Table 5: Comparison between approximate and analytic solution Problem 2 at J=6.

t	ρ_{exact}	ρ_{approx}	$\lvert\rho_{exact} - \rho_{approx}\rvert$
0.1	$9.4912e-02$	$9.4912e-02$	$2.8670e-07$
0.2	$1.6832e-01$	$1.6832e-01$	$4.7186e-07$
0.3	$2.2265e-01$	$2.2265e-01$	$5.7505e-07$
0.4	$2.6907e-01$	$2.6907e-01$	$6.2395e-07$
0.5	$3.0444e-01$	$3.0444e-01$	$6.1674e-07$
0.6	$3.3048e-01$	$3.3048e-01$	$5.6142e-07$
0.7	$3.4864e-01$	$3.4864e-01$	$4.6471e-07$
0.8	$3.5953e-01$	$3.5953e-01$	$3.4350e-07$
0.9	$3.6601e-01$	$3.6601e-01$	$1.8183e-07$

Table 6: Comparison between approximate and analytic solution Problem 3 at J=6.

t	ρ_{exact}	ρ_{approx}	$\lvert\rho_{exact} - \rho_{approx}\rvert$
0.1	$8.9490e-01$	$8.9490e-01$	$3.0232e-07$
0.2	$7.9563e-01$	$7.9563e-01$	$4.9807e-07$
0.3	$7.0701e-01$	$7.0701e-01$	$6.0542e-07$
0.4	$6.1535e-01$	$6.1535e-01$	$6.5268e-07$
0.5	$5.2907e-01$	$5.2907e-01$	$6.3862e-07$
0.6	$4.4879e-01$	$4.4879e-01$	$5.7344e-07$
0.7	$3.7491e-01$	$3.7491e-01$	$4.6661e-07$
0.8	$3.1255e-01$	$3.1255e-01$	$3.3853e-07$
0.9	$2.5139e-01$	$2.5139e-01$	$1.7510e-07$

Table 7: Comparison between approximate and analytic solution Problem 4 at J=6.

t	ρ_{exact}	ρ_{approx}	$\lvert\rho_{exact} - \rho_{approx}\rvert$
0.1	$9.0459e-01$	$9.0459e-01$	$7.0007e-07$
0.2	$8.2848e-01$	$8.2848e-01$	$1.0092e-06$
0.3	$7.6877e-01$	$7.6877e-01$	$1.1088e-06$
0.4	$7.1309e-01$	$7.1309e-01$	$1.0937e-06$
0.5	$6.6494e-01$	$6.6493e-01$	$9.9756e-07$
0.6	$6.2287e-01$	$6.2287e-01$	$8.4864e-07$
0.7	$5.8581e-01$	$5.8581e-01$	$6.6355e-07$
0.8	$5.5531e-01$	$5.5531e-01$	$4.6913e-07$
0.9	$5.2567e-01$	$5.2567e-01$	$2.3844e-07$

Table 8: Comparison between approximate and analytic solution Problem 5 at J=6.

t	ρ_{exact}	ρ_{approx}	$\lvert\rho_{exact} - \rho_{approx}\rvert$
0.1	1.1066	1.1066	$7.4140e-07$
0.2	1.2159	1.2159	$1.3630e-06$
0.3	1.3280	1.3280	$1.8446e-06$
0.4	1.4675	1.4675	$2.2490e-06$
0.5	1.6319	1.6319	$2.5062e-06$
0.6	1.8274	1.8274	$2.5806e-06$
0.7	2.0605	2.0605	$2.4239e-06$
0.8	2.3143	2.3143	$2.0191e-06$
0.9	2.6371	2.6371	$1.2202e-06$

6 Results and Discussions:

The above problems are solved using the Haar wavelet series method. Maximum absolute error and experimental convergence rate are computed for each problem at different resolutions (J). These results are presented in Tables 1-8. Table 1 demonstrate that error decreases with increase in resolution(J). In Table 2, we have calculated the rate of convergence by using the formula

$$R_c = \frac{\log(\frac{E_{J-1}}{E_J})}{\log 2},$$

where E_J is the maximum absolute errors at resolution J. Also, from Table 2 one can observe that the experimental convergence rates in each problem tends to be 2, as described, in [29]. The approximate and analytical solution curves of each problem are plotted in figures 1-5. We observed that both curves coincide and also, a comparison drawn in Table 3 shows the method is more accurate and efficient.

7 Conclusion:

In this work, we have successfully employed the Haar wavelet series method to obtain the approximate solution of Dirichlet Boundary value problems with proportional delay. The accuracy and convergence rate of the method has been validated by solving sample problems of both linear and nonlinear nature. MATLAB package is utilized to carry out computer simulations. Each obtained solution is compared with the analytical solution. Maximum absolute errors decrease with increase in levels of resolution($O(10^{-4}) - O(10^{-9})$). This fact is shown in Fig.6 and Fig.7. Also, the calculated rate of convergence is in good agreement with the theoretical result proved in [29]. Overall, the method is computationally attractive, simple and convenient for dealing with Dirichlet boundary value problems of proportional delay nature.

Data Availability: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Code availability: Not applicable

References

[1] A. Abdullah and M. Rafiq, *A new numerical scheme based on haar wavelets for the numerical solution of the chen–lee–liu equation*, Optik **226** (2021), 165847.

[2] B.J. Adigun, A.G. Buchan, A. Adam, S. Dargaville, M.A. Goffin and C. C. Pain, *A Haar wavelet method for angularly discretising the Boltzmann transport equation*, Prog. Nucl. Energy **108** (2018), 295–309.

[3] K. Ahmad and Abdullah, *Wavelet Packets and Their Statistical Applications*, Springer, 2018.

[4] R.P. Agarwal and Y.M. Chow, *Finite-difference methods for boundary-value problems of differential equations with deviating arguments*, Comp. Math. Appl **12** (1986), no. 11, 1143–1153.

[5] M.O. Aibinu, S.C. Thakur and S. Moyo, *Solving delay differential equations via Sumudu transform*, Int. J. Nonlinear Anal. Appl. **13** (2022), no. 2, 563–575.

[6] W.G. Aiello, H.I. Freedman and J. Wu, *Analysis of a model representing stage-structured population growth with state-depended time delay*, SIAM J. Appl. Math. **52** (1992), no. 3, 855–869.

[7] B.M.R. Amin, S. Taghizadeh, S. Maric, M. Hossain and R. Abbas, *Smart grid security enhancement by using belief propagation*, IEEE Syst. J. **15** (2020), no. 2, 2046–2057.

[8] C.T. Baker, C.A. Paul and D.R. Willé, *Issues in the numerical solution of evolutionary delay differential equations*, Adv. Comput. Math., **3** (1995), no. 1, 171–196.

[9] D. Baleanu, S. Etemad, H. Mohammadi, and S. Rezapour, *A novel modeling of boundary value problems on the glucose graph*, Commun. Nonlinear Sci. Numer. Simul. **100** (2021), 105844.

[10] A. Bellen and M. Zennaro, *Numerical Methods for Delay Differential Equations*, Oxford university Press, 2013.

[11] A.H. Bhrawy, L.M. Assas, E. Tohidi and M.A. Alghamdi, *A Legendre-Gauss collocation method for neutral functional-differential equations with proportional delays*, Adv. Differ. Equ. **63** (2013).

[12] A.M. Bica, M. Curila and S. Curila, *About a numerical method of successive interpolations for two-point boundary value problems with deviating argument*, Appl. Math. Comput. **217** (2011), no. 19, 7772–7789.

[13] J.C. Butcher, *Numerical methods for ordinary differential equations*, John Wiley & Sons, 2016.

[14] L.M. Castro, A.J.M. Ferreira, S. Bertoluzza, R.C. Batra and J.N. Reddy, *A wavelet collocation method for the static analysis of sandwich plates using a layerwise theory*, Compos. Struct. **92** (2010), no. 8, 1786–1792.

[15] C.F. Chen and C.H. Hsiao, *Haar wavelet method for solving lumped and distributed-parameter systems*, IEE Proc. Contr. Theor. Appl. **144** (1997), 87–94.

[16] C. Corduneanu, Y. Li and M. Mahdavi, *Functional Differential Equations: Advances and Applications*, John Wiley & Sons, 2016.

[17] Q. Dai, Q. Cao and Y.Chen, *Frequency analysis of rotating truncated conical shells using the Haar wavelet method*, Appl. Math. Model. **57** (2018), 603–613.

[18] C. Dineshkumar, K.S. Nisar, R. Udhayakumar and V. Vijayakumar, *A discussion on approximate controllability of Sobolev-type Hilfer neutral fractional stochastic differential inclusions*, Asian J. Control, In Press, Doi: https://doi.org/10.1002/asjc.2650.

[19] E.H. Doha, A.H. Bhrawy, D. Baleanu and R.M. Hafez, *A new Jacobi rational-Gauss collocation method for numerical solution of generalized pantograph equations*, Appl. Numer. Math., **77** (2014), 43–54.

[20] R.D. Driver, *Ordinary and Delay Differential Equations*, Springer, 1977.

[21] L. Fox, D.F. Mayers, J.R. Ockendon and A.B. Tayler, *On a functional differential equation*, IMA J. Appl. Math., **8** (1971), no. 3, 271–307.

[22] M. A.Grekov, *Mathematical models of boundary value problems in nanomechanics*, In AIP Conference Proceedings **1863** (2017), no. 1, 170006.

[23] J.L. Henderson, *Boundary Value Problems For Functional Differential Equations*, World Scientific, 1995.

[24] V. Hlaváček, M. Marek and M. Kubíček, *Modelling of chemical reactors—X Multiple solutions of enthalpy and mass balances for a catalytic reaction within a porous catalyst particle*, Chem. Engin. Sci. **23** (1968), no. 9, 1083–1097.

[25] S. Langer and R.C. Swanson, *On Boundary-Value Problems for RANS Equations and Two-Equation Turbulence Models*, J. Sci. Comput. **85** (2020), no. 20.

[26] Ü. Lepik and H. Hein, *Haar Wavelets: With Applications*, Springer, 2014.

[27] D. Li and M. Liu, *Runge-Kutta methods for the multi-pantograph delay equation*, Appl. Math. Comput. **163** (2005), no. 1, 383–395.

[28] C.P. Li and F.R. Zhang, *A survey on the stability of fractional differential equations*, Eur. Phys. J. Special Topics **93** (2011), no. , 27–47.

[29] J. Majak, B.S. Shvartsman, M. Kirs, M. Pohlak and H. Herranen, *Convergence theorem for the haar wavelet based discretization method*, Compos. Struct. **126** (2015), 227–232.

[30] G. V. Malinin, V. G. Malinin, N. A. Malinina and O. A. Malukhina, *Mathematical model, the solution of boundary value problems for alloys with shape memory effect by analytical, numerical-analytical and numerical methods*, IOP Conf. Ser.: Materials Sci. Engin. **441** (2018), no. 1, 012031.

[31] R. Reisenhofer, S. Bosse, G. Kutyniok and T. Wiegand, *A Haar wavelet-based perceptual similarity index for image quality assessment*, Signal Process. Image Commun. **61** (2018), 33–43.

[32] A. Saadatmandi and M. Dehghan, *Variational iteration method for solving a generalized pantograph equation*, Comp. Math. Appl., **58** (2009), no. 11-12, 2190–2196.

[33] S. Sedaghat, Y. Ordokhani and M. Dehghan, *Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials*, Commun. Nonlinear Sci. Numer. Simul. **17** (2012), no. 12, 4815–4830.

[34] F.A. Shah, R. Abass and L. Debnath, *Numerical Solution of Fractional Differential Equations Using Haar Wavelet Operational Matrix Method*, Int. J. Appl. Comput. Math. **3** (2017), 2423—2445.

[35] F. Shakeri and M. Dehghan, *Solution of delay differential equations via a homotopy perturbation method*, Math. Comput. Model., **48** (2008), no. 3-4, 486–498.

[36] F. Shakeri and M. Dehghan, *Application of the decomposition method of Adomian for solving the pantograph equation of order m*, Z. Naturforschung A, **65** (2010), no. 5, 453–460.

[37] R. Singh, H. Garg and V. Guleria, *Haar wavelet collocation method for Lane-Emden equations with Dirichlet, Neumann and Neumann-Robin boundary conditions*, J. Comput. Appl. Math. **346** (2019), 150–161.

[38] E. Tadmor, *A review of numerical methods for nonlinear partial differential equations*, Bull. Amer. Math. Soc. **49** (2012), no. 4, 507–554.

[39] V.Vijayakumar and R. Murugesu, *Controllability for a class of second-order evolution differential inclusions without compactness*,Applicable Analysis, **98** (2019), no. 7, 1367–1385.

[40] V. Vijayakumar, *Approximate controllability for a class of second-order stochastic evolution inclusions of Clarke's subdifferential type*, Results Math. **73** (2018), no. 1, 1–23.

[41] V. Vijayakumar and R. Murugesu, *Controllability for a class of second-order evolution differential inclusions without compactness*, Appl. Anal. **98** (2019), no. 7, 1367-1385.

[42] V. Vijayakumar, C. Ravichandran, K.S. Nisar and K.D. Kucche, *New discussion on approximate controllability results for fractional Sobolev type Volterra-Fredholm integro-differential systems of order* $1 < r < 2$, Numer. Meth. Partial Differ. Equ. In Press, Doi: https://doi.org/10.1002/num.22772.

[43] C. Xu, P. Li and S. Yuan, *New findings on exponential convergence of a Nicholson's blowflies model with proportional delay*, Adv. Differ. Equ. **2019** (2019), no. 358.

[44] S. Xueli, Z. Pan, X. Zhiwei and P. Jigen, *Global asymptotic stability of CNNs with impulses and multi-proportional delays*, Math. Methods Appl. Sci. **39** (2016), no. 4, 722–733.

[45] S. Yalçinbaş, M. Aynigül and M. Sezer, *A collocation method using Hermite polynomials for approximate solution of pantograph equations*, J. Frankl. Inst. **348** (2011), no. 6, 1128–1139.

[46] Ş. Yüzbaşi, N. Şahin and M. Sezer, *A Bessel collocation method for numerical solution of generalized pantograph equations*, Numer. Meth. Partial Differ. Equ. **28** (2011), no. 4, 1105–1123.