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PERTURBATIONS OF HIGHER JORDAN DERIVATIONS IN BANACH TERNARY ALGEBRAS :AN ALTERNATIVE FIXED POINT APPROACH

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ABSTRACT. Using fixed point methods, we investigate approximately higher Jordan ternary derivations in Banach ternaty algebras via the functional equation

$$D_f(x_1,..,x_m) := \sum_{k=2}^{m}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{m-k+1}=i_{m-k}+1}^{m})f(\sum_{i=1,i\neq i_1,\ldots,i_{m-k+1}}^{m}x_i-\sum_{r=1}^{m-k+1}x_{i_r})$$
$$+f(\sum_{i=1}^{m}x_i)-2^{m-1}f(x_1)=0$$

where $m\geq 2$ is an integer number.

1. Introduction and preliminaries

A ternary algebra $\mathcal{A}$ is a real or complex linear space, endowed with a linear mapping, the so-called a ternary product $(x,y,z)\to[x,y,z]$ of $\mathcal{A}\times\mathcal{A}\times\mathcal{A}$ into $\mathcal{A}$ such that $[[x,y,z],w,v]=[x,[y,z,w],v]=[x,y,[z,w,v]]$ for all $x,y,z,w,v\in\mathcal{A}$.

If $(\mathcal{A},\odot)$ is a usual (binary) algebra, then $[x,y,z]:=(x\odot y)\odot z$ makes $\mathcal{A}$ into a ternary algebra. Hence the ternary algebra is a natural generalization of the binary case. In particular, if a ternary algebra $(\mathcal{A},[\ ])$ has a unit, i.e., an element $e\in\mathcal{A}$ such that $x=[x,e,e]=[e,e,x]$ for all $x\in\mathcal{A}$, then $\mathcal{A}$ with the binary product $x\odot y:=[x,e,y]$, is a usual algebra. By a normed ternary algebra we mean a ternary algebra with a norm $\|.\|$ such that $\|[x,y,z]\|\leq\|x\|\|y\|\|z\|$ for all $x,y,z\in\mathcal{A}$. A Banach ternary algebra is a normed ternary algebra such that the normed linear space with norm $\|.\|$ is complete.

Ternary algebraic operations were considered in the XIX-th century by several mathematicians such that as A. Cayley [6] who first introduced in 1840 the notion of "cubic matrices" and a generalization of the determinant, called the "hyperdeterminant", then were found again and generalized by M. Kapranov, I. M. Gelfand and A. Zelevinskii in 1990 [19].

As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called "Nambu mechanics" which has been proposed by Y. Nambu [23] in 1973, is based on such structures. There are also some

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applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the "anyons"), supersymmetric theories, etc, (cf.[1, 20, 34]).

Throughout this paper, we assume that $\mathcal{A}$ and $\mathcal{B}$ are real or complex ternary algebras. For the sake of convenience, we use the same symbol $[\ ]$ (resp. $\|.\|$) in order to represent the ternary products (resp. norms) on ternary algebras $\mathcal{A}$ and $\mathcal{B}$.

A linear mapping $h : \mathcal{A} \to \mathcal{B}$ is said to be a ternary Jordan homomorphism if $h([x, x, x]) = [h(x), h(x), h(x)]$ holds for all $x \in \mathcal{A}$. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a ternary Jordan derivation if $d([x, x, x]) = [d(x), x, x] + [x, d(x), x] + [x, x, d(x)]$ holds for all $x \in \mathcal{A}$ (see [4]).

Let $\mathbb{N}$ be the set of natural numbers. For $m \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$, a sequence $H = \{h_0, h_1, ..., h_m\}$ (resp. $H = \{h_0, h_1, ..., h_n, ...\}$) of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a higher Jordan ternary derivation of rank m (resp. infinite rank) from $\mathcal{A}$ into $\mathcal{B}$ if

$$h_n([x, x, x]) = \sum_{i+j+k=n} [h_i(x), h_j(x), h_k(x)]$$

holds for each $n \in \{0, 1, ..., m\}$ (resp. $n \in \mathbb{N}_0$) and all $x \in \mathcal{A}$. The higher Jordan ternary derivation H from $\mathcal{A}$ into $\mathcal{B}$ is said to be onto if $h_0 : \mathcal{A} \to \mathcal{B}$ is onto. The higher Jordan ternary derivation H on $\mathcal{A}$ is called be strong if h_0 is an identity mapping on $\mathcal{A}$. Of course, a higher Jordan ternary derivation of rank 0 from $\mathcal{A}$ into $\mathcal{B}$ (resp. a strong higher Jordan ternary derivation of rank 1 on $\mathcal{A}$) is a Jordan ternary homomorphism (resp. a Jordan ternary derivation). So a higher Jordan ternary derivation is a generalization of both a Jordan ternary homomorphism and a Jordan ternary derivation.

Here let us consider an approximately higher Jordan ternary derivation which is not an exactly higher Jordan ternary derivation in Banach ternary algebras.

The following remark is a slight modification of an example which is due to B. E. Johnson [16] (see also [21, Example 1.1]).

Remark 1.1. Let X be a compact Hausdorff space and let $(\mathcal{A}, [\ ])$ be the Banach ternary algebra of complex-valued continuous functions on X under the usual addition of complex-valued continuous functions, the ternary operation $[\rho_1, \rho_2, \rho_3] = \rho_1 * \rho_2 * \rho_3$ and the supremum norm $\|.\|_\infty$, where $*$ denotes the usual multiplication of complex-valued continuous functions. Assume that $\tau : \mathcal{A} \to \mathcal{A}$ is a continuous ternary homomorphism. We define $f : \mathcal{A} \to \mathcal{A}$ by

$$f(x)(a) = \begin{cases} \tau(x)(a) \log|\tau(x)(a)| & if\ \tau(x)(a) \neq 0, \\ 0 & if\ \tau(x)(a) = 0 \end{cases}$$

for all $x \in \mathcal{A}$ and all $a \in X$. It is easy to see that $f([x, x, x]) = [f(x), \tau(x), \tau(x)] + [\tau(x), f(x), \tau(x)] + [\tau(x), \tau(x), f(x)]$ for all $x \in \mathcal{A}$. Let $h_0 = \tau$, $h_n = 0, 1 \leq n \leq m-1$ and $h_m = f$. Then we see that the sequence $H = [h_0, h_1, ..., h_m]$ satisfies the relation

$$h_n([x, x, x]) = \sum_{i+j+k=n} [h_i(x), h_j(x), h_k(x)]$$

for all $x \in \mathcal{A}$. It is proved in [27, Remark 1.1] that for all $u, v, w \in \mathbb{C}\backslash\{0\}$ with $u + v + w \neq 0$, where $\mathbb{C}$ is a complex field,

$$|(u + v + w)log|u + v + w| - ulog|u| - vlog|v| - wlog|w|| \leq 2(|u| + |v| + |w|).$$

This yields

$$\|h_n(3x) - 3h_n(x)\|_\infty \leq 6\|\tau\|\|x\|_\infty$$

for each $n = 0, 1, ..., m$ and all $x \in \mathcal{A}$. Hence H is not an exactly higher Jordan ternary derivation on $\mathcal{A}$ since h_n is not exactly linear for each $n \in \mathbb{N}_0$. That is, we may regard H as an approximately higher Jordan ternary derivation of rank m on $\mathcal{A}$.

In 1940, S. M. Ulam [33] gave a talk concerning approximate mappings before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems:"what condition does there exists a homomorphism near an approximate homomorphism ?" In 1941, D. H. Hyers [14] answered affirmatively the question of Ulam for Banach spaces, which states that if $\varepsilon > 0$ and $f : X \to Y$ is a mapping with X a normed space, Y a Banach space such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \to X$. This stability phenomenon is called the Hyers-Ulam stability of the additive functional equation $g(x + y) = g(x) + g(y)$.

A generalized version of the theorem of Hyers for approximately additive mappings was given by Th. M. Rassias [30] in 1978 by considering the case when the above inequality is unbounded: if there exist $\theta \geq 0$ and $0 \leq p < 1$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in X$. From this fact, several authors say that the additive functional equation $g(x + y) = g(x) + g(y)$ has the Hyers-Ulam-Rassias stability property. Since then, a great deal of work of Rassias type has been done by a number of authors (cf. [17, 25, 26, 28] and reference therein).

In 1949, D. G. Bourgin [5] proved the following result, which is sometimes called the superstability of ring homomorphisms: suppose that A and B are Banach algebras with unit. If $f : A \to B$ is a surjective mapping such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon,$$

$$\|f(xy) - f(x)f(y)\| \leq \delta$$

for some $\varepsilon \geq 0$, $\delta \geq 0$ and all $x, y \in A$, then f is a ring homomorphism.

Recently, R. Badora [3] and T. Miura *et al.* [21] proved the Hyers-Ulam sta- bility, the Isac and Rassias-type stability [15], the Hyers-Ulam-Rassias stability and the Bourgin-type superstability of ring derivations on Banach algebras.

On the other hand, C. Park [24] and M. S. Moslehian [22] have contributed works on the stability problem of ternary homomorphisms and ternary derivations and M. Bavand Savadkouhi [4] investigated stability problem of ternary Jordan homomorphisms and ternary Jordan derivations.

K.-H. Park and Y.-S. Jung [27] proved the existence of an exact higher ternary derivation near to an approximately higher ternary derivation by investigating the Hyers-Ulam stability for higher ternary derivations in Banach ternary algebras.

Cădariu and Radu applied the fixed point method to the investigation of stability of the functional equations. (see also [7, 8, 9, 10, 29, 31, 32]).

In this paper, we consider the m-dimensional additive functional equation

$$\sum_{k=2}^{m}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{m-k+1}=i_{m-k}+1}^{m}\Big)f\Big(\sum_{i=1,i\neq i_1,..,i_{m-k+1}}^{m}x_i-\sum_{r=1}^{m-k+1}x_{i_r}\Big)+f\Big(\sum_{i=1}^{m}x_i\Big)=2^{m-1}f(x_1) \tag{1.1}$$

where $m \geq 2$ is an integer number. It is easy to see that the function $f(x) = ax$ is a solution of the functional equation (1.1).

As a special case, if $m = 2$ in (1.1), then the functional equation (1.1) reduces to

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1)$$

also by putting $m = 3$ in (1.1), we obtain

$$\sum_{i_1=2}^{2}\sum_{i_2=i_1+1}^{3} f\Big(\sum_{i=1,i\neq i_1,i_2}^{3} x_i - \sum_{r=1}^{2} x_{i_r}\Big) + \sum_{i_1=2}^{3} f\Big(\sum_{i=1,i\neq i_1}^{3} x_i - x_{i_1}\Big) + f\Big(\sum_{i=1}^{3} x_i\Big) = 2^2 f(x_1)$$

that is,

$$f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) = 4f(x_1).$$

In this paper, we will adopt the fixed point alternative of Cădariu and Radu to show the existence of an exact higher Jordan ternary derivation near to an approximately higher Jordan ternary derivation by investigating the Hyers-Ulam stability for higher Jordan ternary derivations in Banach ternary algebras. Furthermore, we are going to examine the Isac and Rassias-type stability [15] and the Bourgin-type superstability for higher Jordan ternary derivations in Banach ternary algebras.

2. Main results

We start our work with a general solution for equation 1.1.

Lemma 2.1. [11] *Let X and Y be real vector spaces. A function $f : X \to Y$ with $f(0) = 0$ satisfies 1.1 if and only if $f : X \to Y$ is additive.*

From now, if X and Y are linear spaces, for convenience, we use the following abbreviation for a given function $f : X \to Y$:

$$D_f(x_1, .., x_m) = \sum_{k=2}^{m}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{m-k+1}=i_{m-k}+1}^{m}\Big)f\Big(\sum_{i=1,i\neq i_1,...,i_{m-k+1}}^{m}x_i-\sum_{r=1}^{m-k+1}x_{i_r}\Big)$$
$$+ f\Big(\sum_{i=1}^{m}x_i\Big) - 2^{m-1}f(x_1)$$

for all $x_1, ..., x_m \in X$, where $m \geq 2$ is an integer number.

Before proceeding to the main results, we will state the following theorem.

Theorem 2.2. *(the alternative of fixed point [10]). Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T:\Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$d(T^m x, T^{m+1}x) = \infty$ for all $m \geq 0$,

or other exists a natural number m_0 such that

- *$d(T^m x, T^{m+1}x) < \infty$ for all $m \geq m_0$;*
- *the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;*
- *$y^*$is the unique fixed point of T in the set $\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\}$;*
- *$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.*

By a similar to in [2], we first obtain the Hyers-Ulam stability result.

Theorem 2.3. *Let $\mathcal{A}$ be a normed ternary algebra and $\mathcal{B}$ a Banach ternary algebra. Suppose that $F = \{f_0, f_1, ..., f_n, ...\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for some $\delta \geq 0$, $\varepsilon \geq 0$ and each $n \in \mathbb{N}_0$,*

$$\|D_{f_n}(x_1, .., x_m) + f_n(\lambda x) - \lambda f_n(x)\| \leq \varepsilon \tag{2.1}$$

and

$$\|f_n([x,x,x]) - \sum_{i+j+k=n} [f_i(x), f_j(x), f_k(x)]\| \leq \delta \tag{2.2}$$

hold for all $x, y, z \in \mathcal{A}$ and all $\lambda \in \mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$. Then there exists a unique higher Jordan ternary derivation $H = \{h_0, h_1, ..., h_n, ...\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$,

$$\|f_n(x) - h_n(x)\| \leq \frac{\varepsilon}{2^{m-2}} \tag{2.3}$$

holds for all $x \in \mathcal{A}$. Moreover, we have

$$\sum_{i+j+k=n} [h_i(x), h_j(x), \{h_k(x) - f_k(x)\}] = 0 \tag{2.4}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$.

Proof. Putting $x_i = 0$ $(i = 3, ..., m)$ and $x = 0$ in 2.1 implies

$$\|(1 + \sum_{\ell=1}^{m-2} \binom{m-2}{\ell})(f_n(x_1 + x_2) + f_n(x_1 - x_2)) - 2^{m-1} f_n(x_1)\| \leq \varepsilon \tag{2.5}$$

for each $n \in \mathbb{N}_0$ and all $x_1, x_2 \in \mathcal{A}$. Setting $x_1 = x_2 = x$ in 2.5. On the other hand, we have the relation

$$1 + \sum_{\ell=1}^{m-\jmath} \binom{m-\jmath}{\ell} = \sum_{\ell=0}^{m-\jmath} \binom{m-\jmath}{\ell} = 2^{m-\jmath} \tag{2.6}$$

for all $m > \jmath$. Hence we obtain from 2.6 and $f(0) = 0$ that

$$\|2^{m-2} f_n(2x) - 2^{m-1} f_n(x)\| \leq \varepsilon$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$, or

$$\|\frac{f_n(2x)}{2} - f_n(x)\| \leq \frac{\varepsilon}{2^{m-1}} \tag{2.7}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$.

Consider the set $X_n := \{g \mid g : \mathcal{A} \to \mathcal{B}\}$ and introduce the generalized metric on X_n for all $n \in \mathbb{N}_0$:

$$d(h, g) := inf\{c \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq c\varepsilon, \quad \forall x \in \mathcal{A}\}.$$

It is easy to show that (X_n, d) is complete for all $n \in \mathbb{N}_0$. Now we define the linear mapping $J : X_n \to X_n$ for all $n \in \mathbb{N}_0$ by

$$J(h)(x) = \frac{1}{2}h(2x)$$

for all $x \in \mathcal{A}$. By Theorem 2.2,

$$d(J(g), J(h)) \leq \frac{1}{2}d(g, h)$$

for all $g, h \in X_n$.

It follows from 2.7 that

$$d(f_n, J(f_n)) \leq \frac{1}{2^{m-1}}.$$

By Theorem 2.2, J has a unique fixed point in the set $X_{n_1} := \{h \in X : d(f, h) < \infty\}$ for all $n \in \mathbb{N}_0$. Let h_n be the fixed point of J for all $n \in \mathbb{N}_0$. h_n is the unique mapping with

$$h_n(2x) = 2h_n(x)$$

for all $x \in \mathcal{A}$ and for all $n \in \mathbb{N}_0$ satisfying there exists $c \in (0, \infty)$ such that

$$\|h_n(x) - f_n(x)\| \leq c\varepsilon$$

for all $x \in \mathcal{A}$ and for all $n \in \mathbb{N}_0$. On the other hand we have $\lim_r d(J^r(f_n), h_n) = 0$ for all $n \in \mathbb{N}_0$. It follows that

$$\lim_{r\to\infty} \frac{1}{2^r} f_n(2^r x) = h_n(x) \tag{2.8}$$

for all $x \in \mathcal{A}$ and for all $n \in \mathbb{N}_0$. It follows from $d(f_n, h_n) \leq \frac{1}{1-\frac{1}{2}} d(f_n, J(f_n))$, that

$$d(f_n, h_n) \leq \frac{1}{2^{m-2}}$$

for each $n \in \mathbb{N}_0$.This implies the inequality 2.3.

If we replace $x_1, ..., x_m$ with $2^k x_1, ..., 2^k x_m$ in 2.1, respectively, and $x = 0$ and then divide by 2^k, we get

$$\|\frac{1}{2^k} D_{f_n}(2^k x_1, ..., 2^k x_m)\| \leq \frac{\varepsilon}{2^k} \tag{2.9}$$

for each $n \in \mathbb{N}_0$. By letting $k \to \infty$ in 2.9, it follows from 2.8 that for each $n \in \mathbb{N}_0$, $D_{h_n}(x_1, ..., x_m) = 0$, thus for each $n \in \mathbb{N}_0$, h_n satisfies 1.1. Hence by Lemma 2.1, the function $h_n : \mathcal{A} \to \mathcal{B}$, is additive.

If we put $x_i = 0$, $(i = 1, 2, ..., m)$ and replace x with $2^k x$, and then divide by 2^k, in 2.1, for each $n \in \mathbb{N}_0$ yields

$$\|f_n(\lambda x) - \lambda f_n(x)\| \leq \varepsilon$$

for each $\lambda \in \mathbb{U}$ and all $x \in \mathcal{A}$. By letting $k \to \infty$, we get $h_n(\lambda x) = \lambda h_n(x)$, for each $x \in \mathcal{A}$ and all $\lambda \in \mathbb{U}$. Then it follows that the additive mapping h_n is $\mathbb{C}$-linear.

Next, we need to show that the sequence $H = \{h_0, h_1, ..., h_n, ...\}$ satisfies the identity

$$h_n([x,x,x]) = \sum_{i+j+k=n} [h_i(x), h_j(x), h_k(x)]$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. The inequality 2.2 implies that the function $\triangle_n : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ defined by

$$\triangle_n(x, , x, x) = f_n([x,x,x]) - \sum_{i+j+k} [f_i(x), f_j(x), f_k(x)] \tag{2.10}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$, is bounded. Hence we see that

$$\lim_{r\to\infty} \frac{\triangle_n(2^r x, x, x)}{2^r} = 0 \tag{2.11}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Now, using 2.8, 2.10 and 2.11, we have

$$\begin{aligned}
h_n([x,x,x]) &= \lim_{r\to\infty} \frac{f_n(2^r[x,x,x])}{2^r} = \lim_{r\to\infty} \frac{f_n([2^r x,x,x])}{2^r} \\
&= \lim_{r\to\infty} \frac{\sum_{i+j+k=n}[f_i(2^r x), f_j(x), f_k(x)] + \triangle_n(2^r x, x, x)}{2^r} \\
&= \lim_{r\to\infty} \sum_{i+j+k=n} \left[\frac{1}{2^r} f_i(2^r x), f_j(x), f_k(x)\right] + \lim_{r\to\infty} \frac{\triangle_n(2^r x, x, x)}{2^r} \\
&= \sum_{i+j+k=n} \left\{ \lim_{r\to\infty} \left[\frac{1}{2^r} f_i(2^r x), f_j(x), f_k(x)\right] \right\} \\
&= \sum_{i+j+k=n} [h_i(x), f_j(x), f_k(x)].
\end{aligned}$$

That is, we obtain that

$$h_n([x,x,x]) = \sum_{i+j+k=n} [h_i(x), f_j(x), f_k(x)] \tag{2.12}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Let $m \in \mathbb{N}$ be fixed. Then, applying 2.12 and the additivity of each h_n, $n \in \mathbb{N}_0$, we get

$$\begin{aligned}
\sum_{i+j+k=n} [h_i(x), f_j(2^r x), f_k(x)] &= h_n([x, 2^r x, x]) = h_n([2^r x, x, x]) \\
&= \sum_{i+j+k=n} [h_i(2^r x), f_j(x), f_k(x)] \\
&= 2^r \sum_{i+j+k=n} [h_i(x), f_j(x), f_k(x)].
\end{aligned}$$

Hence we have

$$\sum_{i+j+k=n} [h_i(x), f_j(x), f_k(x)] = \sum_{i+j+k=n} \left[h_i(x), \frac{1}{2^r} f_j(2^r x), f_k(x)\right] \tag{2.13}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Letting $r \to \infty$ in 2.13, it follows that

$$\sum_{i+j+k=n} [h_i(x), f_j(x), f_k(x)] = \sum_{i+j+k=n} [h_i(x), h_j(x), f_k(x)] \tag{2.14}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Therefore, we obtain that

$$h_n([x,x,x]) = \sum_{i+j+k=n} [h_i(x), h_j(x), f_k(x)] \tag{2.15}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Again, using 2.15 and the additivity of each h_n, $n \in \mathbb{N}_0$, we get

$$\begin{aligned}\sum_{i+j+k=n} [h_i(x), h_j(x), f_k(2^r x)] &= h_n([x,x,2^r x]) = h_n([2^r x, x, x]) \\ &= \sum_{i+j+k=n} [h_i(2^r x), h_j(x), f_k(x)] \\ &= 2^r \sum_{i+j+k=n} [h_i(x), h_j(x), f_k(x)].\end{aligned}$$

So we have

$$\sum_{i+j+k=n} [h_i(x), h_j(x), f_k(x)] = \sum_{i+j+k=n} \left[h_i(x), h_j(x), \frac{1}{2^r} f_k(2^r x)\right] \tag{2.16}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Taking $r \to \infty$ in 2.16, we have

$$\sum_{i+j+k=n} [h_i(x), h_j(x), f_k(x)] = \sum_{i+j+k=n} [h_i(x), h_j(x), h_k(x)] \tag{2.17}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$ which implies 2.4. Combining 2.15 with 2.17, it follows that $H = \{h_0, h_1, ..., h_n, ...\}$ satisfies the relation

$$h_n([x,x,x]) = \sum_{i+j+k=n} [h_i(x), h_j(x), h_k(x)]$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Thus H is a higher Jordan ternary derivation from $\mathcal{A}$ into $\mathcal{B}$.

To show the uniqueness property of H, assume that $H^* = \{h_0^*, h_1^*, ..., h_n^*, ...\}$ is another higher Jordan ternary derivation from $\mathcal{A}$ into $\mathcal{B}$ satisfying

$$\|f_n(x) - h_n^*(x)\| \leq \frac{\varepsilon}{2}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Let $m \in \mathbb{N}$. Since h_n and h_n^* are additive, we deduce that

$$2^r \|h_n(x) - h_n^*(x)\| = \|h_n(2^r x) - h_n^*(2^r x)\| \leq \varepsilon,$$

so that

$$\|h_n(x) - h_n^*(x)\| \leq \frac{\varepsilon}{2^r}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Setting $r \to \infty$, we find that

$$h_n(x) = h_n^*(x)$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. This completes the proof of the theorem.

□

Let $\mathbb{R}^+$ be the set of positive real numbers. G. Isac and Th. M. Rassias [15] generalized the Hyers theorem by introducing a mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ subject to the conditions

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0, \tag{2.18}$$

$$\psi(ts) \leq \psi(t)\psi(s) \quad for\ all\ t, s \in \mathbb{R}^+, \tag{2.19}$$

$$\psi(t) < t \quad for\ all\ t > 1. \tag{2.20}$$

Theorem 2.4. *Let $\mathcal{A}$ be a normed ternary algebra, $\mathcal{B}$ a Banach ternary algebra and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a mapping with properties 2.18, 2.19 and 2.20. In addition, let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping satisfying the condition*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0. \tag{2.21}$$

Suppose that $F = \{f_0, f_1, ..., f_n, ...\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for some $\varepsilon \geq 0$ and each $n \in \mathbb{N}_0$,

$$\|D_{f_n}(x_1, .., x_m) + f_n(\lambda x) - \lambda f_n(x)\| \leq \varepsilon\psi(\sum_{i=1}^{m} \|x_i\| + \|x\|) \tag{2.22}$$

and

$$\|f_n([x, x, x]) - \sum_{i+j+k=n} [f_i(x), f_j(x), f_k(x)]\| \leq \varphi(\|x\|^3) \tag{2.23}$$

hold for all $x_i, x \in \mathcal{A}$ and all $\lambda \in \mathbb{U}$. Then there exist a unique higher Jordan ternary derivation $H = \{h_0, h_1, ..., h_n, ..\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ and a constant $k \in \mathbb{R}$ such that for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$,

$$\|f_n(x) - h_n(x)\| \leq k\varepsilon\psi(\|x\|). \tag{2.24}$$

Moreover, the relation 2.4 is fulfilled.

Proof. Putting $x_i = 0$ $(i = 3, ..., m)$ and $x = 0$ in 2.22 implies

$$\|(1 + \sum_{\ell=1}^{m-2} \binom{m-2}{\ell})(f_n(x_1 + x_2) + f_n(x_1 - x_2)) - 2^{m-1} f_n(x_1)\| \leq \varepsilon\psi(\|x_1\| + \|x_2\|) \tag{2.25}$$

for each $n \in \mathbb{N}_0$ and all $x_1, x_2 \in \mathcal{A}$. Setting $x_1 = x_2 = x$ in 2.25. Hence we obtain from 2.6 and $f(0) = 0$ that

$$\|2^{m-2} f_n(2x) - 2^{m-1} f_n(x)\| \leq \varepsilon\psi(2\|x\|) \leq 2\varepsilon\psi(\|x\|)$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$, or

$$\|\frac{f_n(2x)}{2} - f_n(x)\| \leq \frac{\varepsilon\psi(\|x\|)}{2^{m-2}} \tag{2.26}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$.

Consider the set $X_n := \{g \mid g : \mathcal{A} \to \mathcal{B}\}$ and introduce the generalized metric on X_n for all $n \in \mathbb{N}_0$:

$$d(h, g) := inf\{c \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq c\psi(\|x\|), \quad \forall x \in \mathcal{A}\}.$$

It is easy to show that (X_n, d) is complete for all $n \in \mathbb{N}_0$. Now we define the linear mapping $J : X_n \to X_n$ for all $n \in \mathbb{N}_0$ by

$$J(h)(x) = \frac{1}{2}h(2x)$$

for all $x \in \mathcal{A}$. It is easy to show that,

$$d(J(g), J(h)) \leq \frac{1}{2}d(g, h)$$

for all $g, h \in X_n$.

It follows from 2.26 that

$$d(f_n, J(f_n)) \leq \frac{\varepsilon}{2^{m-2}}.$$

By Theorem 2.2, J has a unique fixed point in the set $X_{n_1} := \{h \in X : d(f, h) < \infty\}$ for all $n \in \mathbb{N}_0$. Let h_n be the fixed point of J for all $n \in \mathbb{N}_0$. h_n is the unique mapping with

$$h_n(2x) = 2h_n(x)$$

for all $x \in \mathcal{A}$ and for all $n \in \mathbb{N}_0$ satisfying there exists $c \in (0, \infty)$ such that

$$\|h_n(x) - f_n(x)\| \leq c\psi(\|x\|)$$

for all $x \in \mathcal{A}$ and for all $n \in \mathbb{N}_0$. On the other hand, we have $\lim_r d(J^r(f_n), h_n) = 0$ for all $n \in \mathbb{N}_0$. It follows that

$$\lim_{r\to\infty} \frac{1}{2^r} f_n(2^r x) = h_n(x)$$

for all $x \in \mathcal{A}$ and for all $n \in \mathbb{N}_0$. It follows from $d(f_n, h_n) \leq \frac{1}{1-\frac{1}{2}} d(f_n, J(f_n))$, that

$$d(f_n, h_n) \leq \frac{\varepsilon}{2^{m-3}}$$

for all $n \in \mathbb{N}_0$. This implies the inequality 2.24. The further of the proof is similar to the proof of Theorem 2.3. □

Remark 2.5. The typical example of the mapping ψ fulfilling 2.18, 2.19 and 2.20 is given by $\psi(t) = t^p$, where $p < 1$. The example of the mapping φ satisfying 2.21 is $\varphi(t) = t^q$, where $q < 1$. If we intend to consider the case of $p, q > 1$, then we adopt the method given by Z. Gajda in [12] to obtain the Isac and Rassias-type stability result for the mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ fulfilling the conditions

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0, \tag{2.27}$$

$$\psi(ts) \leq \psi(t)\psi(s) \quad for\ all\ t, s \in \mathbb{R}^+, \tag{2.28}$$

$$\psi(t) < t \quad for\ all\ t \in (0, 1). \tag{2.29}$$

In the proof of Theorem 2.3, if we replace 2.8 by

$$h_n(x) = \lim_{r\to\infty} 2^r f_n(\frac{1}{2^r}x)$$

and 2.10 in by

$$\lim_{r\to\infty} 2^r \triangle_n(\frac{1}{2^r}x, x, x) = 0,$$

then Theorem 2.4 is still true under the conditions 2.27, 2.28 and 2.29.

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