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Existence of three solutions for a boundary value problem with impulsive effects

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Abstract

In this paper, we are concerned with the existence of multiplicity solutions for a Dirichlet impulsive differential equation. The approach is based on variational methods.

Keywords: Multiple solutions, impulsive differential equation, Variational methods, Critical point theory
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1 Introduction

The purpose of this paper is to provide sufficient conditions for the existence at least three solutions for the following nonlinear impulsive differential problem

$$\begin{cases} -u''(t) + a(t)u'(t) + b(t)u(t) = \lambda g(t, u(t)), & t \in [0, T], t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = \mu I_j(u(t_j)), & j = 1, 2, ..., n, \end{cases} \tag{1.1}$$

where $\lambda \in]0, +\infty[$, $\mu \in]0, +\infty[$, $g : [0, T] \times \mathbb{R} \to \mathbb{R}$, $a, b \in L^\infty([0, T])$ with $\operatorname{ess\,inf}_{t\in[0,T]} a(t) \geq 0$ and $\operatorname{ess\,inf}_{t\in[0,T]} b(t) \geq 0$, $0 = t_0 < t_1 < t_2 < .. < t_n < t_{n+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim_{t\to t_j^+} u'(t) - \lim_{t\to t_j^-} u'(t)$ and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous functions for every $j = 1, 2, ..., n$.
Dynamic of many evolutionary processes from various fields undergo abrupt changes at certain instants during the evolution process such as earthquake, harvesting, shoke, and so forth. This models are studied in physics, population dynamics, ecology, control theory, biotechnology, medicine, economics, industrial robotics, and optimal control. Associated with this development, the study of impulsive boundary-value problems has captured special attention in the last years. There are some techniques to approach these problems: the fixed point theorems [15, 17, 19, 25, 27], the method of upper and lower solutions [11, 18, 20, 24], or the topological degree theory [16]. For example, Jia and Liu [15] established the existence of at least three nonnegative solutions to a type of three-point boundary value problem for second-order impulsive differential equations, and one obtained the sufficient conditions for existence of three nonnegative solutions by using the Leggett-William fixed point theorem. In [24], Shen and Wang employing the method of upper and lower solutions to solve impulsive differential equations with nonlinear boundary conditions. In [20], based on the method of upper and lower solutions together with Leray-Schauder degree theory, the authors

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investigated different set of assumptions which guarantee the existence of a solution of the impulsive BVP

$$\begin{cases} -x''(t) + f(t, x(t), x'(t)) = 0, & t \in J^*, \\ \Delta x(t_k) = I_k(x(t_j)), & k = 1, 2, ..., p, \\ \Delta x'(t_k) = J_k(x(t_j), x'(t_j)), & k = 1, 2, ..., p, \\ x(0) = x(1) = \int_0^1 g(s)x(s)ds, \end{cases}$$

Xu and Ding in [27] considered the existence of three positive solutions to the following boundary value problem

$$\begin{cases} -u''(t) + f(u(t-\tau)) = 0, & t \in [a, b], t \neq t_k, \\ \Delta u'(t_k) = I_k(u(t_k)), & k = 1, 2, ..., m, \\ u(a) = \mu u(\eta), \quad u^{'}(b) = 0, \quad \mu \in (0, 1), \quad \eta \in [a, \frac{a+b}{2}] \\ u(t) = 0, & a - \tau \leq t < a, \end{cases} \tag{1.2}$$

where $a = t_1 < t_2 < ... < t_k < ... < t_m < b$, $I_k \in C[p \times p, p]$, $\Delta u'(t_k)$ denotes the jump of $u'(t)$ at t_k, i. e. $eltau'(t_j) = u'(t_j^+) - u'(t_j^-)$ where $u'(t_k^+)$ and $u'(t_k^-)$ represent the right-hand limit and left-hand limit of $u'(t)$ at $t = t_k$, respectively. They impose growth conditions on f to apply the Leggett-Williams fixed point theorem in finding three positive solutions of (1.2).

Recently, the existence and multiplicity of solutions for impulsive boundary value problems by using variational methods has been considered. Let us mention some recent paper on impulsive boundary-value problems. Many interesting results are obtained see for examples [2, 3, 5, 6, 7, 8, 9, 10, 21, 26] and the references therein. For example, Chen and Li [8] by using variational methods and critical point theory studied the existence of n distinct pairs of nontrivial solutions to the Dirichlet boundary problem for the second-order impulsive differential equations. In [3] Bonanno et al. based on critical points theorem, established existence of infinitely many solutions for the nonlinear impulsive differential problem (1.1). In [7] the authors studied the existence of solutions for following second-order impulsive differential equation

$$\begin{cases} -u''(t) + cu'(t) = \lambda g(t, u(t)), & a.e. t \in [0, \infty), \\ \Delta u'(t_j) = I_j(u(t_j)), & j = 1, 2, ..., p, \\ u'(0+) = h(u(0)), & u'(+\infty) = 0, \end{cases}$$

where c and λ are two positive parameters, $0 = t_0 < t_1 < t_2 < .. < t_p < +\infty$, $u'(0^+) = \lim_{t\to 0^+} u'(t)$, and $u'(+\infty) = \lim_{t\to+\infty} u'(t)$, $h, I_j \in C(\mathbb{R}, \mathbb{R})$, and $g \in C([0, +\infty) \times \mathbb{R}, \mathbb{R})$. Applying variational methods, they give some new criteria to guarantee that the impulsive problem has at least one classical solution, three classical solutions and infinitely many classical solutions, respectively. Wang and Zhao [26], via some critical point theory and the variational method, studied the existence and multiplicity of solutions for the following nonlinear impulsive problem

$$\begin{cases} -u''(t) + r(t)u'(t) + \lambda u(t) = f(t, u(t)), & a.e. t \in J, \\ \Delta u'(t_i) = u'(t_i^+) - u'(t_i^-) = I_i(u(t_i)), & j = 1, 2, ..., p, \\ u(0) = u(T) = 0, \end{cases}$$

where $J = [0, T], 0 = t_0 < t_1 < t_2 < .. < t_p < t_{p+1} = T, r \in C[0, T], I_i \in C[\mathbb{R}, \mathbb{R}]$, λ is parameter, $f \in C[J \times \mathbb{R}, \mathbb{R}]$, with $F(t, u) = \int_0^u f(t, \xi)d\xi$.

Motivated by the above facts, the paper is organized as follows. In section 2, we introduce some preliminary results, including some properties ,the variational structure and several important lemmas. In section 3, will be devoted to existence result for Impulsive boundary value problem. We give an example to demonstrate the application.

2 Preliminaries

In this present paper X denotes a finite dimensional real Banach space and $I_\lambda : X \to \mathbb{R}$ is a functional satisfying the following structure hypothesis: $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$ where $\Phi, \Psi : X \to \mathbb{R}$ are two functions of class C^1 on X with Φ coercive, i.e. $\lim_{\|u\|\to+\infty} \Phi(u) = +\infty$, and λ is a positive real parameter. For all r, r_1, r_2 with $r_2 > r_1$ and $r_2 > \inf_X \Phi$, and all $r_3 > 0$, we define

$$\phi(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1,r_2)=\inf_{u\in\Phi^{-1}(-\infty,r_1)}\sup_{v\in\Phi^{-1}[r_1,r_2)}\frac{\Psi(v)-\Psi(u)}{\Phi(v)-\Phi(u)},$$

$$\gamma(r_2,r_3)=\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3},$$

$$\alpha(r_1,r_2,r_3)=\max\{\phi(r_1),\phi(r_2),\gamma(r_2,r_3)\}.$$

Theorem 2.1. [4, Theorem 3.3] Assume that

(1) Φ is convex and $\inf_X\Phi=\Phi(0)=\Psi(0)=0$;

(2) for every $u_1,u_2\in X$ such that $\Psi(u_1)\geq 0$ and $\Psi(u_2)\geq 0$, one has

$$\inf_{s\in[0,1]}\Psi(su_1+(1-s)u_2)\geq 0.$$

Assume that there are three positive constants r_1,r_2,r_3 with $r_1<r_2$ such that

(i) $\phi(r_1)<\beta(r_1,r_2)$;

(ii) $\phi(r_2)<\beta(r_1,r_2)$;

(iii) $\gamma(r_2,r_3)<\beta(r_1,r_2)$.

Then, for each $\lambda\in]\frac{1}{\beta(r_1,r_2)},\frac{1}{\alpha(r_1,r_2,r_3)}[$ the functional $\Phi-\lambda\Psi$ admits three distinct critical points u_1,u_2,u_3 such that $u_1\in\Phi^{-1}(]-\infty,r_1[)$, $u_2\in\Phi^{-1}([r_1,r_2[)$ and $u_3\in\Phi^{-1}(]-\infty,r_2+r_3[)$.

Theorem 2.1 is a counter-part of general result (three critical point theorem) of Ricceri [22, 23].

We refer the interested reader to the papers [1, 12, 13, 14] in which Theorem 2.1 has been successfully employed to obtain the existence of at least three solutions for boundary value problems.

By a classical solution of (1.1) we mean a function

$$u\in\{w\in C([0,T]):w\,|_{[t_j,t_{j+1}]}\in H^2([t_j,t_{j+1}])\}$$

that satisfies the equation in (1.1) a.e. on $[0,T]\setminus\{t_1,...,t_n\}$, the limits $u'(t_j^+),u'(t_j^-)$, $j=1,...,n$, exist, that satisfies the impulsive conditions $\Delta u'(t_j)=\mu I_j(u(t_j))$ and the boundary conditions $u(0)=u(T)=0$. Clearly if a,b and g are continuous, then a classical solution $u\in C^2([t_j,t_{j+1}]),j=0,...,n$, satisfies the equation in (1.1) for all $t\in[0,T]\setminus\{t_1,...,t_n\}$.

We consider the following slightly different form of problem (1.1):

$$\begin{cases}-(p(t)u'(t))'+q(t)u(t)=\lambda f(t,u(t)), & t\in[0,T],t\neq t_j,\\ u(0)=u(T)=0,\\ \Delta u'(t_j)=\mu I_j(u(t_j)), & j=1,2,...,n,\end{cases}\tag{2.1}$$

where $p\in C^1([0,T],]0,+\infty[)$, and $q\in L^\infty([0,T])$ with $\operatorname{ess\,inf}_{t\in[0,T]}q(t)\geq 0$.

It is easy to see that, by choosing

$$p(t)=e^{-\int_0^t a(\zeta)d\zeta},\qquad q(t)=b(t)e^{\int_0^t a(\zeta)d\zeta},\qquad f(t,u)=g(t,u)e^{\int_0^t a(\zeta)d\zeta},$$

the solutions of (2.1) are solutions of (1.1).

In the Sobolev space $X:=H_0^1(0,T)$, consider the inner product

$$(u,v):=\int_0^T p(t)u'(t)v'(t)dt+\int_0^T q(t)u(t)v(t)dt,$$

which induces the norm

$$\|u\|:=\left(\int_0^T p(t)(u'(t))^2dt+\int_0^T q(t)(u(t))^2dt\right)^{\frac{1}{2}}.$$

Then the following Poincaré-type inequality holds:

$$\left[\int_0^T u^2(t)dt\right]^{\frac{1}{2}}\leq\frac{T}{\pi}\left[\int_0^T (u')^2(t)dt\right]^{\frac{1}{2}}.\tag{2.2}$$

Theorem 2.2. Let $u \in X$. Then

$$\|u\|_\infty \leq \frac{1}{2}\sqrt{\frac{T}{p^*}}\|u\|. \tag{2.3}$$

where $p^* := \min_{t\in[0,T]} p(t)$.

Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be nonnegative L^1-Carathéodory function. We recall $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory function if

(a) the mapping $t \mapsto f(t,x)$ is measurable for every $x \in \mathbb{R}$,
(b) the mapping $x \mapsto f(t,x)$ is continuous for almost every $t \in [0,T]$,
(c) for every $\nu > 0$ there exists a function $l_\nu \in L^1([0,T])$ such that

$$\sup_{|x|\leq\nu} |f(t,x)| \leq l_\nu(t)$$

for almost every $t \in [0,T]$.

We say that a functional $u \in X$ is a weak solution of problem (2.1) if u satisfies

$$\int_0^T p(t)u'(t)v'(t)dt + \int_0^T q(t)u(t)v(t)dt$$
$$-\lambda\int_0^T f(t,u(t))v(t)dt + \mu\sum_{j=1}^n p(t_j)I_j(u(t_j))v(t_j) = 0$$

for every any $v \in X$.

Lemma 2.3. $u \in X$ is a weak solution of (2.1) if and only if u is a classical solution of (2.1).

Now, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined as follows

$$\Phi(u) := \frac{1}{2}\|u\|^2 \tag{2.4}$$

and

$$\Psi(u) := \int_0^T F(t,u(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx, \tag{2.5}$$

for each $u \in X$, where $F(t,\xi) = \int_0^\xi f(t,x)dx$ for each $(t,\xi) \in [0,T] \times \mathbb{R}$. Clearly, Φ and Ψ are well defined and continuously differentiable whose differential at the point $u \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_0^T p(t)u'(t)v'(t)dt + \int_0^T q(t)u(t)v(t)dt,$$

$$\Psi'(u)(v) = \int_0^T f(t,u(t))v(t)dt + \frac{\mu}{\lambda}\sum_{j=1}^n p(t_j)I_j(u(t_j))v(t_j),$$

for every $v \in X$.

3 Main results

In this section, we present the main abstract result of this paper.
Put

$$k = \frac{6p^*}{12\|p\|_\infty + T^2\|q\|_\infty} \tag{3.1}$$

and let $\mathcal{I}_j(\xi)=\int_0^\xi I_j(x)dx$ for all $\xi\in\mathbb{R}$ such that $\sup_{\xi\geq 0}\mathcal{I}_j(\xi)=0$. Moreover, set

$$\mathcal{I}^\rho := \|p\|_\infty \sum_{j=1}^{n} \max_{|\xi|\leq\rho}(-\mathcal{I}_j(\xi)) \text{ for all } \rho>0$$

and

$$\mathcal{I}_d := p^* \sum_{j=1}^{n} \min_{|\xi|\leq d}(-\mathcal{I}_j(\xi)) \text{ for all } d>0.$$

Our first result is as follows. Fixing four positive constants ρ_1,ρ_2,ρ_3 and d, put

$$\begin{aligned}\delta_{\lambda,I} := \min\Bigg\{\frac{1}{T}\min\Bigg\{&\frac{2p^*\rho_1^2-\lambda T\int_0^T F(t,\rho_1)dt}{\mathcal{I}^{\rho_1}},\frac{2p^*\rho_2^2-\lambda T\int_0^T F(t,\rho_2)dt}{\mathcal{I}^{\rho_2}}\\ &,\frac{2p^*(\rho_3^2-\rho_2^2)-\lambda T\int_0^T F(t,\rho_3)dt}{\mathcal{I}^{\rho_3}}\Bigg\},\frac{\frac{2p^*}{Tk}d^2-\lambda(\int_{T/4}^{3T/4}F(t,d)dt-\int_0^T F(t,\rho_1)dt)}{\mathcal{I}_d-\mathcal{I}^{\rho_1}}\Bigg\}.\end{aligned} \tag{3.2}$$

We formulate our main result as follows.

Theorem 3.1. Assume that there exist four positive constants ρ_1,ρ_2,ρ_3 and d with $\frac{\rho_1}{\sqrt{2}}<d<\sqrt{k}\rho_2<\sqrt{k}\rho_3$ where k as given by (3.1), such that

(j_1) $F(t,x)\geq 0$ for all $(t,x)\in[0,T]\times\mathbb{R}$,

(j_2)

$$\max\left\{\frac{\int_0^T F(t,\rho_1)dt}{\rho_1^2},\frac{\int_0^T F(t,\rho_2)dt}{\rho_2^2},\frac{\int_0^T F(t,\rho_3)dt}{\rho_3^2-\rho_2^2}\right\}<k\frac{\int_{T/4}^{3T/4}F(t,d)dt-\int_0^T F(t,\rho_1)dt}{d^2}.$$

Then, for every

$$\lambda\in\Lambda:=\left(\frac{\frac{2p^*}{Tk}d^2}{\int_{T/4}^{3T/4}F(t,d)dt-\int_0^T F(t,\rho_1)dt},\frac{2p^*}{T}\min\left\{\frac{\rho_1^2}{\int_0^T F(t,\rho_1)dt},\frac{\rho_2^2}{\int_0^T F(t,\rho_2)dt},\frac{\rho_3^2-\rho_2^2}{\int_0^T F(t,\rho_3)dt}\right\}\right)$$

and for every non-negative continuous function $I:\mathbb{R}\to\mathbb{R}$, there exists $\delta_{\lambda,I}>0$ given by (3.2) such that, for each $\mu\in[0,\delta_{\lambda,I})$, the problem (2.1) admits at least three generalized solutions $u_i (i=1,2,3)$ such that

$$\|u_1\|_\infty\leq\rho_1, \|u_2\|_\infty\leq\rho_2 \text{ and } \|u_3\|_\infty\leq\rho_3.$$

Proof . Our aim is to apply Theorem 2.1 to problem (2.1) in the Banach space $X=H_0^1(0,T)$ endowed with the norm

$$\|u\| := \left(\int_0^T p(t)(u'(t))^2dt+\int_0^T q(t)(u(t))^2dt\right)^{\frac{1}{2}}$$

For every $u\in X$. Let Φ , Ψ be the the functionals defined in (2.4) and (2.5), respectively. Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem (2.1). clearly Φ is convex and and sequentially weakly lower semicontinuous and Ψ is sequentially weakly upper semicontinuous. Also, Φ' admits a continuous inverse on X^* and Ψ' is compact.
Now, let us put

$$\overline{v}(t)=\begin{cases}\dfrac{4dx}{T}, & x\in[0,\frac{T}{4}],\\ d, & x\in]\frac{T}{4},\frac{3T}{4}],\\ \frac{4d}{T}(T-x), & x\in]\frac{3T}{4},T].\end{cases}$$

Clearly, one has

$$\frac{4p^*}{T}d^2\leq\Phi(\overline{v})\leq\frac{2p^*}{kT}d^2.$$

Put $r_1 = \frac{2p^*}{T}\rho_1^2$, $r_2 = \frac{2p^*}{T}\rho_2^2$ and $r_3 = \frac{2p^*}{T}(\rho_3^2 - \rho_2^2)$. From the assumption, we obtain $r_1 < \Phi(\overline{v}) < r_2$ and $r_3 > 0$. Moreover, due to definition of r_1, we have

$$\Phi^{-1}(]-\infty, r_1[) \subseteq \{u \in X : \max_{t\in[0,T]} |u(t)| \leq \rho_1\}, \tag{3.3}$$

therefore, since f is nonnegative in $[0,T]\times\mathbb{R}$, one has

$$\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \int_0^T F(t,u(t))dt \leq \int_0^T \max_{|\xi|\leq\rho_1} F(t,\xi)dt \leq \int_0^T F(t,\rho_1)dt.$$

In a similar way, it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \int_0^T F(t,u(t))dt \leq \int_0^T F(t,\rho_2)dt$$

and

$$\sup_{u\in\Phi^{-1}(]-\infty,r_3[)} \int_0^T F(t,u(t))dt \leq \int_0^T F(t,\rho_3)dt.$$

Hence, exploiting (3.3) and since $0 \in \Phi^{-1}(]-\infty, r_1])$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}
\phi(r_1) &= \inf_{u\in\Phi^{-1}(]-\infty,r_1])} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_1])}\Psi(v) - \Psi(u)}{r_1 - \Phi(u)} \\
&\leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_1])}\Psi(v)}{r_1} \\
&= \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_1])}\left(\int_0^T F(t,u(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx\right)}{r_1} \\
&\leq \frac{T}{2p^*}\frac{\int_0^T \max_{|\xi|\leq\rho_1} F(t,u(t))dt + \frac{\mu}{\lambda}\|p\|_\infty \sum_{j=1}^n \max_{|\xi|\leq\rho_1}(-\mathcal{I}_j(\xi))}{\rho_1^2} \\
&\leq \frac{T}{2p^*}\left(\frac{\int_0^T F(t,\rho_1)dt}{\rho_1^2} + \frac{\mu}{\lambda}\frac{\mathcal{I}^{\rho_1}}{\rho_1^2}\right),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\phi(r_2) &= \inf_{u\in\Phi^{-1}(]-\infty,r_2])} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_2])}\Psi(v) - \Psi(u)}{r_2 - \Phi(u)} \\
&\leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_2])}\Psi(v)}{r_2} \\
&= \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_2])}\left(\int_0^T F(t,u(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx\right)}{r_2} \\
&\leq \frac{T}{2p^*}\frac{\int_0^T \max_{|\xi|\leq\rho_2} F(t,u(t))dt + \frac{\mu}{\lambda}\|p\|_\infty \sum_{j=1}^n \max_{|\xi|\leq\rho_2}(-\mathcal{I}_j(\xi))}{\rho_2^2} \\
&\leq \frac{T}{2p^*}\left(\frac{\int_0^T F(t,\rho_2)dt_1}{\rho_2^2} + \frac{\mu}{\lambda}\frac{\mathcal{I}^{\rho_2}}{\rho_2^2}\right)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\gamma(r_2,r_3) &\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_2+r_3])}\Psi(u)}{r_3}\\ &= \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_2+r_3])}\left(\int_0^T F(t,u(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx\right)}{r_3}\\ &\le \frac{T}{2p^*}\frac{\int_0^T \max_{|\xi|\le\rho_3} F(t,u(t))dt + \frac{\mu}{\lambda}\|p\|_\infty \sum_{j=1}^n \max_{|\xi|\le\rho_3}(-\mathcal{I}_j(\xi))}{\rho_3^2-\rho_2^2}\\ &\le \frac{T}{2p^*}\left(\frac{\int_0^T F(t,\rho_3)dt_1}{\rho_3^2-\rho_2^2} + \frac{\mu}{\lambda}\frac{\mathcal{I}^{\rho_3}}{\rho_3^2-\rho_2^2}\right). \end{aligned} \tag{3.6}$$

On other hand, since

$$\int_0^T F(t,\overline{v})dt \ge \int_{T/4}^{3T/4} F(t,\overline{v})dt = \int_{T/4}^{3T/4} F(t,d)dt,$$

for each $u \in \Phi^{-1}(]-\infty,r_1[)$, one has

$$\Psi(\overline{v}) - \Psi(u) \ge \int_{T/4}^{3T/4} F(t,d)dt - \int_0^T F(t,\rho_1)dt + \frac{\mu}{\lambda}(\mathcal{I}_d - \mathcal{I}^{\rho_1}).$$

Consequently, we have

$$\begin{aligned}\beta(r_1,r_2) &\ge \inf_{u\in\Phi^{-1}(]-\infty,r_1[)} \frac{\Psi(\overline{v})-\Psi(u)}{\Phi(\overline{v})-\Phi(u)}\\ &\ge \frac{\int_{T/4}^{3T/4} F(t,d)dt - \int_0^T F(t,\rho_1)dt + \frac{\mu}{\lambda}(\mathcal{I}_d - \mathcal{I}^{\rho_1})}{\frac{1}{2}\|\overline{v}\|^2 - \frac{1}{2}\|u\|^2}\\ &\ge \frac{\int_{T/4}^{3T/4} F(t,d)dt - \int_0^T F(t,\rho_1)dt + \frac{\mu}{\lambda}(\mathcal{I}_d - \mathcal{I}^{\rho_1})}{\frac{2p^*d^2}{Tk}}\\ &= \frac{Tk}{2p^*}\frac{\int_{T/4}^{3T/4} F(t,d)dt - \int_0^T F(t,\rho_1)dt + \frac{\mu}{\lambda}(\mathcal{I}_d - \mathcal{I}^{\rho_1})}{d^2}. \end{aligned} \tag{3.7}$$

Thanks to (j_2) and inequalities (3.3)-(3.7), we get

$$\alpha(r_1,r_2,r_3) < \beta(r_1,r_2).$$

Now, the conclusion of Theorem (2.1) can be used. It follows that, for every

$$\lambda \in \left(\frac{\frac{2p^*}{Tk}d^2}{\int_{T/4}^{3T/4} F(t,d)dt - \int_0^T F(t,\rho_1)dt}, \frac{2p^*}{T}\min\left\{\frac{\rho_1^2}{\int_0^T F(t,\rho_1)dt}, \frac{\rho_2^2}{\int_0^T F(t,\rho_2)dt}, \frac{\rho_3^2-\rho_2^2}{\int_0^T F(t,\rho_3)dt}\right\}\right)$$

and $\mu \in [0,\delta_{\lambda,I})$, the functional $\Phi - \lambda\Psi$ has three critical points $u_i, i = 1,2,3$ in X. Further, one has $\Phi(u_1) < r_1, \Phi(u_2) < r_2, \Phi(u_3) < r_2 + r_3$, that is,

$$\max_{k\in[0,T]} |u_1(k)| < \rho_1, \max_{k\in[0,T]} |u_2(k)| < \rho_2, \max_{k\in[0,T]} |u_3(k)| < \rho_3,$$

which completes the proof. □ An immediate consequence of Theorem 3.1 is the following.
For positive constants ρ_1, ρ_4 and d, set

$$\delta'_{\lambda,I} := \min\left\{\frac{1}{T}\min\left\{\frac{2p^*\rho_1^2 - \lambda T\int_0^T F(t,\rho_1)dt}{\mathcal{I}^{\rho_1}}, \frac{p^*\rho_4^2 - \lambda T\int_0^T F(t,\frac{1}{\sqrt{2}}\rho_4)dt}{\mathcal{I}^{\frac{1}{\sqrt{2}}\rho_4}}, \frac{p^*\rho_4^2 - \lambda T\int_0^T F(t,\rho_4)dt}{\mathcal{I}^{\rho_4}}\right\}, \frac{\frac{2p^*}{Tk}d^2 - \lambda(\int_{T/4}^{3T/4} F(t,d)dt - \int_0^T F(t,\rho_1)dt)}{\mathcal{I}_d - \mathcal{I}^{\rho_1}}\right\}. \tag{3.8}$$

Theorem 3.2. Assume that there exist three constants ρ_1, ρ_4 and d with $\rho_1 < \sqrt{2}d < \sqrt{k}\rho_4$, such that

(j$_1$) $F(t,x) \geq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}$,

(j$_3$)

$$\max\left\{\frac{\int_0^T F(t,\rho_1)dt}{\rho_1^2}, \frac{2\int_0^T F(t,\rho_4)dt}{\rho_4^2}\right\} < \frac{1}{2}\frac{k}{1+k}\frac{\int_{T/4}^{3T/4} F(t,d)dt}{d^2}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{2p^*(1+k)}{Tk}d^2}{\int_{T/4}^{3T/4} F(t,d)dt}, \frac{2p^*}{T}\min\left\{\frac{\rho_1^2}{\int_0^T F(t,\rho_1)dt}, \frac{\rho_4^2}{2\int_0^T F(t,\rho_4)dt},\right\}\right)$$

and for every non-negative continuous function $I : \mathbb{R} \to \mathbb{R}$, there exists $\delta'_{\lambda,I} > 0$ given by (3.8) such that, for each $\mu \in [0, \delta'_{\lambda,I})$, the problem (2.1) admits at least three generalized solutions $u_i (i = 1,2,3)$ such that

$$\|u_1\|_\infty \leq \rho_1, \|u_2\|_\infty \leq \frac{1}{\sqrt{2}}\rho_4 \text{ and } \|u_3\|_\infty \leq \rho_4.$$

Proof . Put

$$\rho_2 = \frac{1}{\sqrt{2}}\rho_4, \quad \rho_3 = \rho_4.$$

Therefore, using (j_3), we have

$$\frac{\int_0^T F(t,\rho_2)dt}{\rho_2^2} = \frac{2\int_0^T F(t,\frac{1}{\sqrt{2}}\rho_4)dt}{\rho_4^2} \leq \frac{2\int_0^T F(t,\rho_4)dt}{\rho_4^2} < \frac{k}{1+k}\frac{\int_{T/4}^{3T/4} F(t,d)dt}{d^2}, \tag{3.9}$$

and

$$\frac{\int_0^T F(t,\rho_3)dt}{(\rho_3^2 - \rho_2^2)} = \frac{2\int_0^T F(t,\rho_4)dt}{\rho_4^2} < \frac{k}{1+k}\frac{\int_{T/4}^{3T/4} F(t,d)dt}{d^2}. \tag{3.10}$$

Furthermore, from (j_3) and condition $\rho_1 < \sqrt{2}d$ we get

$$\begin{aligned} k\frac{\int_{T/4}^{3T/4} F(t,d)dt - \int_0^T F(t,\rho_1)dt}{d^2} &> k\frac{\int_{T/4}^{3T/4} F(t,d)dt}{d^2} - 2k\frac{\int_0^T F(t,\rho_1)dt}{\rho_1^2} \\ &> k\frac{\int_{T/4}^{3T/4} F(t,d)dt}{d^2} - \frac{k^2}{1+k}\frac{\int_{T/4}^{3T/4} F(t,d)dt}{d^2} \\ &= \frac{k}{1+k}\frac{\int_{T/4}^{3T/4} F(t,d)dt}{d^2}. \end{aligned}$$

Hence, using again (j_3) and (3.9), (3.10), the assumption (j_2) of theorem 3.1 are fulfilled. □

Theorem 3.3. Let $f_1 \in C([0,T])$ and $f_2 \in C(\mathbb{R})$ be two functions. Set $\widehat{F}(\xi) = \int_0^\xi f_2(x)dx$ for all $\xi \in \mathbb{R}$, and assume that there exists three positive constants ρ_1, ρ_4 and d with $\rho_1 < \sqrt{2}d < \sqrt{k}\rho_4$, such that

(j$_4$) $\int_0^T f_1(t)dt \geq 0$ for each $t \in [0,T]$ and $\widehat{F}(x) \geq 0$ for each $x \in \mathbb{R}$,

(j$_5$)

$$\max\left\{\frac{\widehat{F}(\rho_1)\int_0^T f_1(t)dt}{\rho_1^2}, \frac{\widehat{F}(\rho_4)\int_0^T f_1(t)dt}{\rho_4^2}\right\} < \frac{1}{2}\frac{k}{1+k}\frac{\widehat{F}(d)\int_{T/4}^{3T/4} f_1(t)dt}{d^2}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{2p^*(1+k)}{Tk}d^2}{\widehat{F}(d)\int_{T/4}^{3T/4} f_1(t)dt}, \frac{2p^*}{T\int_0^T f_1(t)dt}\min\left\{\frac{\rho_1^2}{\widehat{F}(\rho_1)}, \frac{\rho_4^2}{2\widehat{F}(\rho_4)},\right\}\right)$$

and for every non-negative continuous function $I : \mathbb{R} \to \mathbb{R}$ for each

$$\mu \in \Bigg(0, \min\Bigg\{\frac{1}{T}\min\Bigg\{\frac{2p^*\rho_1^2 - \lambda T\widehat{F}(\rho_1)\int_0^T f_1(t)dt}{\mathcal{I}^{\rho_1}}, \frac{p^*\rho_4^2 - \lambda T\widehat{F}(\frac{1}{\sqrt{2}}\rho_4)\int_0^T f_1(t)dt}{\mathcal{I}^{\frac{1}{\sqrt{2}}\rho_4}}, \frac{p^*\rho_4^2 - \lambda T\widehat{F}(\rho_4)\int_0^T f_1(t)dt}{\mathcal{I}^{\rho_4}}\Bigg\}, \frac{\frac{Tk}{2p^*}d^2 - \lambda(\widehat{F}(d)\int_{T/4}^{3T/4} f_1(t)dt - \widehat{F}(\rho_1)\int_0^T f_1(t)dt)}{\mathcal{I}_d - \mathcal{I}^{\rho_1}}\Bigg\}\Bigg),$$

the problem

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = \lambda f_1(t)f_2(u(t)), & t \in [0,T], t \neq t_J, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = \mu I_j(u(t_j)), & j = 1,2,...,n, \end{cases}$$

admits at least three generalized solutions u_1, u_2 and u_3 such that

$$\max_{t\in[0,T]}|u_1(t)| \leq \rho_1, \max_{t\in[0,T]}|u_2(t)| \leq \frac{1}{\sqrt{2}}\rho_4 \text{ and } \max_{t\in[0,T]}|u_3(t)| \leq \rho_4.$$

Proof . Set $f(t,x) = f_1(t)f_2(x)$ for each $(t,x) \in [0,T] \times \mathbb{R}$. Since

$$F(t,x) = \int_0^x f(t,\xi)d\xi = f_1(t)\int_0^x f_2(\xi)d\xi = f_1(t)\widehat{F}(x),$$

from (j_5), we obtain (j_3). The conclusion follows from Theorem 3.2. □

Remark 3.4. When f does not depend on t, hypotheses (j_4) and (j_5) become the following simpler forms;

(j_4') $f(x) \geq 0$ for each $x \in \mathbb{R}$,

(j_5')

$$\max\left\{\frac{F(\rho_1)}{\rho_1^2}, \frac{F(\rho_4)}{\rho_4^2}\right\} < \frac{1}{2}\frac{k}{1+k}\frac{F(d)}{d^2},$$

and the intervals become

$$\lambda \in \left(\frac{4p^*(1+k)d^2}{kF(d)}, \frac{2p^*}{T^2}\min\left\{\frac{\rho_1^2}{F(\rho_1)}, \frac{\rho_4^2}{2F(\rho_4)},\right\}\right)$$

and

$$\mu \in \left(0, \min\left\{\frac{1}{T}\min\left\{\frac{2p^*\rho_1^2 - \lambda T^2F(\rho_1)}{\mathcal{I}^{\rho_1}}, \frac{p^*\rho_4^2 - \lambda T^2F(\frac{1}{\sqrt{2}}\rho_4)}{\mathcal{I}^{\frac{1}{\sqrt{2}}\rho_4}}, \frac{p^*\rho_4^2 - \lambda T^2F(\rho_4)}{\mathcal{I}^{\rho_4}}\right\}, \frac{\frac{2p^*}{Tk}d^2 - \lambda T(\frac{F(d)}{2} - F(\rho_1))}{\mathcal{I}_d - \mathcal{I}^{\rho_1}}\right\}\right).$$

Example 3.5. Consider the following boundary value problem:

$$\begin{cases} -(\frac{\sqrt{4t+1}}{t+1}u'(t))' + (1+\sqrt{t})u(t) = \lambda f_1(t)f_2(u(t)), & t \in [0,1], t \neq t_j, \\ u(0) = u(1) = 0, \\ \Delta u'(t_j) = \mu(-e^u(u^2+2u)), & j = 1,2,...,n, \end{cases} \tag{3.11}$$

where $f_1(t) = 2t$ for each $t \in [0,1]$ and

$$f_2(x) := \begin{cases} 6x^5, & \text{if } x < 1, \\ 6, & \text{if } x = 1, \\ \frac{6}{x}, & \text{if } x > 1. \end{cases}$$

By expression of f_2, we have

$$\widehat{F}(x) := \begin{cases} x^6, & \text{if } x < 1, \\ 6x - 5, & \text{if } x = 1, \\ 1 + 6\ln(x), & \text{if } x > 1. \end{cases}$$

Choosing $\rho_1 = 10^{-8}, d = 1$ and $\rho_4 = 10^6$, we clearly see that all assumptions of Theorem 3.3 are satisfied. Then, for every

$$\lambda \in \left(15, \frac{10^{12}}{1+\ln(10^6)}\right)$$

and for every

$$\mu \in \Bigg(0, \min\Bigg\{\min\Bigg\{\frac{6-3\lambda\times 10^{-32}}{2\sqrt{3}e^{10^{-8}}}, \frac{3-3\lambda\times 10^{-12}(1+6\ln(\frac{10^6}{\sqrt{2}}))}{\sqrt{3}e^{\frac{10^6}{\sqrt{2}}}}, \frac{3-3\lambda\times 10^{-12}(1+6\ln(10^6))}{2\sqrt{3}e^{10^6}}\Bigg\}, \frac{\frac{9}{2(4\sqrt{3}+1)}-3\lambda(\frac{1}{2}-10^{-48})}{2\sqrt{3}e^{10^{-8}}10^{-16}}\Bigg\}\Bigg),$$

the problem (3.11) possesses at least three solutions u_1, u_2 and u_3 such that

$$\max_{t\in[0,1]}|u_1(t)| \leq 10^{-8}, \max_{t\in[0,1]}|u_2(t)| \leq \frac{1}{\sqrt{2}}10^6 \text{ and } \max_{t\in[0,1]}|u_3(t)| \leq 10^6.$$

Let $A(t)$ be a primitive of $a(t)$, $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ an L^1-Carathéodory function. Put

$$G(t,\xi) = \int_0^{\xi} g(t,x)dx \quad , \quad \overline{k} := \frac{6}{e^{\|a\|_1}(12+T^2\|b\|_\infty)}.$$

Moreover, let

$$\mathcal{I}^{\rho} := \sum_{j=1}^{n}\max_{|\xi|\leq\rho}(-\mathcal{I}_j(\xi)) \text{ for all } \rho > 0$$

and

$$\mathcal{I}_d := e^{-\|a\|_1}\sum_{j=1}^{n}\min_{|\xi|\leq d}(-\mathcal{I}_j(\xi)) \text{ for all } d > 0.$$

In virtue of Theorem 3.1 and 3.2, we obtain the following results for problem (1.1).

Theorem 3.6. Assume that there exist four positive constants ρ_1, ρ_2, ρ_3 and d with $\frac{\rho_1}{\sqrt{2}} < d < \sqrt{k}\rho_2 < \sqrt{k}\rho_3$ where k as given by (3.1), such that

(i_1) $G(t,x) \geq 0$ for all $(t,x) \in [0,T]\times\mathbb{R}$.

(i_2)

$$\max\left\{\frac{\int_0^T e^{-A(t)}F(t,\rho_1)dt}{\rho_1^2}, \frac{\int_0^T e^{-A(t)}F(t,\rho_2)dt}{\rho_2^2}, \frac{\int_0^T e^{-A(t)}F(t,\rho_3)dt}{\rho_3^2-\rho_2^2}\right\} < \overline{k}\frac{\int_{T/4}^{3T/4} e^{-A(t)}F(t,d)dt - \int_0^T e^{-A(t)}F(t,\rho_1)dt}{d^2}.$$

Then, for every

$$\lambda \in \Lambda := \left(\frac{\frac{2e^{-\|a\|_1}}{T\overline{k}}d^2}{\int_{T/4}^{3T/4} e^{-A(t)}F(t,d)dt - \int_0^T e^{-A(t)}F(t,\rho_1)dt}, \frac{2e^{-\|a\|_1}}{T}\min\left\{\frac{\rho_1^2}{\int_0^T e^{-A(t)}F(t,\rho_1)dt}, \frac{\rho_2^2}{\int_0^T e^{-A(t)}F(t,\rho_2)dt}, \frac{\rho_3^2-\rho_2^2}{\int_0^T e^{-A(t)}F(t,\rho_3)dt}\right\}\right)$$

and for every non-negative continuous function $I : \mathbb{R} \to \mathbb{R}$ for each

$$\mu \in \Bigg(0, \min\Bigg\{\frac{1}{T}\min\Bigg\{\frac{2e^{-\|a\|_1}\rho_1^2 - \lambda T\int_0^T e^{-A(t)}F(t,\rho_1)dt}{\mathcal{I}^{\rho_1}}, \frac{2e^{-\|a\|_1}\rho_2^2 - \lambda T\int_0^T e^{-A(t)}F(t,\rho_2)dt}{\mathcal{I}^{\rho_2}}, \frac{2e^{-\|a\|_1}(\rho_3^2-\rho_2^2) - \lambda T\int_0^T e^{-A(t)}F(t,\rho_3)dt}{\mathcal{I}^{\rho_3}}\Bigg\}, \frac{\frac{2e^{-\|a\|_1}}{T\overline{k}}d^2 - \lambda(\int_{T/4}^{3T/4} e^{-A(t)}F(t,d)dt - \int_0^T e^{-A(t)}F(t,\rho_1)dt)}{\mathcal{I}_d - \mathcal{I}^{\rho_1}}\Bigg\}\Bigg),$$

the problem (1.1) admits at least three solutions u_1, u_2 and u_3 such that

$$\|u_1\|_\infty \leq \rho_1, \|u_2\|_\infty \leq \rho_2 \text{ and } \|u_3\|_\infty \leq \rho_3.$$

Proof . As see in section 2, we put $p(t) = e^{-A(t)}, q(t) = b(t)e^{-A(t)}$ and $f(t,x) = g(t,x)e^{-A(t)}$ where $A(t) = \int_0^t a(\tau)d\tau$ for all $t \in [0,T]$. It is clear that $F(t,x) = e^{-A(t)}G(t,x)$ and $p^* = e^{-\|a\|_1}$. Hence, from Theorem 3.1 the conclusion is achieved. □

Theorem 3.7. Assume that there exist three constants ρ_1, ρ_4 and d with $\rho_1 < \sqrt{2}d < \sqrt{k}\rho_4$, such that

(i_1) $G(t,x) \geq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}$.

(i_3)
$$\max\left\{\frac{\int_0^T e^{-A(t)}F(t,\rho_1)dt}{\rho_1^2}, \frac{\int_0^T e^{-A(t)}F(t,\rho_4)dt}{\rho_4^2}\right\} < \frac{1}{2}\frac{\overline{k}}{1+\overline{k}}\frac{\int_{T/4}^{3T/4} e^{-A(t)}F(t,d)dt}{d^2}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{2e^{-\|a\|_1}(1+\overline{k})}{T\overline{k}}d^2}{\int_{T/4}^{3T/4} e^{-A(t)}F(t,d)dt}, \frac{2e^{-\|a\|_1}}{T}\min\left\{\frac{\rho_1^2}{\int_0^T e^{-A(t)}F(t,\rho_1)dt}, \frac{\rho_4^2}{2\int_0^T e^{-A(t)}F(t,\rho_4)dt},\right\}\right)$$

and for every non-negative continuous function $I : \mathbb{R} \to \mathbb{R}$ for each

$$\mu \in \left(0, \min\left\{\frac{1}{T}\min\left\{\frac{2e^{-\|a\|_1}\rho_1^2 - \lambda T\int_0^T e^{-A(t)}F(t,\rho_1)dt}{\mathcal{I}^{\rho_1}}\right.\right.\right.$$
$$\left., \frac{e^{-\|a\|_1}\rho_4^2 - \lambda T\int_0^T e^{-A(t)}F(t,\frac{1}{\sqrt{2}}\rho_2 4)dt}{\mathcal{I}^{\frac{1}{\sqrt{2}}\rho_4}}, \frac{e^{-\|a\|_1}\rho_4^2 - \lambda T\int_0^T e^{-A(t)}F(t,\rho_4)dt}{\mathcal{I}^{\rho_4}}\right\}$$
$$\left.\left., \frac{\frac{2e^{-\|a\|_1}}{Tk}d^2 - \lambda(\int_{T/4}^{3T/4} e^{-A(t)}F(t,d)dt - \int_0^T e^{-A(t)}F(t,\rho_1)dt)}{\mathcal{I}_d - \mathcal{I}^{\rho_1}}\right\}\right)$$

the problem (1.1) admits at least three solutions $u_i (i = 1,2,3)$ such that

$$\|u_1\|_\infty \leq \rho_1, \|u_2\|_\infty \leq \frac{1}{\sqrt{2}}\rho_4 \text{ and } \|u_3\|_\infty \leq \rho_4.$$

Now, we give an application of Theorem 3.7.

Example 3.8. Consider the following problem

$$\begin{cases} -u''(t) + u'(t) + u(t) = \lambda f(t,u(t)), & t \in [0,1], t \neq t_j, \\ u(0) = u(1) = 0, \\ \Delta u'(t_j) = \mu(-2u), & j = 1,2,...,n, \end{cases} \tag{3.12}$$

where

$$f(t,x) = f(x) = \begin{cases} 4x^3, & \text{if } x < 1, \\ 4, & \text{if } x = 1, \\ \frac{4}{x}, & \text{if } x > 1. \end{cases}$$

By expression of f_2, we have

$$F(t,x) = F(x) := \begin{cases} x^4, & \text{if } x < 1, \\ 4x - 3, & \text{if } x = 1, \\ 1 + 4\ln(x), & \text{if } x > 1. \end{cases}$$

Choosing $\rho_1 = \frac{1}{10}, d = 1$ and $\rho_4 = 100$, It is easy to see that all assumptions of Theorem 3.7 are satisfied. Then, for every

$$\lambda \in \left(54, \frac{400}{\sqrt{e}-1}\right)$$

and for every

$$\mu \in \left(0, \min\left\{\min\left\{\frac{400 - \lambda(\sqrt{e}-1)}{100\sqrt{e}}, \frac{200 - \lambda(1 + 4\ln(\frac{10}{\sqrt{2}}))(\sqrt{e}-1)}{50\sqrt{e}}\right.\right.\right.$$
$$\left.\left.\left., \frac{200 - \lambda(1 + 4\ln(10))(\sqrt{e}-1)}{100\sqrt{e}}\right\}, \frac{300\lambda((e^{-\frac{-1}{8}} - e^{\frac{-3}{8}}) - 10^{-4}(1 - e^{\frac{-1}{2}})) - 50}{6}\right\}\right),$$

the problem (3.12) possesses at least three solutions u_1, u_2 and u_3 such that

$$\max_{t\in[0,\frac{1}{2}]} |u_1(t)| \leq \frac{1}{10}, \max_{t\in[0,\frac{1}{2}]} |u_2(t)| \leq \frac{10}{\sqrt{2}} \text{ and } \max_{t\in[0,\frac{1}{2}]} |u_3(t)| \leq 10.$$

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