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Remarks on the paper "Best proximity point theorem in higher dimensions with an application"

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Abstract

Very recently S. Mondal et al. [Best proximity point theorem in higher dimensions with an application, Int. J. Nonlinear Anal. Appl. (2022)] introduced the concept of F_n-contractions ($n \geq 2$) and investigated the existence and uniqueness of an n-tuple best proximity point for this class of mappings in the framework of metric spaces. In this paper we prove that their main result is a straightforward consequence of the Banach contraction principle.

Keywords: Best proximity point, F_n-contraction, weak P-property, n-tupled fixed point, n-tupled best proximity point
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1 Introduction

Metric fixed point theory is an essential part of Mathematics as it gives sufficient conditions which will ensure the existence of solutions of the equation $U(x) = x$ where U is a self mapping defined on a metric space (X, d). Banach contraction principle [1] for standard metric spaces is one of the important results in metric fixed point theory and it has lot of applications in differential equations as well as integral equations for the existence of solutions.

Let A, B be two non-empty subsets of a metric space (X, d) and $Q : A \to B$ be a non-self mapping. A necessary condition, to guarantee the existence of solutions of the equation $Qx = x$, is $Q(A) \cap A \neq \phi$. If $Q(A) \cap A = \phi$ then the mapping Q has no fixed points. In this case, one seek for an element in the domain space whose distance from its image is minimum i.e, one interesting problem is to minimize $d(x, Qx)$ such that $x \in A$. Since $d(x, Qx) \geq dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$, so, one search for an element $x \in A$ such that $d(x, Qx) = dist(A, B)$. Best proximity point problems deal with this situation. Authors usually discover best proximity point theorems to generalize the corresponding fixed point theorems.

In [4] the authors introduced the concept of an F_n-contraction ($n \geq 2$) in metric spaces for mapping $T : A^n \to B$ (where A and B are non-empty closed subsets of a metric space (X, d)) and proved a best proximity point result [4, Theorem 3.5] for such class of mappings.

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In this article we first present a simple generalization of the Banach contraction principle and then we show that the main existence and uniqueness result of an n-tuple best proximity point for non-self F_n-contraction ([4, Theorem 3.5]) can be obtained from the corresponding fixed point result.

2 Preliminaries

We first recall some definitions from [4] which will be needed throughout this paper.

Definition 2.1. [4] Let A, B be two non-empty closed subsets of a metric space (X, d) and $T : A^n \to B$ be a non-self mapping. Then T is said to be an F_n-contraction ($n \geq 2$) if there exists $0 \leq k < 1$ such that

$$d\big(T(x^1, x^2, \dots, x^n), T(y^1, y^2, \dots, y^n)\big) \leq \frac{k}{n}\Big\{d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)\Big\},$$

holds for all $(x^1, x^2, \dots, x^n), (y^1, y^2, \dots, y^n) \in A^n$. In special case if $A = B$, then we say that T is an F_n-contraction self-mapping.

Definition 2.2. [4] Let A, B be two non-empty subsets of a metric space (X, d) and $T : A^n \to A$ be a mapping. A point $(x^1, x^2, \dots, x^n)$ is called an n-tupled fixed point of the mapping T if

$$T(x^i, x^{i+1}, \dots, x^{i-1}) = x^i$$

holds for all $i = 1, 2, \dots, n$.

Definition 2.3. [4] Let A, B be two non-empty subsets of a metric space (X, d) and $T : A^n \to B$ be a mapping. A point $(x^1, x^2, \dots, x^n)$ is called an n-tupled best proximity point of the mapping T if

$$d(x^i, T(x^i, x^{i+1}, \dots, x^{i-1})) = dist(A, B)$$

holds for all $i = 1, 2, \dots, n$.

Let A, B be two non-empty subsets of a metric space (X, d). Recall the following notations from [4].

$$A_0 = \{x \in A : d(x, y) = dist(A, B), \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = dist(A, B), \text{ for some } x \in A\}.$$

The following notion was first introduced in [2].

Definition 2.4. Let A, B be two non-empty subsets of a metric space (X, d). The pair (A, B) with $A_0 \neq \emptyset$ is said to have the weak P-property if for every $u_1, u_2 \in A_0$ and $v_1, v_2 \in B_0$, we have

$$\left.\begin{aligned} d(u_1, v_1) = dist(A, B), \\ d(u_2, v_2) = dist(A, B), \end{aligned}\right\} \Longrightarrow d(u_1, u_2) \leq d(v_1, v_2).$$

It was announced in [3] that every nonempty, bounded, closed and convex pair of subsets of a reflexive and Busemann convex space X has the weak P-property (see Lemma 4.3 of [3]).

Here we state the main result of [4].

Theorem 2.5. [4, Theorem 3.5] Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d) such that A_0 is non-empty and $T : A^n \to B$ is a mapping. Assume that the following conditions are satisfied:
(i) $T(A_0^n) \subseteq B_0$;
(ii) the pair (A, B) satisfies the weak P-property;
(iii) T is an F_n-contraction.
Then T has a unique n-tuple best proximity point in A^n.

3 Main results

We begin our main conclusions with the following extension of the Banach contraction principle for an n-tuple fixed point of F_n-contraction self-mappings.

Theorem 3.1. Let (X,d) be a complete metric space and $S : X^n \to X$ be an F_n-contraction in the sense of Definition 2.1. Then S has a unique n-tuple fixed point.

Proof . It is easy to see that if $(x^1, x^2, \ldots, x^n) \in X^n$ is an n-tuple fixed point of the mapping S then we must have $x^1 = x^2 = \cdots = x^n$ (see Lemma 3.4 of [4] for more details). Since the metric space (X,d) is complete, so the product space X^n is also complete with respect to the product metric d_∞ defined by

$$d_\infty\big((x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n)\big) := \max\{d(x^i, y^i) : 1 \le i \le n\}.$$

Let us define a mapping $f : X \to X^n$ by

$$f(x) = (x, x, \ldots, x), x \in X.$$

Now consider a self-mapping $foS : X^n \to X^n$. Then for any $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$ we have

$$d_\infty\big((foS)(x^1, x^2, \ldots, x^n), (foS)(y^1, y^2, \ldots, y^n)\big) = d\big(S(x^1, x^2, \ldots, x^n), S(y^1, y^2, \ldots, y^n)\big).$$

Thus

$$\begin{aligned} &d_\infty\big((foS)(x^1, x^2, \ldots, x^n), (foS)(y^1, y^2, \ldots, y^n)\big) \\ &\le \frac{k}{n}\Big\{d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)\Big\}, \end{aligned}$$

and so

$$\begin{aligned} &d_\infty\big((foS)(x^1, x^2, \ldots, x^n), (foS)(y^1, y^2, \ldots, y^n)\big) \\ &\le k d_\infty((x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n)). \end{aligned}$$

This implies that the self-mapping foS is a contraction with the contractive constant $k \in [0, 1)$. It now follows from the Banach contraction principle that foS has a unique fixed point, call $(e^1, e^2, \ldots, e^n) \in X^n$, that is,

$$(foS)(e^1, e^2, \ldots, e^n) = (e^1, e^2, \ldots, e^n).$$

Hence

$$\big(S(e^1, e^2, \ldots, e^n), S(e^1, e^2, \ldots, e^n), \ldots, S(e^1, e^2, \ldots, e^n)\big) = (e^1, e^2, \ldots, e^n),$$

which ensures that $x := e^1 = e^2 = \cdots = e^n = S(e^1, e^2, \ldots, e^n)$. Therefore, $S(x, \ldots, x) = x$ and so $(x, \ldots, x) \in X^n$ is an n-tuple fixed point of S. The uniqueness of an n-tuple fixed point of S is trivial. □

We are now ready to state the main result of this paper.

Theorem 3.2. Theorem 2.5 is a straightforward consequence of Theorem 3.1.

Proof . Let $y \in \bar{B_0}$. Then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subseteq B_0$ such that $y_n \to y$ as $n \to \infty$. Since $y_n \in B_0$, so, there exists $x_n \in A_0$ such that $d(x_n, y_n) = dist(A, B)$. Let $m, n \in \mathbb{N}$. Since,

$$d(x_m, y_m) = dist(A, B),$$

$$d(x_n, y_n) = dist(A, B)$$

and the pair (A, B) has the weak P-property, so, we have

$$d(x_m, x_n) \le d(y_m, y_n).$$

By the fact that the sequence (y_n) is Cauchy, so the sequence (x_n) is also Cauchy. As A is complete, there exists $x \in A$ such that $x_n \to x$ as $n \to \infty$. From the continuity property of the metric d it can be seen that $d(x, y) = dist(A, B)$. Thus $y \in B_0$. This shows that the set B_0 is closed and hence, complete. Also, from the weak P-property of the pair (A, B), it can be easily seen that for every $s \in B_0$ there exists unique $t \in A_0$ such that $d(t, s) = dist(A, B)$.

Let $(p^1, p^2, \ldots, p^n) \in B_0^n$. Now, for every $p^i \in B_0$ there exists unique $t^i \in A_0$ such that $d(t^i, p^i) = dist(A, B)$ for all $i = 1, 2, \ldots, n$. Let us define a mapping $S : B_0^n \to B_0$ by

$$S(p^1, p^2, \ldots, p^n) = T(t^1, t^2, \ldots, t^n),$$

where for all $i = 1, 2, \ldots, n$, there exists unique $t^i \in A_0$ such that $d(t^i, p^i) = dist(A, B)$. So, the mapping S is well defined. Let $(p^1, p^2, \ldots, p^n), (q^1, q^2, \ldots, q^n) \in B_0^n$. Corresponding to $(p^1, p^2, \ldots, p^n)$, $(q^1, q^2, \ldots, q^n) \in B_0^n$ there exists unique $(t^1, t^2, \ldots, t^n), (s^1, s^2, \ldots, s^n) \in A_0^n$ such that $d(t^i, p^i) = dist(A, B)$ and $d(s^i, q^i) = dist(A, B)$ for all $i = 1, 2, \ldots, n$. From the weak P-property of the pair (A, B) it implies that $d(t^i, s^i) \leq d(p^i, q^i)$ for all $i = 1, 2, \ldots, n$. We now have

$$\begin{aligned} d\big(S(p^1, p^2, \ldots, p^n), S(q^1, q^2, \ldots, q^n)\big) &= d\big(T(t^1, t^2, \ldots, t^n), T(s^1, s^2, \ldots, s^n)\big) \\ &\leq \frac{k}{n}\Big\{d(t^1, s^1) + d(t^2, s^2) + \cdots + d(t^n, s^n)\Big\} \\ &\leq \frac{k}{n}\Big\{d(p^1, q^1) + d(p^2, q^2) + \cdots + d(p^n, q^n)\Big\}, \end{aligned}$$

where $0 \leq k < 1$. This shows that the mapping S is an F_n-contraction. So, by Theorem 3.1, S has a unique n-tuple fixed point, say, $(x^1, x^2, \ldots, x^n) \in B_0^n$. So,

$$S(x^i, x^{i+1}, \ldots, x^{i-1}) = x^i, \text{ for all } i = 1, 2, \ldots, n.$$

Corresponding to $(x^1, x^2, \ldots, x^n) \in B_0^n$, consider a unique element $(z^1, z^2, \ldots, z^n) \in A_0^n$ for which $d(z^i, x^i) = dist(A, B)$ for all $i = 1, 2, \ldots, n$. This implies that

$$d(z^i, S(x^i, x^{i+1}, \ldots, x^{i-1})) = dist(A, B) \text{ for all } i = 1, 2, \ldots, n,$$

and hence,

$$d(z^i, T(z^i, z^{i+1}, \ldots, z^{i-1})) = dist(A, B) \text{ for all } i = 1, 2, \ldots, n.$$

This shows that $(z^1, z^2, \ldots, z^n) \in A_0^n$ is an n-tuple best proximity point of the mapping T. Uniqueness of n-tuple best proximity point is already shown in [4, Theorem 3.5], so omitted. □

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