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Generalized Euler-Lagrange (m, n)-cubic functional equation and its stability

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Abstract

In this article, we introduce a new generalized Euler-Lagrange (m, n)-cubic functional equation with its general solution. Also, we investigate its various stabilities related to the Ulam problem in fuzzy normed linear spaces using Felbin's concept. Also, we provide a counter-example for the non-existence of stability of a generalized Euler-Lagrange (m, n)-cubic functional equation.

Keywords: Euler-Lagrange cubic functional equation, Hyers-Ulam stability, generalized Hyers-Ulam- stability, Fuzzy normed linear space
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1 Introduction and preliminaries

One of the famous question concerning the stability of homomorphisms was raised by Ulam [21] in 1940. Hyers [5] provided a partial answer to Ulam's question in 1941 and then a generalized solution to Ulam's question was given by Th.M. Rassias [17] in 1978, which is called Hyers-Ulam-Rassias stability or generalized Hyers-Ulam stability. A generalization of Hyers stability result was given by J.M. Rassias. Later, J.M. Rassias [18] discussed the stability using mixed powers of norms.

Ravi *et al.* [19] introduced the following special cubic functional equation

$$3q(u + 3v) - q(3u + v) = 12[q(u + v) + q(u - v)] + 80q(v) - 48q(u)$$

with its general solution and various stabilities. Also, Ravi *et al.* [20] studied the stability of a generalized radical reciprocal quadratic functional equation in Felbin's space. Rassias [15, 14] investigated the stability of Euler-Lagrange type quadratic functional equation

$$q(ru + sv) + q(su - rv) = (r^2 + s^2)\,[q(u) + q(v)] \tag{1.1}$$

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for fixed real numbers r, s with $r \neq 0, s \neq 0$. Jun and Kim [6] introduced the generalized Euler-Lagrange type cubic functional equation

$$q(au+bv)+q(bu+av)=(a+b)(a-b)^2\,[q(u)+q(v)]+ab(a+b)q(u+v) \tag{1.2}$$

for fixed integers a, b with $a \neq 0, b \neq 0, a \pm b \neq 0$ and its solution of an Euler-Lagrange type cubic mappings [15, 14].

Felbin [3] introduced the concept of a fuzzy normed linear space and then Xiao and Zhu [24] studied linear topological structures and some basic properties of a fuzzy normed linear space.

Definition 1.1. [3] A fuzzy subset η on $\mathbb{R}$ is called a fuzzy real number whose α-level set is denoted by $[\eta]_\alpha$, i.e., $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$ if it satisfies two axioms:

(N_1) there exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$;

(N_2) for each $\alpha \in (0,1], [\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$ where $-\infty < \eta_\alpha^- \leq \eta_\alpha^+ < +\infty$.

The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a nonnegative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all nonnegative fuzzy real numbers.

The number $\bar{0}$ stands for a fuzzy real number defined as

$$\bar{0}(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

Definition 1.2. [3] Fuzzy arithmetic operations $\oplus, \ominus, \otimes, \oslash$ on $F(\mathbb{R}) \times F(\mathbb{R})$ can be defined as:

(1) $(\eta \otimes \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(t-s)\}$, $t \in \mathbb{R}$,

(2) $(\eta \ominus \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(s-t)\}$, $t \in \mathbb{R}$,

(3) $(\eta \otimes \delta)(t) = \sup_{s \in \mathbb{R}, s \neq 0} \{\eta(s) \wedge \delta(t/s)\}$, $t \in \mathbb{R}$,

(4) $(\eta \oslash \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(st) \wedge \delta(s)\}$, $t \in \mathbb{R}$.

The additive and multiplicative identities in $F(\mathbb{R})$ are $\bar{0}$ and $\bar{1}$, respectively. Let $\ominus\eta$ be defined as $\bar{0} - \eta$. It is clear that $\eta \ominus \delta = \eta \oplus (\ominus\eta)$.

Definition 1.3. [3] For $k \in \mathbb{R}/0$, fuzzy scalar multiplication $k \odot \eta$ is defined as $(k \odot \eta)(t) = \eta(t/k)$ and $0 \odot \eta$ is defined to be $\bar{0}$.

Definition 1.4. [24] Let X be a real linear space and L and R (respectively, the left norm and the right norm) be symmetric and non-decreasing mappings from $[0;1] \times [0;1]$ into $[0;1]$ satisfying $L(0;0) = 0$, $R(1;1) = 1$. Then $\|\cdot\|$ is called a fuzzy norm and $(X, \|\cdot\|, L, R)$ is a fuzzy normed linear space (abbreviated to FNLS) if the mapping $\|\cdot\|$ from X into $F^*(R)$ satisfies the following axioms, where $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ for $x \in X$ and $\alpha \in (0;1]$:

$(N1)$ $\|x\| = \bar{0}$ if and only if $x = 0$,

$(N2)$ $\|rx\| = |r| \odot \|x\|$ for all $x \in X$ and $r \in (-\infty, \infty)$,

$(N3)$ For all $x, y \in X$,

$(N3L)$ if $s \leq \|x\|_1^-, t \leq \|y\|_1^-$ and $s+t \leq \|x+y\|_1^-$, then $\|x+y\|(s+t) \geq L\big(\|x\|(s), \|y\|(t)\big)$,

$(N3R)$ if $s \geq \|x\|_1^-, t \geq \|y\|_1^-$ and $s+t \geq \|x+y\|_1^-$, then $\|x+y\|(s+t) \leq R\big(\|x\|(s), \|y\|(t)\big)$.

Lemma 1.5. [25] Let $(X, \|\cdot\|, L, R)$ be an FNLS, and suppose that

$(R1)$ $R(a,b) \leq \max(a,b)$,

$(R2)$ $\forall\alpha \in (0,1], \exists\beta(0,\alpha]$ such that $R(\beta, y) \leq \alpha$ for all $y \in (0,\alpha)$,

$(R3)$ $\lim_{a\to 0^+} R(a,a) = 0$.

Then $(R1) \Rightarrow (R2) \Rightarrow (R3)$, but not conversely.

Lemma 1.6. [25] Let $(X, \|\cdot\|, L, R)$ be an FNLS. Then we have the following:

(1) If $R(a,b) \leq \max(a,b)$, then for all $\alpha \in (0,1]$, $\|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$ for all $x, y \in X$.

(2) If $(R2)$, then for each $\alpha \in (0,1]$, there is $\beta \in (0,\alpha]$ such that $\|x+y\|_\alpha^+ \leq \|x\|_\beta^+ + \|y\|_\alpha^+$ for all $x, y \in X$.

(3) If $\lim_{a\to 0^+} R(a,a) = 0$, then for each $\alpha \in (0,1]$ there is $\beta \in (0,\alpha]$ such that $\|x+y\|_\alpha^+ \leq \|x\|_\beta^+ + \|y\|_\beta^+$ for all $x, y \in X$.

Lemma 1.7. [25] Let $(X, \|\cdot\|, L, R)$ be an FNLS, and suppose that

$(L1)$ $L(a,b) \geq \min(a,b)$,

$(L2)$ $\forall\alpha \in (0,1], \exists\beta(\alpha,1]$ such that $L(\beta,\gamma) \geq \alpha$ for all $\gamma \in (\alpha,1)$,

$(L3)$ $\lim_{a\to 1^-} L(a,a) = 1$.

Then $(L1) \Rightarrow (L2) \Rightarrow (L3)$, but not conversely.

Lemma 1.8. [25] Let $(X, \|\cdot\|, L, R)$ be an FNLS. Then we have the following:

(1) If $L(a,b) \geq \min(a,b)$, then for all $\alpha \in (0,1]$, $\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$ for all $x, y \in X$.

(2) If $(L2)$ holds, then for each $\alpha \in (0,1]$ there is $\beta \in (\alpha,1]$ such that $\|x+y\|_\alpha^- \leq \|x\|_\beta^- + \|y\|_\alpha^-$ for all $x, y \in X$.

(3) If $\lim_{a\to 1^-} L(a,a) = 1$, then for each $\alpha \in (0,1]$ there is $\beta \in (\alpha,1]$ such that $\|x+y\|_\alpha^- \leq \|x\|_\beta^- + \|y\|_\beta^-$ for all $x, y \in X$.

Lemma 1.9. [24] Let $(X, \|\cdot\|, L, R)$ be an FNLS.

(1) If $R(a,b) \geq \max(a,b)$ and for all $\alpha \in (0,1]$, $\|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$ for all $x, y \in X$, then (N3R) holds.

(2) If $L(a,b) \leq \min(a,b)$ and for all $\alpha \in (0,1]$, $\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$ for all $x, y \in X$, then (N3L) holds.

Definition 1.10. [24] Let $(X, \|\cdot\|, L, R)$ be an FNLS and $\lim_{a\to 0^+} R(a,a) = 0$. A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ converges to $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$ for every $\alpha \in (0,1]$, and is called a Cauchy sequence if $\lim_{m,n\to\infty} \|x_m - x_n\|_\alpha^+ = 0$ for every $\alpha \in (0,1]$. A subset $A \subseteq X$ is said to be complete if every Cauchy sequence in A converges in A. A fuzzy normed space $(X, \|\cdot\|, L, R)$ is said to be a fuzzy Banach space if it is complete.

Lemma 1.11. Let $(X, \|\cdot\|, L, R)$ be an FNLS satisfying $(R2)$.

(1) For each $\alpha \in (0,1]$, $\|\cdot\|_\alpha^+$ is a continuous function from X into $\mathbb{R}$ at $x \in X$.

(2) For any $n \in \mathbb{Z}^+$ and $\{x_i\}_{i=1}^n$ and for all $\alpha \in (0,1]$, there exists $\beta \in (0,\alpha]$; $\|\sum_{i=1}^n x_i\|_\alpha^+ \leq \sum_{i=1}^n \|x_i\|_\beta^+$.

Eskandani and Rassias [2] investigated the generalized Hyers-Ulam stability of a general cubic functional equation in Felbin's type fuzzy normed linear spaces and some applications. Narasimman *et al.* [10] derived a general solution of a new n-dimensional quintic and sextic functional equations with its various stabilities in Felbin type fuzzy normed linear spaces. Wang [22] introduced the notion of Felbin's type fuzzy normed linear spaces, and studied the more general cubic functional equation

$$q(u+kv) + q(u-kv) + q(ku) = k^2 q(u+v) + k^2 q(u-v) + (k^3 - 2k^2 + 2)q(u)$$

for $k \geq 2$ in Felbin's type fuzzy normed linear spaces. Recently, the stability problems of several functional equations have been extensively investigated by a number of authors [1, 4, 7, 8, 9, 11, 12, 13, 16].

From the above literetures, we introduce a new generalized Euler-Lagrange $(m, n)-$cubic functional equation of the form

$$m\left[q(mu+nv)+q(nu+mv)\right]+n\left[q(mu-nv)+q(mv-nu)\right] \\ =8mn(m^2+n^2)q\left(\frac{u+v}{2}\right)+(m^4-n^4)\left[q(u)+q(v)\right] \tag{1.3}$$

for integers $m \neq 0, n \neq 0, mn \neq 0, m \neq n$ and $t = m+n \neq 0, \pm 1$ and investigate various stabilities related to Ulam problem in fuzzy normed linear spaces using Felbin's concept with a suitable counter-example for non-stability of generalized Euler-Lagrange (m, n)-cubic functional equation.

This paper is organized as follows: In Section 2, we obtain the general solution of the functional equation (1.3). In Section 3 and in Section 4, we discuss various stabilities in fuzzy normed linear spaces using Felbin concept in direct method and fixed point method, respectively. Finally, the conclusion is given in Section 5.

2 General solution of generalized Euler-Lagrange (m, n)-cubic functional equation

Theorem 2.1. Let U and V be real vector spaces. A mapping $q : U \to V$ satisfies the generalized cubic Euler-Lagrange-Jensen functional equation (1.3) if and only if it satisfies the functional equation (1.2).

Proof . Suppose a mapping $q : U \to V$ satisfies (1.3). Putting $u = v = 0$ in (1.3), we get $q(0) = 0$. Letting $v = 0$ in (1.3), we obtain

$$q(mu) - m^3 q(u) = n^3 q(u) - q(nu),$$

which implies

$$q(mu) = m^3 q(u) \text{ if and only if } q(nu) = n^3 q(u)$$

for all $u \in U$. Setting $u = 0$ in (1.3), we get $q(-v) = q(v)$ and hence q is odd. Replacing u by v and v by u in (1.3) and again adding the resultant with (1.3) and using the oddness of q, we get (1.2).

Conversely, assume q satisfies the functional equation (1.2). Replacing v by $-v$ in (1.2) and again adding the resultant with (1.2) and using the oddness of q, we get (1.3). □

Theorem 2.2. [26] A mapping $q : U \to V$ is a solution of the functional equation (1.3) if and only if q is of the form $q(u) = A_3(u, u, u)$ for all $u \in U$, where $A_3 : U^3 \to V$ is additive for each variable and is symmetric for each fixed variable.

3 Generalized Hyers-Ulam stability of a generalized Euler-Lagrange (m, n)-cubic functional equation: Direct method

Let us denote

$$D_q(u,v) = q(mu+nv)+q(mu-nv)+q(nu+mv)+q(nu-mv) \\ -2(m+n)(m-n)^2 q(u) - mn(m+n)\left[q(u+v)+q(u-v)\right]$$

for all $u, v \in U$ and integers $m, n \neq 0, mn \neq 0, m \neq n,\ m+n \neq 0, \pm 1$.

Theorem 3.1. Let U be a linear space and $(V, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R2)$. Let $q : U \to V$ be a mapping for which there exists a function $\varphi : U \times U \to F^*(\mathbb{R})$ such that

$$\sum_{i=0}^{\infty} \frac{\left(\phi(m^i u, m^i v)\right)_{\alpha}^{+}}{m^{3i}} \prec \infty \tag{3.1}$$

for all $u, v \in U$,

$$\|D_q(u,v)\| \leq \phi(u,v) \tag{3.2}$$

for all $u,v\in U$ and all $\alpha\in(0,1]$. Then there exists a unique cubic mapping $T:U\to V$ and for all $\alpha\in(0,1]$, there exists $\beta\in(0,\alpha]$ such that

$$\|q(u)-T(u)\|_{\alpha}^{+}\leq\frac{1}{2m^3}\sum_{i=0}^{\infty}\frac{\left(\phi(m^iu,0)\right)_{\beta}^{+}}{m^{3i}},\quad\forall\ u\in U\tag{3.3}$$

if and only if

$$\|q(u)-D(u)\|_{\alpha}^{+}\leq\frac{1}{2n^3}\sum_{i=0}^{\infty}\frac{\left(\phi(n^iu,0)\right)_{\beta}^{+}}{n^{3i}},\quad\forall\ u\in U,\tag{3.4}$$

where

$$T(u):=\lim_{k\to\infty}\frac{q(m^ku)}{m^{3k}},\quad D(u):=\lim_{k\to\infty}\frac{q(n^ku)}{n^{3k}}.$$

Proof . Setting $v=0$ in (3.2), we get

$$\left\|2q\left(mu\right)-2m^3q\left(u\right)\right\|\leq\phi\left(u,0\right)\tag{3.5}$$

if and only if

$$\left\|2q\left(nu\right)-2n^3q\left(u\right)\right\|\leq\phi\left(u,0\right)\tag{3.6}$$

for all $u\in U$. Multiplying both sides of (3.5) by $\frac{1}{2}$ in the fuzzy scalar multiplication sense, we have

$$\left\|q\left(mu\right)-m^3f\left(u\right)\right\|\leq\frac{1}{2}\odot\phi\left(u,0\right)\tag{3.7}$$

for all $u\in U$. Replacing u by m^ku and multiplying by $\frac{1}{m^{3k+3}}$ in (3.7) in the fuzzy scalar multiplication sense, we obtain

$$\left\|\frac{q\left(m^{k+1}u\right)}{m^{3(k+1)}}-\frac{q\left(m^ku\right)}{m^{3k}}\right\|\leq\frac{1}{2m^3}\frac{1}{m^{3k}}\odot\phi\left(m^ku,0\right)\tag{3.8}$$

for all $u\in U$ and all nonnegative integers $m\in\mathbb{N}$. By Lemma 1.11 and (3.8), we conclude that for all $\alpha\in(0,1]$ there exists $\beta\in(0,\alpha]$ such that

$$\left\|\frac{q\left(m^{k+1}x\right)}{m^{3(k+1)}}-\frac{q\left(m^lu\right)}{m^{3l}}\right\|_{\alpha}^{+}\leq\frac{1}{2m^3}\sum_{i=l}^{k}\frac{1}{m^{3i}}\left(\phi\left(m^iu,0\right)\right)_{\beta}^{+}\tag{3.9}$$

for all $u\in U$ and all nonnegative integers l and k with $k\geq l$. Now (3.1) and (3.9) imply that $\{\frac{q(m^ku)}{m^{3k}}\}$ is a fuzzy Cauchy sequence in V for all $u\in U$. Since V is a fuzzy Banach space, the sequence $\{\frac{q(m^ku)}{m^{3k}}\}$ converges for all $u\in U$. So we can define the mapping $T:U\to V$ by

$$T(u):=\lim_{k\to\infty}\frac{q(m^ku)}{m^{3k}}$$

for all $u\in U$. Letting $l=0$ and passing to the limit as $k\to\infty$ in (3.9), by the continuity of $\|\cdot\|_{\alpha}^{+}$, we have

$$\|q(u)-T(u)\|_{\alpha}^{+}\leq\frac{1}{2m^3}\sum_{i=0}^{\infty}\frac{\left(\phi(m^iu,0)\right)_{\beta}^{+}}{m^{3i}}$$

for all $u\in U$. Therefore we obtain (3.3). Starting the process from (3.6) instead (3.5), we obtain (3.4). Now we show that T is cubic and unique. Applying (3.1), (3.2) and the continuity of $\|\cdot\|_{\alpha}^{+}$, we have

$$\begin{aligned}&\big|\big|T(mu+nv)+T(mu-nv)+T(nu+mv)+T(nu-mv)\\&\quad-2(m+n)(m-n)^2T(u)-mn(m+n)\left[T(u+v)+T(u-v)\right]\big|\big|_{\alpha}^{+}\\&\leq\lim_{k\to\infty}\frac{\left(\phi(m^ku,m^kv)\right)_{\alpha}^{+}}{m^{3k}}=0\end{aligned}$$

for all $u, v \in U$. Therefore the mapping $T : U \to V$ is cubic. To prove the uniqueness of T, let $T' : U \to V$ be a cubic mapping satisfying (3.3). Then by Lemma 1.11

$$\begin{aligned}\|T(u) - T'(u)\| &\leq \lim_{k\to\infty} \frac{1}{m^{3k}} \frac{1}{2m^3} \sum_{i=0}^{\infty} \frac{\big(\phi(m^i m^k u, 0)\big)^+_\beta}{m^{3i}} \\ &\leq \lim_{k\to\infty} \frac{1}{2m^3} \sum_{i=k}^{\infty} \frac{\big(\phi(m^i u, 0)\big)^+_\beta}{m^{3i}} = 0\end{aligned}$$

for all $u \in U$, $T = T'$. Hence T is unique. This completes the proof. □

Remark 3.2. The above theorem is also true if $\|\cdot\|^+_\alpha$ is replaced by $\|\cdot\|^-_\alpha$ in (3.1) and the fuzzy Banach space V satisfies $(L2)$ and $(R2)$.

The following theorem is an alternative result of Theorem 3.1.

Theorem 3.3. Let U be a linear space and $(V, \|\cdot\|, L, R)$ be a fuzzy Banach space such that $R(m,n) \leq \max(m,n)$ and $L(m,n) \geq \min(m,n)$. Let $q : U \to V$ be a mapping for which there exists a function $\varphi : U \times U \to F^*(\mathbb{R})$ satisfying (3.1) and (3.2) for all $u, v \in U$ and all $\alpha \in (0,1]$. Then there exists a unique cubic mapping $T : U \to V$ such that

$$\|q(u) - T(u)\| \leq \bar{\varphi}(u, 0)$$

for all $u \in U$, where $\bar{\varphi}(u,0)$ is a fuzzy real number generated by the families of nested bounded closed intervals $[t_\alpha, l_\alpha]$ such that

$$\begin{aligned} t_\alpha &= \frac{1}{2m^3} \sum_{i=0}^{\infty} \frac{\big(\varphi(m^i u, 0)\big)^-_\alpha}{m^{3i}}, \\ l_\alpha &= \frac{1}{2m^3} \sum_{i=0}^{\infty} \frac{\big(\varphi(m^i u, 0)\big)^+_\alpha}{m^{3i}} \end{aligned}$$

for all $u \in U$.

Theorem 3.4. Let U be a linear space and $(V, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R2)$. Let $q : U \to V$ be a mapping for which there exists a function $\varphi : U \times U \to F^*(\mathbb{R})$ such that

$$\sum_{i=1}^{\infty} m^{3i} \Big(\phi\Big(\frac{u}{m^i}, \frac{v}{m^i}\Big)\Big)^+_\alpha \prec \infty \tag{3.10}$$

for all $u, v \in U$,

$$\|D_q(u, v)\| \leq \phi(u, v) \tag{3.11}$$

for all $u, v \in U$ and all $\alpha \in (0,1]$. Then there exists a unique cubic mapping $T : U \to V$ and for all $\alpha \in (0,1]$, there exists $\beta \in (0, \alpha]$ such that

$$\|q(u) - T(u)\|^+_\alpha \leq \frac{1}{2m^3} \sum_{i=1}^{\infty} m^{3i} \Big(\varphi\Big(\frac{u}{m^i}, 0\Big)\Big)^+_\beta$$

if and only if

$$\|q(u) - D(u)\|^+_\alpha \leq \frac{1}{2n^3} \sum_{i=1}^{\infty} n^{3i} \Big(\varphi\Big(\frac{u}{n^i}, 0\Big)\Big)^+_\beta$$

for all $u \in U$, where

$$T(u) := \lim_{k\to\infty} \left\{ m^{3k} q\left(\frac{u}{m^k}\right) \right\}, \quad D(u) := \lim_{k\to\infty} \left\{ n^{3k} q\left(\frac{u}{n^k}\right) \right\}.$$

Proof . Setting $v=0$ in (3.11), we get

$$\left\|2q\left(mu\right)-2m^{3}q\left(u\right)\right\|\le\varphi\left(u,0\right) \tag{3.12}$$

if and only if

$$\left\|2q\left(nu\right)-2n^{3}q\left(u\right)\right\|\le\varphi\left(u,0\right)$$

for all $u\in U$. Multiplying both sides of (3.12) by $\frac{1}{2}$ in the fuzzy scalar multiplication sense, we have

$$\left\|q\left(mu\right)-m^{3}q\left(u\right)\right\|\le\frac{1}{2}\odot\varphi\left(u,0\right) \tag{3.13}$$

for all $u\in U$. Replacing u by $\frac{u}{m^{k+1}}$ and multiplying both sides by m^{3k} in (3.13) in the fuzzy scalar multiplication sense, we obtain

$$\left\|m^{3k}q\left(\frac{u}{m^{k}}\right)-m^{3(k+1)}q\left(\frac{u}{m^{k+1}}\right)\right\|\le\frac{m^{3k}}{2}\odot\varphi\left(\frac{u}{m^{k+1}},0\right) \tag{3.14}$$

for all $u\in U$ and all nonnegative integers $m\in\mathbb{N}$. By Lemma 1.11 and (3.14), we conclude that for all $\alpha\in(0,1]$ there exists $\beta\in(0,\alpha]$ such that

$$\left\|m^{3(k+1)}q\left(\frac{u}{m^{k+1}}\right)-m^{3l}q\left(\frac{u}{m^{l}}\right)\right\|_{\alpha}^{+}\le\sum_{i=l}^{k}\frac{m^{3i}}{2}\left(\varphi\left(\frac{u}{m^{i+1}},0\right)\right)_{\beta}^{+} \tag{3.15}$$

for all $u\in U$ and all nonnegative integers l and k with $k\ge l$. Now (3.10) and (3.15) imply that $\{m^{3k}q(\frac{u}{m^{k}})\}$ is a fuzzy Cauchy sequence in V for all $u\in U$. Since V is a fuzzy Banach space, the sequence $\{m^{3k}q(\frac{u}{m^{k}})\}$ converges for all $u\in U$. The rest of this proof is similar to the proof of Theorem 3.1. □

The same discussion in Remark 3.2 does hold for the above theorem. Also, the following theorem is an alternative result of Theorem 3.4.

Theorem 3.5. Let U be a linear space and $(V,\|\cdot\|,L,R)$ be a fuzzy Banach space such that $R(m,n)\le\max(m,n)$ and $L(m,n)\ge\min(m,n)$. Let $q:U\to V$ be a mapping for which there exists a function $\varphi:U\times U\to F^{*}(\mathbb{R})$ satisfying (3.10) and (3.11) for all $u,v\in U$ and all $\alpha\in(0,1]$. Then there exists a unique cubic mapping $T:U\to V$ such that

$$\|q(u)-T(u)\|\le\bar{\varphi}(u,0)$$

for all $u\in U$, where $\bar{\varphi}(u,0)$ is a fuzzy real number generated by the families of nested bounded closed intervals $[t_{\alpha},l_{\alpha}]$ such that

$$t_{\alpha}=\frac{1}{2m^{3}}\sum_{i=1}^{\infty}m^{3i}\Big(\varphi\Big(\frac{u}{m^{i}},0\Big)\Big)_{\alpha}^{-},$$
$$l_{\alpha}=\frac{1}{2m^{3}}\sum_{i=1}^{\infty}m^{3i}\Big(\varphi\Big(\frac{u}{m^{i}},0\Big)\Big)_{\alpha}^{+}$$

for all $u\in U$.

The following corollaries are the immediate consequences of Theorems 3.1 and 3.4, which give the Hyers-Ulam and generalized Hyers-Ulam stabilities of the functional equation (1.3).

Corollary 3.6. Let ϵ be a nonnegative fuzzy real number and U be a linear space and $(V,\|\cdot\|,L,R)$ be a fuzzy Banach space such that $R(t,l)\le\max(t,l)$ and $L(t,l)\ge\min(t,l)$. Suppose that a mapping $q:U\to V$ satisfies the inequality

$$\|D_{q}(u,v)\|\le\epsilon$$

for all $u,v\in U$. Then there exists a unique cubic mapping $T:U\to V$ satisfying

$$\|q(u)-T(u)\|\le\frac{\epsilon}{2(m^{3}-1)}$$

for all $u\in U$.

Proof . Let $\varphi(u,v) := \epsilon$ for all $u,v \in U$. By Theorem 3.1, we get the desired result. $\square$

Corollary 3.7. Let ϵ be a nonnegative fuzzy real number and p,q be nonnegative real numbers such that $p,q > 3$ or $0 < p,q < 3$. Let U be a fuzzy normed linear space and $(V, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R2)$. Suppose that a mapping $q : U \to V$ satisfies the inequality

$$\|D_q(u,v)\| \le \epsilon \otimes (\|u\|_U^p \oplus \|v\|_U^q) \tag{3.16}$$

for all $u,v \in U$. Then there exists a unique cubic mapping $T : U \to V$ and for all $\alpha \in (0,1]$, there exists $\beta \in (0,\alpha]$ such that

$$\|q(u) - T(u)\|_\alpha^+ \le \frac{\epsilon_\beta^+ (\|u\|^p)_\beta^+}{2|m^p - m^3|}$$

for all $u \in U$.

Proof . The result follows from Theorems 3.1 and 3.4 by taking

$$\varphi(u,v) := \epsilon \otimes \left(\|u\|_U^p \oplus \|v\|_U^q\right)$$

for all $u,v \in U$. $\square$

Now we provide an example to illustrate that the functional equation (1.3) is not stable for $p = 3$ in Corollary 3.7.

Example 3.8. Let U be a linear space and $(V, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R2)$. Let $\varphi : U \times U \to F^*(\mathbb{R})$ be a function defined by

$$\phi(u) = \begin{cases} \epsilon \otimes u^3, & \text{if } |u| < 1, \\ \epsilon, & \text{otherwise,} \end{cases}$$

where $\epsilon > 0$ is a nonnegative fuzzy real number and a function $f : X \to Y$ is defined by

$$q(u) = \sum_{k=0}^{\infty} \frac{\varphi(m^k u)}{(m^3)^k}$$

for all $u \in U$. Then q satisfies the functional inequality

$$\begin{aligned} &\big|q(mu+nv) + q(mu-nv) + q(nu+mv) + q(nu-mv) \\ &\quad - 2(m+n)(m-n)^2 q(u) - mn(m+n)\left[q(u+v) + q(u-v)\right]\big| \\ &\quad \le \left(\frac{2m^9 + 4m^6 + 2m^6 n^3}{m^3 - 1}\right) \epsilon \left(|u|^3 + |v|^3\right) \end{aligned} \tag{3.17}$$

for all $u,v \in U$. Then there do not exist a cubic mapping $T : U \to V$ and a constant $\beta > 0$ such that

$$|q(u) - T(u)| \le \beta \otimes |u|^3 \quad \text{for all } x \in U. \tag{3.18}$$

Proof .

$$|q(u)| \le \sum_{k=0}^{\infty} \frac{|\phi(m^k u)|}{|m^{3k}|} = \sum_{k=0}^{\infty} \frac{\epsilon}{m^{3k}} = \frac{m^3\ \epsilon}{m^3 - 1}.$$

So we see that q is bounded.

Now we prove that q satisfies (3.17). If $u = v = 0$, then (3.17) is trivial. If $|u|^3 + |v|^3 \ge \frac{1}{m^3}$, then the left-hand side of (3.17) is less than $\left(\frac{2m^6 + 4m^3 + 2m^3 n^3}{m^3 - 1}\ \epsilon\right)$. Now suppose that $0 < |u|^3 + |v|^3 < \dfrac{1}{m^3}$. Then there exists a positive integer r such that

$$\frac{1}{(m^3)^{r+1}} \le |u|^3 + |v|^3 < \frac{1}{(m^3)^r} \tag{3.19}$$

and so

$$(m^3)^{r-1} u^3 < \frac{1}{m^3},\ (m^3)^{r-1} v^3 < \frac{1}{m^3}$$

and therefore for each $k = 0, 1, \cdots, r-1$, we have $m^k(u),\ m^k(u+v),\ m^k(u-v),\ m^k(mu+nv),\ m^k(mu-nv),\ m^k(nu+mv),\ m^k(nu-mv) \in (-1, 1)$ and

$$\begin{aligned}
&\varphi\big(m^k(mu+nv)\big) + \varphi\big(m^k(mu-nv)\big) + \varphi\big(m^k(nu+mv)\big) + \varphi\big(m^k(nu-mv)\big) \\
&- 2(m+n)(m-n)^2\varphi\big(m^k(u)\big) - mn(m+n)\left[\varphi\big(m^k(u+v)\big) + \varphi\big(m^k(u-v)\big)\right] = 0
\end{aligned}$$

for $k = 0, 1, \ldots, r-1$. From the definition of q and (3.19), we obtain that

$$\begin{aligned}
&\Big| q(mu+nv) + q(mu-nv) + q(nu+mv) + q(nu-mv) \\
&\quad - 2(m+n)(m-n)^2 q(u) - mn(m+n)\left[q(u+v) + q(u-v)\right] \Big| \\
&\leq \sum_{k=0}^{\infty} \frac{1}{m^{3k}} \Big| \varphi\big(m^k(mu+nv)\big) + \varphi\big(m^k(mu-nv)\big) + \varphi\big(m^k(nu+mv)\big) + \varphi\big(m^k(nu-mv)\big) \\
&\quad - 2(m+n)(m-n)^2\varphi\big(m^k(u)\big) - mn(m+n)\left[\varphi\big(m^k(u+v)\big) + \varphi\big(m^k(u-v)\big)\right] \Big| \\
&\leq \sum_{k=r}^{\infty} \frac{1}{m^{3k}} \Big| \varphi\big(m^k(mu+nv)\big) + \varphi\big(m^k(mu-nv)\big) + \varphi\big(m^k(nu+mv)\big) + \varphi\big(m^k(nu-mv)\big) \\
&\quad - 2(m+n)(m-n)^2\varphi\big(m^k(u)\big) - mn(m+n)\left[\varphi\big(m^k(u+v)\big) + \varphi\big(m^k(u-v)\big)\right] \Big| \\
&= \left(\frac{2m^6 + 4m^3 + 2m^3n^3}{m^3 - 1}\ \epsilon \right) \times \frac{1}{m^{3r}} \\
&= \frac{(2m^9 + 4m^6 + 2m^6n^3)\ \epsilon}{m^3 - 1} \left(|u|^3 + |v|^3\right).
\end{aligned}$$

Thus q satisfies (3.17) for all $u, v \in U$ with $0 < |u|^3 + |v|^3 < \dfrac{1}{m^3}$.

We claim that the cubic functional equation (1.3) is not stable for $p = 3$ in Corollary 3.7. Suppose on the contrary, there exist a cubic mapping $T : U \to V$ and a constant $\beta > 0$ satisfying (3.18). Since q is bounded and continuous for all $u \in U$, T is bounded on any open interval containing the origin and continuous at the origin. In view of Corollary 3.7, $T(u)$ must have the form $T(u) = ru^3$ for any u in U. Thus we obtain

$$|q(u)| \leq (\beta \oplus |r|) \otimes |u|^3. \tag{3.20}$$

But we can choose a positive integer l with $l\epsilon > \beta \oplus |r|$.

If $u \in \left(0, \frac{1}{m^{l-1}}\right)$, then $m^k u \in (0, 1)$ for all $k = 0, 1, \ldots, l-1$. For this u, we get

$$q(u) = \sum_{k=0}^{\infty} \frac{\phi(m^k u)}{m^{3k}} \geq \sum_{k=0}^{l-1} \frac{\epsilon(m^k u)^3}{m^{3k}} = l\epsilon u^3 > (\beta \oplus |r|) \otimes u^3,$$

which contradicts (3.20). Therefore the cubic functional equation (1.3) is not stable in the sense of Ulam, Hyers and Rassias if in the inequality (3.16) it is assumed that $p = 3$. □

We obtain the following corollary for Theorems 3.1 and 3.4 for the functional equation (1.3).

Corollary 3.9. Let ϵ be a nonnegative fuzzy real number and p, q be nonnegative real numbers such that $p, q > \frac{3}{2}$ or $0 < p, q < \frac{3}{2}$. Let U be a fuzzy normed linear space and $(V, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R2)$. Suppose that a mapping $q : U \to V$ satisfies the inequality

$$\|D_q(u, v)\| \leq \epsilon \otimes \left(\|u\|_U^p \otimes \|v\|_U^p \oplus \left[\|u\|_U^{2p} \oplus \|v\|_U^{2p} \right] \right)$$

for all $u, v \in U$. Then there exists a unique cubic mapping $T : U \to V$ and for all $\alpha \in (0, 1]$, there exists $\beta \in (0, \alpha]$ such that

$$\|q(u) - T(u)\|_\alpha^+ \leq \frac{\epsilon_\beta^+ \left(\|u\|^{2p}\right)_\beta^+}{2|m^{2p} - m^3|}$$

for all $u \in U$.

Proof . The result follows from Theorems 3.1 and 3.4 by taking

$$\varphi(u,v) := \epsilon \otimes \left(\|u\|_U^p \otimes \|v\|_U^p \oplus \left[\|u\|_U^{2p} \oplus \|v\|_U^{2p} \right] \right)$$

for all $u, v \in U$. □

4 Generalized Hyers-Ulam stability of a generalized Euler-Lagrange (m,n)-cubic functional equation: Fixed point method

Theorem 4.1. Let U be a linear space and $(V, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R2)$. Let $q : U \to V$ be a mapping for which there exists a function $\varphi : U \times U \to F^*(\mathbb{R})$ such that

$$\lim_{t\to\infty} \frac{1}{d_i^{3t}} \varphi(d_i^t u, d_i^t v)_\alpha^+ = 0$$

where

$$d_i = \begin{cases} m & if \quad i = 0, \\ \frac{1}{m} & if \quad i = 1 \end{cases}$$

such that the functional inequality

$$\|D_q(u,v)\|_\alpha^+ \preceq \varphi(u,v)_\beta^+$$

holds for all $u, v \in G$. Assume that there exists $L = L(i)$ such that the function

$$u \to T(u,0)_\beta^+ = \frac{1}{2}\varphi\left(\frac{u}{m}, 0\right)_\beta^+$$

with the property

$$\frac{1}{d_i^3} \odot T(d_i u, 0) = L \odot T(u,0)_\beta^+$$

for all $u \in G$. Then there exists a unique cubic mapping $T : U \to V$ satisfying the functional equation (1.3) and

$$\| q(u) - T(u) \| \preceq \left(\frac{L^{1-i}}{1-L}\right) T(u,0)_\beta^+$$

for all $u \in G$.

Proof . The proof is similar to the proofs of the theorems given in [23]. So we omit the proof. □

Corollary 4.2. Let ϵ be a nonnegative fuzzy real number and U be a linear space and $(V, \|\cdot\|, L, R)$ be a fuzzy Banach space such that $R(t,l) \le \max(t,l)$ and $L(t,l) \ge \min(t,l)$. Suppose that a mapping $q : U \to V$ satisfies the inequality

$$\|D_q(u,v)\|_\alpha^+ \preceq \begin{cases} \epsilon, & \\ \epsilon \otimes \{||u||^p \oplus ||v||^p\}, & p \ne 3; \\ \epsilon \otimes \left\{||u||^p \otimes ||v||^p \oplus \left\{||u||^{2p} \oplus ||v||^{2p}\right\}\right\}, & 2p \ne 3; \end{cases}$$

for all $u, v \in U$. Then there exists a unique cubic mapping $T : U \to V$ satisfying

$$\|q(u) - T(u)\|_\alpha^+ \preceq \begin{cases} \dfrac{\epsilon}{2|m^3 - 1|}, \\ \dfrac{\epsilon_\beta^+ (||u||^p)_\beta^+}{2|m^p - m^3|}, \\ \dfrac{\epsilon_\beta^+ (||u||^{2p})_\beta^+}{2|m^{2p} - m^3|} \end{cases}$$

for all $u \in U$.

5 Conclusion

In this paper, we have obtained the general solution of a new generalized Euler-Lagrange $(a, b)-$cubic functional equation and studied its generalized Hyers-Ulam and Hyers-Ulam stabilities in fuzzy normed linear space using Felbin's concept in both direct and fixed point methods. Furthermore, we gave a counter-example to illustrate the fuzzy version of the Hyers-Ulam-Rassias stability of the functional equation (1.3) for some cases.

Furthermore, one can easily observe that the functional equation (1.3) is the more generalized version of the functional equations (1.1) and (1.2).

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