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On various types of cone metric spaces and some applications in fixed point theory

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Abstract

This paper gives a study of various types of cone metric spaces and their topological characterizations. Contrarily to the case of cone metric space X, the paper shows with examples that the limit of a sequence may not be unique in the topology generated by partial cone metric T_p and (X, T_p) is not generally Hausdorff topological space and also the cone valued partial metric mapping p may not generally be continuous. Hence T_p is not equivalent to any topology generated by any metric on X. Furthermore, the paper considers some generalized contraction types of mappings on θ-complete cone metric-like spaces and then generalizes some coupled fixed point theorems of some previous results in this setting.

Keywords: Partial Metric Spaces, Cone Metric Like Spaces, Cone Metric Spaces, Partial Cone Metric Space, coupled Fixed Point Theorems
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1 Introduction

Some construction of topological models in the study of denotational semantics of programming languages (dataflow networks) is needed in such a way that a model need not be a Hausdorff space but a T_0-space. Therefore; the conditions of metric space can be reduced to result an ideal candidate of generalized metric space called partial metric space that gave the required T_0-space, see [26, 23].

In 1994, Matthews [24] used the notion of a partial metric space as a part of this study and showed that the Banach's contraction mapping theorem can be generalized to the partial metric context for applications and verification in this theoretical computer science field, the self-distance of a data presents the amount of information included in this data.

In 1997, Zabrejko [31] and afterwards in 2007 Huang and Zhang [17] introduced cone metric spaces as generalization of metric spaces and they gave a generalized fixed point theorem for contraction type mapping on a cone metric space provided that the given cone is normal.

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In 2008, Rezapour and Hamlbarani [27] improved the results of [17] by omitting the normality condition of the given cone induced the partial relation.

After that, in 2009 and then in 2010, the characterizations of the topologies that induced by cone metric spaces have been listed and surveyed by Rezapour and DuWS, as given in [16, 11] respectively; and the references therein. They linked these characterizations to fixed point theorems and have been studied by many authors.

In 2010, Khamsi [22] showed that any cone metric space is a metric like space provided that the underlying cone is normal, in his proof the metric parameter were given to equal the normal constant of the underlying normal cone. Specifically he showed the following:

Theorem 1.1. Let (X, C, q) be a cone metric space, C be a normal cone in a normed space $(\mathbb{A}, \|.\|)$. Then the composition $D(x, y) := \|d(x, y)\|$ is a metric like space with metric parameter equals to the normal constant M of C.

In 2010, Amini-Harandi and Fakhar [9] used some scalarization method and showed the following:

Theorem 1.2. Let (X, C, q) be a complete cone metric space, C be a solid cone. Then there exists a metric D on X such that (X, D) is a complete metric space and a sequence in (X, C, q) is convergent if and only if it is convergent in (X, D). Moreover; any Geraghty type mapping in (X, C, q) is Geraghty mapping in (X, D).

In 2010 Feng and Mao [13] showed that the cone metric space and metric space are equivalent and gave the following theorem:

Theorem 1.3. Let (X, C, q) be a cone metric space, C be a solid cone in a normed space $(\mathbb{A}, \|.\|)$. Then $D(x, y) := \inf\Big\{\|u\| : q(x, y) \preceq u,\ u \in C\Big\}$ is a metric on X. Moreover; the metric space (X, D) is complete if and only if (X, C, q) is complete cone metric space and any contractive mapping in (X, C, d) is contractive mapping in (X, D).

In 2011, Jankovic et al [19] and Kadelburg et al [21] studied more equivalences and characterizations of cone metric spaces.

In 2012, Cakalli et al [10] worked in the case of topological vector spaces and give the following equivalent theorem:

Theorem 1.4. Let (X, C, q) be a cone metric space, C be a solid cone in a topological vector space $(\mathbb{A}, \tau)$ with $e \in Int(C)$. Then $D(x, y) := \inf\Big\{r : r \in \mathbb{R}^+;\ q(x, y) \in r\ e - C\Big\}$ is a metric on X. Moreover; the two generated topologies are equivalent and for neighborhoods, we have

$$U_q(x, r\ e) := \{y \in X : q(x, y) \ll r\ e\} = U_D(x, r) := \{y \in X : D(x, y) \leq r\}.$$

In 2014, 2016, 2017, and then in 2020, further studies have been given to a larger category of metric spaces, like b-cone metric spaces, cone metric spaces over Banach algebra, and theta cone metric spaces, see Xu and Radenovic [30], Huang and Radenovic [18], Neetu Sharma [29], and then Sahar [2] results in fixed point theory's fields. These results are proved by omitting the normality condition of the underlying cone.

Different approaches to fixed point theory have been successfully given along decades. Notably to mention for example the results given in [12, 1, 5, 6, 7] for cone metric spaces over Banach algebra.

In addition, many authors derived fixed point theorems in partial metric spaces. Subsequently, the concept of partial metric has been generalized to the concept of partial cone metric.

Specifically, in 2013, Jiang and Li [20] extend Banach contraction principle to partial cone metric spaces over a non-normal solid cone, hence improve many fixed point results in cone metric spaces and partial metric spaces. They gave example to support the usability of their results.

In 2017, Moshokoa [25] gave a nice study on partial metric spaces and studied some of its characterizations with some fixed point applications.

On the other side, in 1987 the concept of coupled fixed point was initiated by Gue and Lakshmikantham [15], in partially ordered metric spaces, after that in 2006, Bhaskar and Lakshmikantham [14] proved existence of coupled fixed points for mappings having the mixed monotone property.

In 2009, Sabetghadam et al [28] proved some coupled fixed point theorems for mappings satisfying different contractive conditions on complete cone metric spaces. Specifically, they proved the following:

Theorem 1.5. [28] Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:

$$d(F(x, y), F(u, v)) \preceq kd(x, u) + ld(y, v),$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

In 2021, Sahar [4, 3] generalized some coupled fixed point results given by some previous researchers in theta cone metric spaces, the concept which is introduced by the author in [2] and then gave more extension of this result in both b-cone metric and b-theta cone metric spaces, respectively.

2 Preliminaries and Basic Definitions

First, we recall some standard notations and definitions in cone metric spaces.

A subset C of a linear space $\mathbb{A}$ is said to be a cone in $\mathbb{A}$ if and only if

1. C is non-empty closed and $C \neq \{\theta\}$, where θ is the zero (neutral element) of $\mathbb{A}$;
2. $\lambda C + \mu C \subset C$ for all non-negative real numbers $\lambda,\ \mu$;
3. $C \bigcap -C = \{\theta\}$.

If $(\mathbb{A}, \|.\|)$ is a normed space, C is cone in $\mathbb{A}$, and $\operatorname{int} C$ is the set of all interior points of C, then C generates the following ordered relations:

$$u \preceq v \Longleftrightarrow v - u \in C, \quad u < v \Longleftrightarrow (v - u \in C \,\text{and}\, u \neq v),$$

and

$$u \ll v \Longleftrightarrow v - u \in \operatorname{int} C.$$

A sequence $\{\omega_n\}_{n\in N}$ in $\mathcal{A}$ is bounded above by $\omega \in \mathcal{A}$ iff

$$\omega_n \preceq \omega \quad \forall n \in \mathbb{N},$$

and its bounded below by $\omega \in \mathcal{A}$ iff

$$\omega \preceq \omega_n \quad \forall n \in \mathbb{N}.$$

A cone C in normed space is solid cone iff it has a nonempty interior. A cone C in a normed space is called normal if there is a number $M > 0$ (actually this number is shown to be greater than or equal one) such that for all $x, y \in C$,

$$\theta \leq u \leq v \Longrightarrow \|u\| \leq M\|v\|. \tag{2.1}$$

The normal constant of C is defined to be the smallest constant M satisfying (2.1).

Remark 2.1. The simplest example of a normal solid cone is the set of all non-negative real numbers $\mathbb{R}^+$ in $\mathbb{R}$, the normal constant of $\mathbb{R}^+$ equals one.

Remark 2.2. [8] The cones in the spaces, space of all convergent to zero sequences c_0 and the space of all p-summing sequences, $l_p, p \geq 1$ are normal with normal constant 1 and they are not solid cones.

We have the following example.

Example 1. Let $C[0, 1]$ be the Banach space of all real valued continuous functions on the compact interval $[0, 1]$ with the norm

$$\|u\| := \max\{|u(t)| : t \in [0, 1]\} \quad \forall\, u \in C[0, 1],$$

and $C := \{u : u \in C[0,1], u(t) \geq 0 \; \forall\, t \in [0,1]\}$, we have

$$u \preceq v \Longleftrightarrow u(t) \leq v(t) \quad \forall\, t \in [0,1] \Longleftrightarrow \|u\| \leq \|v\|.$$

The cone C is normal and solid.

First: The cone C is normal with normal constant one. Indeed; if $\theta \preceq u \preceq v$, then $0 \leq u(t) \leq v(t)$ for every $t \in [0,1]$, hence $0 \leq \max_{t\in[0,1]} u(t) \leq \max_{t\in[0,1]} v(t)$, this implies $\|u\| \leq \|v\|$.

Second: The cone is solid. Because, if we let $\Psi := \{u : \min_{t\in[0,1]} u(t) > 0\}$, then such a set is not empty; for example $u(t) = \exp^{-t}$ ($\min_{t\in[0,1]} u(t) = \frac{1}{e} > 0$) and $v(t) = \cos(t)$ ($\min_{t\in[0,1]} v(t) = cos(1) > 0$) are belonging to Ψ and we claim that $\Psi \subset Int(C)$. To prove such a claim, we let $u \in \Psi$ be any element, then the neighborhood of u with radius $\min_{t\in[0,1]} u(t)$, $N_{\min_{t\in[0,1]} u(t)}(u)$ is a subset of C. In fact; if $v \in N_{\min_{t\in[0,1]} u(t)}(u)$, we have $\|u - v\| < \min_{t\in[0,1]} u(t)$, thus the inequality

$$|u(x) - v(x)| \leq \max\{|(u-v)(t)| : \; t \in [0,1]\} = \|u - v\| < \min_{t\in[0,1]} u(t) \text{ for every } x \in [0,1]$$

implies

$$|u(x) - v(x)| < \min_{t\in[0,1]} u(t) \quad \forall\, x \in [0,1],$$

hence

$$0 \leq u(x) - \min_{t\in[0,1]} u(t) < v(x) < u(x) + \min_{t\in[0,1]} u(t) \quad \forall\, x \in [0,1],$$

this insures that

$$0 \leq v(x) < u(x) + \min_{t\in[0,1]} u(t) \quad \forall\, x \in [0,1].$$

Hence $v \in C$ and proves that $u \in Int(C)$.

Definition 2.3. Let $(\mathbb{A}, \|.\|)$ be a normed space and C be a solid cone in $\mathbb{A}$. Then a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{A}$ is called V-sequence if and only if it satisfies the following V-condition:

$$\forall\, \theta \ll v \;\exists\, n_0 \in \mathbb{N} \text{ such that } \; u_n \ll v \quad \forall\, n \geq n_0. \tag{2.2}$$

and it is N-convergent to zero if it satisfy the following norm-convergent to zero condition:

$$\lim_{n\to\infty} \|u_n\| = 0; \; \forall\, \epsilon > 0 \;\exists\, n_0 \in \mathbb{N} \text{ such that } \|u_n\| < \epsilon \quad \forall\, n \geq n_0. \tag{2.3}$$

Lemma 2.4. [17] Let $(\mathbb{A}, \|.\|)$ be a normed space and C be a solid cone in $\mathbb{A}$. Then every norm-convergent to zero sequence in $\mathbb{A}$ is V-sequence, if the cone is normal then every V-sequence is norm-convergent to zero. That is; the condition (2.3) is stronger than the V-condition, (2.2) and the two conditions are equivalent in case of normal cones.

There are V-sequences which are not N-convergent to zero sequences. The following gives an example of a solid non-normal cone with infinitely many non-N-convergent to zero but V-sequences.

Example 2. Let $C([0,1])$ be the Banach space of all differentiable real valued functions with the norm

$$\|u\| := \|u\|_\infty + \|\frac{du}{dt}\|_\infty \quad \forall\, u \in C([0,1])$$

and consider the cone $C := \{v : v \in C([0,1]), v(t) \geq 0 \,\forall\, t \in [0,1]\}$.

Using the fact that $|u(t) - v(t)| \leq \|u - v\|_\infty \leq \|u - v\| := \|u - v\|_\infty + \|\frac{d(u-v)}{dt}\|_\infty$ for every $t \in [0,1]$ and every $u, v \in C([0,1])$ insures that Ψ defined in the last example is such that $\Psi \subset Int(C)$ and then C is solid.

Now; we show that C is not normal with respect the given norm. First, Let n be a positive integer, $n \in \mathbb{N}$, p be a real number $p \geq 1$, $0 \neq \lambda \in \mathbb{R}$, and denote $\Lambda_{n,p,\lambda}$ the class of functions,

$$\Lambda_{n,p,\lambda} := \{u : [0,1] \to [0,1], \; \|\frac{du(t)}{dt}\|_{+\infty} = n^p|\lambda| \; \}.$$

Then the class $\Lambda_{n,p,\lambda}$ is a non-empty subset of the cone C. In fact; the functions $u(t) = \exp^{-n^p\lambda t}$, $v(t) = \frac{\exp^{n^p\lambda t}}{exp^{n^p\lambda}}$, $\omega(t) = \sin(n^p\lambda t)$, $\varpi(t) = \frac{\cos(n^p\lambda t)}{|\sin(n^p\lambda)|}$, and $\xi(t) = t^{n^p\lambda}$ with $\lambda > 0$ are examples belonging to $\Lambda_{n,p,\lambda}$.

Now; pick any element $u \in \Lambda_{n,p,\lambda}$ and consider the functions

$$u_n(t) = \frac{u(t)}{n^p} \text{ and } v_n(t) = \frac{1}{n},$$

we have the following:

$$u_n(t) = \frac{u(t)}{n^p} \leq \frac{1}{n} = v_n(t) \quad \forall\, t \in [0,1], \;\; n \in \mathbb{N}.$$

Consequently;

$$u_n \preceq v_n \quad \forall\, n \in \mathbb{N}. \tag{2.4}$$

If C were normal with normal constant M, then we would have

$$\|u_n\| \leq M\, \|v_n\| \quad \forall\, n \in \mathbb{N},$$

hence

$$[\frac{\|u\|_\infty}{n^p} + |\lambda|] \leq M\, [\frac{1}{n}] \quad \forall\, n \in \mathbb{N}.$$

Taking the limit as $n \to \infty$ gives the contradiction $|\lambda| \leq 0$. Hence C can not be normal cone. On the other hand we have

$$\|u_n\| = \frac{\|u\|_\infty}{n^p} + |\lambda|, \; \|v_n\| = \frac{1}{n} \quad \forall\, n \in \mathbb{N},$$

consequently;

$$\lim_{n\to\infty} \|u_n\| = |\lambda| \text{ does not equal to zero.} \tag{2.5}$$

The fact that $\lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \frac{1}{n} = 0$ yields $\lim_{n\to\infty} v_n = \theta$ in the norm topology of the space $C([0,1])$, consequently, it should be a V-sequence, in fact; if $v \in Int(C)$ is an arbitrarily element, then v posses a neighborhood of some radius $\epsilon > 0$, $N_\epsilon(v)$ such that $N_\epsilon(v) \subset C$, this implies $\omega \in Int(C)$ for every ω with $\|v - \omega\| < \epsilon$, for this $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $[\frac{1}{n}] < \epsilon$ for every $n \geq n_0$. Now; the inequality

$$\|v - [v - v_n]\| = \|v_n\| = [\frac{1}{n}] < \epsilon \quad \forall\, n \geq n_0$$

implies that $[v - v_n] \in Int(C)$ for every $n \geq n_0$, hence $v_n \ll v \quad \forall\, n \geq n_0$, in conclusion, we have

$$\forall\, \theta \ll v \;\exists\, n_0 \in \mathbb{N}; \text{such that } v_n \ll v \quad \forall\, n \geq n_0. \tag{2.6}$$

Using (2.4) and (2.6) prove that

$$\forall\, \theta \ll v \;\exists\, n_0 \in \mathbb{N} \text{ such that } u_n \ll v \quad \forall\, n \geq n_0. \tag{2.7}$$

Equation (2.7) proves that $\{u_n\}_{n\in\mathbb{N}}$ is a V-sequence. Using (2.5) and (2.7) proves the existence of a V-sequence which does not tend to zero, hence the condition for a sequence to converge to zero in the norm topology is stronger than the condition for a sequence to be V-sequence.

A cone C is called regular if every monotonically non-increasing (non-decreasing) bounded above (bounded below) sequence has a limit in the norm sense of $\mathbb{A}$.

Remark 2.5. [16] Every regular cone is normal, there are normal cones but not regular, there are cones which are not normal, the normal constant M of any normal cone is such that $M \geq 1$, and for any real number k, $k \geq 1$ there is a cone with normal constant $M = k$.

Cone metric spaces and b-cone metric spaces have been introduced and studied by many authors. Their definitions are as follows:

Definition 2.6. Suppose that X is a non empty set, C is a cone in a normed space $\mathbb{A}$, $r \in \mathbb{R}^+$, $r \geq 1$; and q is a function; $q : X \times X \to C$ satisfying the following:

1. $\theta \preceq q(x,y) \quad \forall x, y \in X$.
2. $q(x,y) = \theta \Longleftrightarrow x = y$ (equality is equivalent to indistancy).
3. $q(x,y) = q(y,x) \quad \forall x, y \in X$.
4. $q(x,y) \preceq r\ [q(x,w) + q(w,y)] \quad \forall x, y, w \in X$.

Then (X, C, q) is defined to be a b-cone metric space over C. If in particular $r = 1$, then (X, C, q) is defined to be cone metric space.

We have the following important remark:

Remark 2.7. If $\{v_n\}_{n\in\mathbb{N}}$ is an N-convergent to θ in the normed space $\mathbb{A}$ and C is a solid cone in $\mathbb{A}$, then it is V-sequence. In particular, we have, if a sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, C, q) is such that $\{q(x_n, x)\}_{n\in\mathbb{N}}$ N-convergent to θ in the norm topology of $\mathbb{A}$ for some $x \in X$, then $\{x_n\}_{n\in\mathbb{N}}$ is convergent to x in the topological sense of cone metric space (X, C, q). [It is shown that in case of normal cone the converse is true [17]].

Definition 2.8. Let (X, C, q) be a cone metric space, where the cone C is solid. Then

1. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, C, q) is Cauchy if and only if for every $v \in C$ with $\theta \ll v$ there is $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) \ll v$ for all $n, m \geq n_0$.
2. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, C, q) is convergent sequence if and only if there is x such that for every $v \in C$ with $\theta \ll v$ there is $n_0 \in \mathbb{N}$ such that $q(x_n, x) \ll v$ for all $n \geq n_0$. Equivalently; $\{q(x_n, x)\}_{n\in\mathbb{N}}$ is V-sequence in C.
3. A cone metric space (X, C, q) is complete whenever every Cauchy sequence in (X, C, q) converges to an element belonging to X.

The notion of partial metric space is as follows:

Definition 2.9. [24] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

1. $0 \leq p(x,x) \leq p(x,y) \quad \forall\, x, y \in X$.
2. If $p(x,y) = p(y,y) = p(x,x)$, then $x = y$.
3. $p(x,y) = p(y,x) \quad \forall\, x, y \in X$.
4. $p(x,y) \leq\ p(x,z) +\ p(z,y) - p(z,z) \quad \forall\, x, y, z \in X$.

The double (X, p) is defined to be a pmetric space or a partial metric space.

The open p-ball of center x and radius $0 < \epsilon$, $B_p(x, \epsilon)$ is defined as $B_p(x, \epsilon) = \{\, y \in X : p(x,y) < \epsilon \,\}$. Note the following:

1. If $B_p(x, \epsilon)$ is not empty, then it contains the element x itself because $p(x,x) \leq p(x,y)$ for every $x, y \in X$.
2. Clearly $x \in B_p(x, p(x,x) + \epsilon)$ for every $x \in X$ and $\epsilon > 0$.
3. $x \notin B_p(x, \epsilon)$ for every $x \in X$ and every ϵ, $0 < \epsilon < p(x,x)$. This means, there are p-balls not containing their centers.

Each partial metric p on X generates a T_0 topology τ_p on X, the family of open p-balls

$$\mathbb{F} := \{\, B_p(x, p(x,x) + \epsilon) : x \in X, \epsilon > 0 \,\}$$

constitutes a base family for τ_p because the set X and the intersection of any of its two members are union of some members of these p-balls.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, p) converges to x_0 in the topology τ_p whenever $\lim_{n\to\infty} p(x_n, x_0) = p(x_0, x_0)$. Since $p(x_0, x_0) \leq p(x_n, x_0)$, we have $0 \leq p(x_n, x_0) - p(x_0, x_0)$, consequently; convergence means

$$\forall\, \epsilon > 0\ \exists\, n_0 \in \mathbb{N} \text{ such that } [p(x_n, x_0) - p(x_0, x_0)] < \epsilon \quad \forall n \geq n_0,$$

i.e,

$$\forall \, \epsilon > 0 \; \exists \, n_0 \in \mathbb{N} \text{ such that } p(x_n, x_0) < p(x_0, x_0) + \epsilon \quad \forall n \geq n_0.$$

This convergent is denoted by $x_n \rightarrow_{n\rightarrow\infty} x_0$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X, p) is Cauchy whenever there is $r \in \mathbb{R}$ such that $\lim_{n,m\rightarrow\infty} p(x_n, x_m) = r$ (i.e. exists and is finite).

A partial metric space (X, p) is complete whenever every Cauchy sequence in (X, p) converges to an element belonging to X.

Remark 2.10. Contrarily to the case of metric space, the limit of a sequence in the topology generated by a partial metric, τ_p generally may not be unique because it is not generally Hausdorff space and the mapping p may not be continuous.

Many ways can be used to generalize the concept of cone metric space, one can reduce the second condition to the one direction implication, if $x = y$, then $q(x, y) = \theta$, that is; equality implies indistancy, this leads to the concept of pseudo cone metric space, other can reduce this condition to the reverse direction of implication, if $q(x, y) = \theta$, then $x = y$, that is; indistancy implies equality, and some others can modify the fourth condition. But in this paper motivated by Matthew's results, we replace the second and the fourth conditions of cone metric space by analogues generalized conditions enabling us to introduce the concept of cone pmetric like and cone metric like spaces those increase the field of applications in computer science and to give some generalization of fixed point and coupled fixed point theorems.

On the other side, we have the following:

Definition 2.11. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if and only if $F(x, y) = x$ and $F(y, x) = y$.

In this paper, contrarily to the case of cone metric space X, we show with examples that the limit of a sequence in the topology generated T_q by a partial cone metric, may not be unique because T_q is not generally Hausdorff space and the cone valued partial metric mapping q may not be continuous. Hence is not equivalent to any topology generated by a metric on X, and consider some generalized contraction type of mappings on θ-complete cone metric like spaces, and generalize the coupled fixed point theorem of Sabet (1.5) in this setting.

3 Main Results

In the following we suppose that C is a cone in a normed space $\mathbb{A}$ and $\preceq$ is the corresponding obtained ordered relation, X is a non-empty set, and $p : X \times X \rightarrow C$. Consider the following conditions:

1. $\theta \preceq p(x, y) \quad \forall \, x, y \in X$ (nonnegativity),
2. $p(x, x) = \theta \quad \forall \, x \in X$ (equality implies indistancy),
3. $p(x, y) = \theta$, then $x = y$ (indistancy implies equality),
4. If $p(x, y) = p(y, y) = p(x, x)$, then $x = y$ (indistancy implies equality),
5. $p(x, x) \preceq p(x, y) \quad \forall \, x, y \in X$ (small self distance),
6. $p(x, y) = p(y, x) \quad \forall \, x, y \in X$ (symmetry),
7. $p(x, y) \preceq \; p(x, z) + \; p(z, y) - p(z, z) \quad \forall \, x, y, z \in X$ (triangularity).
8. $p(x, y) \preceq \; p(x, z) + \; p(z, y) \quad \forall \, x, y, z \in X$ (triangularity).
9. $p(x, y) \preceq \; r \, [p(x, z) + \; p(z, y)] \quad \forall \, x, y, z \in X$ (for some $r \geq 1$).

We focus on some generalizations of the cone metric spaces, namely; pseudo cone metric space, partial cone metric space, cone pmetric like space, and cone metric like space. These generalizations are defined as follows:

If p satisfies the conditions (1), (2), (3), (6), and (8), then p is defined to be a cone metric on X and the triple (X, C, p) is a cone metric space.

If p satisfies the conditions (1), (2), (6), and (8), then p is defined to be a pseudo cone metric on X and the triple (X, C, p) is a cone pseudo metric space.

If p satisfies the conditions (1), (4), (5), (6), and (7), then p is defined to be a partial cone metric on X and the triple (X, C, p) is a partial cone metric space.

If p satisfies the conditions (1), (3), (6), and (7), then p is defined to be a cone pmetric like (reads p metric like or partial cone metric like) on X and the triple (X, C, p) is a cone pmetric like space.

If p satisfies the conditions (1), (3), (6), and (8), then p is defined to be a cone metric like on X and the triple (X, C, p) is a cone metric like space.

If p satisfies the conditions (1), (3), (6), and (9), then p is defined to be a b-cone metric like on X and the triple (X, C, p) is a b-cone metric like space.

If p satisfies the conditions (1), (2), (3), (6), and (9), then p is defined to be a b-cone metric on X and the triple (X, C, p) is a b-cone metric space.

Remark 3.1.

1. A space (X, C, p) is cone metric space if and only if it is both pseudo cone metric and partial cone metric.
2. There are pseudo cone metric spaces which are not partial cone metric spaces, and there are partial cone metric spaces which are not pseudo cone metric spaces.
3. Each of pseudo cone metric space, partial cone metric space, cone pmetric like space, and cone metric like space is a generalization of cone metric space.
4. Every partial cone metric space is a cone pmetric like space. Indeed; given that p is partial metric, we need to verify condition (3). For this reason let $p(x, y) = \theta$, since p is partial metric, we have $p(x, x) \preceq p(x, y)$ and $p(y, y) \preceq p(x, y)$, this implies $p(x, x) = p(y, y) = p(x, y) = 0$, using condition (4) of partial metric gives $x = y$.
5. Every cone pmetric like space is clearly a cone metric like space. These implies that every partial cone metric space is a cone metric like space.
6. Every cone metric like space is clearly a b-cone metric like space.
7. Every b-cone metric space is cone b- metric like space.

Hence we have the following observation and diagram:

- The category of partial cone metric spaces is larger than the category of cone metric spaces and the concept of partial cone metric is a generalization of the concept of cone metric.
- The category of cone pmetric like spaces is larger than the category of partial cone metric spaces and the concept of cone pmetric like is a generalization of the concept of partial cone metric.
- The category of cone metric like spaces is larger than both of the two categories, the category of partial cone metric spaces and the category of cone pmetric like spaces, hence the concept of cone metric like is a generalization of both of the concepts of partial cone metric and cone pmetric like.

$$
\begin{array}{ccccc}
\dfrac{\text{pseudo cone metric space}}{(1,2,6,8)}. & & & & \\
\Uparrow & & & & \\
\dfrac{\text{cone metric space}}{(1,2,3,6,8)} & \Rightarrow & \dfrac{b-\text{cone metric space}}{(1,2,3,6,9)} & \Rightarrow & \dfrac{b-\text{cone metric like space}}{(1,3,6,9)} \\
\Downarrow & & & & \\
\Downarrow & & & & \\
\dfrac{\text{partial cone metric space}}{(1,4,5,6,7)} & \Rightarrow & \dfrac{\text{cone pmetric like space}}{(1,3,6,7)} & \Rightarrow & \dfrac{\text{cone metric like space}}{(1,3,6,8)} \\
 & & & & \Downarrow \\
 & & & & \dfrac{b-\text{cone metric like space}}{(1,3,6,9)},
\end{array}
\tag{3.1}
$$

where each implication means the inclusion of the corresponding spaces, generally the inverses of these inclusions are not hold. Indeed we will give some counter examples at the end of this paper.

For the cone metric like space (X, C, p), we introduce the following:

Let C be a solid cone in the normed space $\mathbb{A}$, denote the open p-ball of center $x \in X$ and radius v, $\theta \ll v$ $(v \in Int(C))$ by $U_p(x, v)$,

$$U_p(x, v) := \{y \in X : p(x, y) \ll v\}.$$

and

$$\mathbb{U} = \{ \ U_p(x, v) : x \in X, \theta \ll v \ \},$$

also, let $B_p(x, v)$ be defined as:

$$B_p(x, v) := U_p(x, p(x, x) + v) = \{y \in X : p(x, y) \ll v + p(x, x)\}.$$

For every $x \in X$ and v, $\theta \ll v$, the open p-ball, $B_p(x, v) := U_p(x, v + p(x, x))$ contains the center x itself because $p(x, x) \ll v + p(x, x)$, $x \in B_p(x, v)$ and therefore it is not empty set, while $U_p(x, v)$ may not contain the center x (if $p(x, x) > v$).
Denote $\mathbb{B}$ as:

$$\mathbb{B} := \{ \ B_p(x, v) : x \in X, \theta \ll v \ \}.$$

The topology T_p on (X, C, p) is defined as follows: $V \in T_p$ iff for every $x \in V$ there is v, $\theta \ll v$ such that $U_p(x, v) \subset V$. A subset $W \subset X$ is defined to be a neighborhood of x iff there is $\theta \ll v$ such that $U_p(x, v) \subset W$.

$$T_p := \{ \ V : V \subset X \text{ such that } \forall \ x \in V \ \exists \ \theta \ll v, U_p(x, v) \subset V \ \},$$

Lemma 3.2. We have the following inclusions:

$$\mathbb{U} \subset T_p \text{ and } \ \mathbb{B} \subset T_p.$$

Equivalently; the open p-balls, $U_p(x, v)$ and $B_p(x, v)$ belong to T_p for every $x \in X$ and every v, $\theta \ll v$.

Proof . Each open p-ball $U_p(x, v)$ in (X, C, p) is an open set in the topology T_p. Indeed; if $y \in U_p(x, v)$, we have $p(y, x) \ll v$, consider the p-ball of center y and radius $v - p(x, y)$, $U_p(y, v - p(x, y))$. Then for every $z \in U_p(y, v - p(x, y))$, we have $p(z, y) \ll v - p(x, y)$, consequently,

$$p(z, x) \preceq p(z, y) + p(y, x) \ll v - p(x, y) + p(y, x) = v,$$

this gives $p(z, x) \ll v$. That is; $z \in U_p(x, v)$ and then $U_p(y, v - p(x, y)) \subset U_p(x, v)$.

On the other side, for every $x \in X$ and v, $\theta \ll v \in C$, let $y \in B_p(x, v)$ be an arbitrarily element, we show that there is $\theta \ll v(y) \in C$ such that $U_p(y, v(y)) \subset B_p(x, v)$. Since $p(x, y) \ll v + p(x, x)$, then $\theta \ll [v + p(x, x) - p(y, x)] \in C$, take $v(y) = v + p(x,x) - p(y,x)$ the open p-ball $U_p(y, v(y))) \in T_p$ is such that $U_p(y, v(y)) \subseteq B_p(x, v)$. Indeed; if $z \in U_p(y, v(y))$, then $p(z, y) \ll [v + p(x,x) - p(x,y)]$, hence

$$\begin{aligned} p(z,x) &\preceq p(z,y) + p(y,x) \\ &\ll [v + p(x,x) - p(x,y)] + p(y,x) = v + p(x,x). \end{aligned}$$

This shows that $p(z,x) \ll v + p(x,x)$, that is; $z \in B_p(x,v)$. Consequently; $U_p(y, v(y)) \subset B_p(x,v)$. □

We have the following characterization:

Theorem 3.3. The family of open p-balls $\mathbb{B}$ constitutes a base family for the topology T_p of cone pmetric like space (X, C, p).

Proof . First; using Lemma (3), we see that $\mathbb{B} \subset T_p$, since $x \in B_p(x,v)$ for every $x \in X$ and v, $\theta < v$ $[B_p(x,v) \neq \emptyset]$, we have $X = \bigcup_{x\in X} B_p(x,v)$.

Second: We show that the intersection of any two members of $\mathbb{B}$ is a union of some members of $\mathbb{B}$, that is; if $B_p(x,u)$ and $B_p(y,v)$ are arbitrarily two open p-balls in $\mathbb{B}$, we show that there is $\mathbb{B}^* \subset \mathbb{B}$ such that $B_p(x,u) \bigcap B_p(y,v) = \bigcup_{B\in\mathbb{B}^*} B$. Equivalently, we show that for any $z \in B_p(x,u) \bigcap B_p(y,v)$ there is some $\theta \ll \omega_z$ such that $B_p(z, \omega_z) \subset B_p(x,u) \bigcap B_p(y,v)$.

Suppose that $z \in B_p(x,u) \bigcap B_p(y,v)$, we have $p(x,z) \ll u + p(x,x)$ and $p(y,z) \ll v + p(y,y)$. Then $\theta \ll u + p(x,x) - p(x,z)$ and $\theta \ll v + p(y,y) - p(y,z)$, hence there is a neighborhood of $u + p(x,x) - p(x,z)$ with some radius δ_1, $N_{\delta_1}([u + p(x,x) - p(x,z)])$ and a neighborhood of $v + p(y,y) - p(y,z)$ with some radius δ_2, $N_{\delta_2}([v + p(y,y) - p(y,z)])$ such that

$$N_{\delta_1}([u + p(x,x) - p(x,z)]) = [u + p(x,x) - p(x,z)] + N_{\delta_1}(\theta) \subset C,$$

and

$$N_{\delta_2}([v + p(y,y) - p(y,z)]) = [v + p(y,y) - p(y,z)] + N_{\delta_2}(\theta) \subset C.$$

Since $u + p(x,x) - p(x,z) \in C$, $v + p(y,y) - p(y,z) \in C$, and C is a cone, we have

$$\frac{[u + p(x,x) - p(x,z)] + [v + p(y,y) - p(y,z)]}{n} \in C \quad \forall n \in \mathbb{N}.$$

Since

$$\frac{[u + p(x,x) - p(x,z)] + [v + p(y,y) - p(y,z)]}{n} \to_{n\to\infty} \theta.$$

For the two non-negative real numbers δ_1 and δ_2, there are two integers $n_1, n_2 \in \mathbb{N}$ such that

$$\pm\frac{[u + p(x,x) - p(x,z)] + [v + p(y,y) - p(y,z)]}{n} \in N_{\delta_1}(\theta) \quad \forall n \geq n_1$$

and

$$\pm\frac{[u + p(x,x) - p(x,z)] + [v + p(y,y) - (y,z)]}{n} \in N_{\delta_2}(\theta) \quad \forall n \geq n_2.$$

Let $N = \max\{n_1, n_2\}$ and set $\omega_z := \frac{[u+p(x,x)-p(x,z)]+[v+p(y,y)-p(y,z)]}{N}$. Then

$$[u + p(x,x) - p(x,z)] - \omega_z \in [u + p(x,x) - p(x,z)] + N_{\delta_1}(\theta) = N_{\delta_1}(u + p(x,x) - p(x,z)) \subset C, \tag{3.2}$$

and

$$[v + p(y,y) - p(y,z)] - \omega_z \in [v + p(y,y) - p(y,z)] + N_{\delta_2}(\theta) = N_{\delta_2}(v + p(y,y) - p(y,z)) \subset C. \tag{3.3}$$

Equations (3.2) and (3.3) prove the following:

$$\theta \ll \omega_z, \;\; \omega_z \ll [u + p(x,x) - p(x,z)], \;\; and \;\; \omega_z \ll [v + p(y,y) - p(y,z)].$$

The open ball $B_p(z, \omega_z)$ contains z and $B_p(z, \omega_z) \subset B_p(x,u) \bigcap B_p(y,v)$. Indeed; if $w \in B_p(z, \omega_z)$, then $p(z,w) \ll p(z,z) + \omega_z$, hence

$$\begin{aligned} p(x,w) &\preceq p(x,z) + p(z,w) - p(z,z) \ll p(x,z) + p(z,z) + \omega_z - p(z,z) \\ &< p(x,z) + [u + p(x,x) - p(x,z)] = u + p(x,x), \end{aligned}$$

hence $p(x,w) \ll u + p(x,x)$, that is; $w \in B_p(x,u)$. Similarly; we have $p(y,w) \ll v + p(y,y)$, hence $w \in B_p(y,v)$. Therefore $w \in B_p(x,u) \bigcap B_p(y,v)$, in conclusion, we have

$$B_p(z,\omega_z) \subset B_p(x,u) \bigcap B_p(y,v) \quad \forall\, z \in B_p(x,u) \bigcap B_p(y,v).$$

Consequently, we have

$$\bigcup \{B_p(z,\omega_z) : z \in B_p(x,u) \bigcap B_p(y,v)\} \subset B_p(x,u) \bigcap B_p(y,v). \tag{3.4}$$

On the other side for every $z \in B_p(x,u) \bigcap B_p(y,v)$, we have $z \in B_p(z,\omega_z)$, hence

$$B_p(x,u) \bigcap B_p(y,v) \subset \bigcup \{B_p(z,\omega_z) : z \in B_p(x,u) \bigcap B_p(y,v)\}. \tag{3.5}$$

Equations (3.4) and (3.5) prove the required claim

$$B_p(x,u) \bigcap B_p(y,v) = \bigcup \{B_p(z,\omega_z) : z \in B_p(x,u) \bigcap B_p(y,v)\}.$$

□

Let (X,C,p) be either partial cone metric space, cone pmetric like, or cone metric like space, the convergence's characterizations of sequences in T_p is as follows:

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,C,p) is T_p-convergent to x iff for every $v \in Int(C)$, $B_p(x,v)$ there is $n_0 \in \mathbb{N}$ such that $x_n \in B_p(x,v)$ for every $n \geq n_0$. Equivalently;

$$\forall\, v \in Int(C)\ \exists\, n_0 \in \mathbb{N} \text{ such that } p(x_n,x) \ll p(x,x) + v \quad \forall\, n \geq n_0.$$

This convergent is denoted by $x_n \longrightarrow_{n\to\infty} x - (T_p)$. Equivalently; the sequence $\{p(x_n,x) - p(x,x)\}_{n\in\mathbb{N}}$ is V-sequence in C. The element x itself is said to be T_p-limit point of $\{x_n\}_{n\in\mathbb{N}}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,C,p) is θ-Cauchy iff for every $v \in Int(C)$ there is $n_0 \in \mathbb{N}$ such that $p(x_n,x_m) \ll v$ for every $n,m \geq n_0$.

A metric (X,C,p) is θ-complete whenever every θ-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,C,p) is T_p-convergent to an element x belonging to X such that $p(x,x) = \theta$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,C,p) is Cauchy iff there is $\theta \preceq \omega$, $\|\omega\| < \infty$ such that $\lim_{n,m\to\infty} p(x_n,x_m) = \omega$, the limit is taken in the normed space $\mathbb{A}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,C,p) is said to be p-strongly convergent to x (or $p - \|.\|$ convergent) if and only if the sequence $\{p(x_n,x) - p(x,x)\}_{n\in\mathbb{N}}$ is norm-convergent to θ, that is; it converges in the norm topology of $(\mathbb{A}, \|.\|)$, $\lim_{n\to\infty} \|p(x_n,x) - p(x,x)\| = 0$ (or $\lim_{n\to\infty} p(x_n,x) = p(x,x)$ in the norm topology of $\mathbb{A}$). This convergent is denoted by $x_n \longrightarrow_{n\to\infty} x - (p - \|.\|)$.

A metric (X,C,p) is complete whenever every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,C,p) is p-strongly convergent to an element x belonging to X such that $p(x,x) = \theta$.

Lemma 3.4. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in (X,C,p). If $x_n \longrightarrow_{n\to\infty} x - (p - \|.\|)$ for some $x \in X$, then $x_n \longrightarrow_{n\to\infty} x - (T_p)$ and the converse is true whenever the cone C is normal.

Proof . Let $\lim_{n\to\infty} p(x_n,x) = p(x,x)$, then

$$\forall \epsilon > 0\ \exists\, n_0 \in \mathbb{N} \text{ such that } \|p(x_n,x) - p(x,x)\| < \epsilon \quad \forall\, n \geq n_0.$$

Let $\theta \ll v$ be arbitrarily element. Then $v \in Int(C)$, hence, there is a neighborhood of v with some radius ϵ_0, $N_{\epsilon_0}(v)$ such that $N_{\epsilon_0}(v) \subset C$, $u \in C$ for every u, $\|u - v\| < \epsilon_0$, for this ϵ_0 there is $n_0 \in \mathbb{N}$ such that $\|p(x_n,x) - p(x,x)\| < \epsilon_0$ for every $n \geq n_0$, we see that $\|v - [v + p(x,x) - p(x_n,x)]\| < \epsilon_0$ for every $n \geq n_0$, hence $[v + p(x,x) - p(x_n,x)] \in N_{\epsilon_0}(v)$

for every $n \geq n_0$, since $N_{\epsilon_0}(v)$ is an open set in the normed space $\mathbb{A}$ there are $N_{\epsilon_n}(v + p(x,x) - p(x_n,x))$ such that $N_{\epsilon_n}(v+p(x,x)-p(x_n,x)) \subset N_{\epsilon_0}(v)$ which is true for every $n \geq n_0$, consequently we have $N_{\epsilon_n}(v+p(x,x)-p(x_n,x)) \subset C$ for every $n \geq n_0$, that is; $[v + p(x,x) - p(x_n,x)] \in Int(C)$ every $n \geq n_0$, this shows that $p(x_n,x) \ll v + p(x,x)$ for every $n \geq n_0$, $x_n \to_{n\to\infty} \to x - (T_p)$.
□

There are T_p-convergent sequences which are not $p - \|.\|$ convergent, the T_p limit point may not be unique, and also the metric function $p : X \times X \to C$ may not be continuous in the sense that given $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, x, y, $x_n \to_{n\to\infty} x - (T_p)$, and $y_n \to_{n\to\infty} y - (T_p)$ implies $p(x_n,y_n) \to_{n\to\infty} p(x,y)(\|.\|)$. The following example supports this intension.

Example 3. Let $C([0,1])$ be the Banach space of all differentiable real valued functions with the norm $\|u\| := \|u\|_\infty + \|\frac{du}{dt}\|_\infty \quad \forall\, u \in C([0,1])$, $C = \{u : u \in C([0,1]), u(t) \geq 0 \;\forall\, t \in [0,1]\ \}$, $X = C$, and $p : C \times C \to C$ be defined by

$$p(x,y) = \begin{cases} x, & \text{if } x = y; \\ x + y, & \text{otherwise.} \end{cases}$$

Then (C, C, p) is (partial cone metric) cone pmetric like and then it is cone metric like which is not a cone metric because $p(x,x) = x \neq \theta$. If $p(x,y) = \theta$, then $x + y = \theta$, $x = -y$, but $x, y \in C$, then $x = y = \theta$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (C, C, p) is θ-Cauchy if and only if it is V-sequence. Indeed if $\{x_n\}_{n\in\mathbb{N}}$ is θ-Cauchy, then for every $v \in Int(C)$ there is $n_0 \in \mathbb{N}$ such that $p(x_n,x_m) \ll v$ for every $n,m \geq n_0$, since $x_n \preceq x_n + x_m = p(x_n,x_m)$ and $x_m \preceq x_n + x_m = p(x_n,x_m)$, we see that $x_n \ll v$ for every $n \geq n_0$ and then it is V-sequence.

Conversely; if $\{x_n\}_{n\in\mathbb{N}}$ is V-sequence, then for every $v \in Int(C)$ there is $n_0 \in \mathbb{N}$ such that $x_n \ll \frac{v}{2}$ for every $n \geq n_0$, hence $x_n + x_m \ll \frac{v}{2} + \frac{v}{2} = v$ for every $n,m \geq n_0$, since $x_n + x_m = p(x_n,x_m)$, we see that $p(x_n,x_m) \ll v$ for every $n,m \geq n_0$ and it is θ-Cauchy.

If $\{x_n\}_{n\in\mathbb{N}}$ is any V-sequence in $X = C$, then it is T_p-convergent and its set of all T_p-limit points is the whole of C, $\overline{\{x_n\}_{n\in\mathbb{N}}}^{T_p} = C$, the closure of the sequence in the topology T_p. Indeed if $x \in C$ is an arbitrarily element, we have $p(x_n,x) - p(x,x) = x_n$, hence $\{p(x_n,x) - p(x,x)\}_{n\in\mathbb{N}}$ is V-sequence in $X = C$, that is $x_n \to_{n\to\infty} x - (T_p)$.

These last two paragraphs prove that every θ-Cauchy sequence is T_p convergent to infinitely many limits in $X = C$, hence (C, C, p) is θ-complete.

Back to example (2) with $p = 1, \lambda = 1$, the sequence $\{x_n\}_{n\in\mathbb{N}}$, where $x_n(t) = \frac{\sin(nt)}{n}$; $t \in [0,1]$ is V-sequence in $X = C$ and then it is T_p-convergent to each x in the space X. On the other side, it is not $p - \|.\|$-convergent to any $x \in X$ because $\{p(x_n,x) - p(x,x)\}_{n\in\mathbb{N}}$ is not strongly convergent to zero. Indeed, if x is any element in C, then

$$\begin{aligned} \|p(x_n,x) - p(x,x)\| = \|x_n\| &= \max_{t\in[0,1]} \frac{\sin(nt)}{n} + \max_{t\in[0,1]} \frac{n\cos(nt)}{n} \\ &= \frac{\sin(n)}{n} + \max_{t\in[0,1]} \cos(nt) = \frac{\sin(n)}{n} + \cos(0) \\ &= \frac{\sin(n)}{n} + 1 \to_{n\to\infty} 1 \neq 0. \end{aligned}$$

If $\{x_n\}_{n\in\mathbb{N}}$ is any V-sequence in $X = C$ which is not $\|.\|$-convergent to zero, hence it is T_p-convergent, we have $x_n \to_{n\to\infty} \theta - (T_p)$ as θ is one of its limit points, hence take $y_n = x_n$, we have

$$x_n \longrightarrow_{n\to\infty} \theta - (T_p), \;\; y_n \longrightarrow_{n\to\infty} \theta - (T_p), \;\; p(x_n,y_n) = x_n, \; p(\theta,\theta) = \theta,$$

and

$$\|p(x_n,y_n) - p(\theta,\theta)\| = \|x_n\|.$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is not $\|.\|$-convergent to zero, the sequence $\{p(x_n,y_n) - p(\theta,\theta)\}_{n\in\mathbb{N}}$ is not $\|.\|$-convergent to zero. That is; p is not continuous.

We claim that the space (C, C, p) is not complete. To prove this claim, we will give example of a Cauchy sequence which is not a $p - \|.\|$ convergent to any element in C. In fact; using example (2) with $p = 1, \lambda = 1$, we see that the sequence $\{y_n\}_{n\in\mathbb{N}}$ where $y_n(t) = \frac{\exp^{-nt}}{n}$; $t \in [0, 1]$ is a V-sequence and for any $u \in C$, we have

$$\begin{aligned}\|p(y_n, u) - p(u, u)\| = \|y_n\| &= \max_{t\in[0,1]} |\frac{\exp^{-nt}}{n}| + \max_{t\in[0,1]} |\frac{-n\exp^{-nt}}{n}| \\ &= \frac{1}{n} + 1 \to_{n\to\infty} 1 \neq 0.\end{aligned}$$

Hence, $\{y_n\}_{n\in\mathbb{N}}$ is not $p - \|.\|$ convergent to any element in X, particularly is not $p - \|.\|$ convergent to θ, $p(\theta, \theta) = \theta$, the only zero self distance element (is not $\|.\|$-convergent to zero). This sequence is Cauchy sequence because the element $w(t) := \frac{2}{\cos(1)} \sin(1 - t)$ is such that

$$\|w\| = \max_{t\in[0,1]} |\frac{2}{\cos(1)} \sin(1 - t)| + \max_{t\in[0,1]} | - \frac{2}{\cos(1)} \cos(1 - t)| = \frac{2\sin(1)}{\cos(1)} + \frac{2}{\cos(1)} < \infty.$$

On the other hand, for any $n, m \in \mathbb{N}$, $n \neq m$we have

$$\begin{aligned}\|p(y_n, y_m) - w\| &= \|p(y_n, y_m) - w\|_\infty + \|\frac{d}{dt}[p(y_n, y_m) - w]\|_\infty \\ &= \|y_n + y_m - w\|_\infty + \|\frac{d}{dt}[y_n + y_m - w]\|_\infty \\ &= \max_{t\in[0,1]} |\frac{\exp^{-nt}}{n} + \frac{\exp^{-mt}}{m} - \frac{2}{\cos(1)} \sin(1 - t)|+ \\ &+ \max_{t\in[0,1]} | - \exp^{-nt} - \exp^{-mt} - [-\frac{2}{\cos(1)} \cos(1 - t)]| \\ &\leq \max_{t\in[0,1]} \frac{\exp^{-nt}}{n} + \frac{\exp^{-mt}}{m} - \min_{t\in[0,1]} \frac{2}{\cos(1)} \sin(1 - t) \\ &+ \max_{t\in[0,1]} \exp^{-nt} + \exp^{-mt} + \max_{t\in[0,1]} [-\frac{2}{\cos(1)} \cos(1 - t)] \\ &= \frac{1}{n} + \frac{1}{m} + 1 + 1 - 2 \\ &= \frac{1}{n} + \frac{1}{m} \to 0 \text{ as } n, m \to \infty.\end{aligned}$$

This proved that $\|.\| - \lim_{n,m\to\infty} p(y_n, y_m) = w$. Consequently; this sequence is Cauchy sequence which is not $p - \|.\|$ convergent to any element in C.

We also have the following example:

Example 4. Let $X := \{ (x_1, x_2) : x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Q} \}$, $\mathbb{A} = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = \max\{ x_1, x_2 \}$, and $C = \{ (x_1, x_2) : x_1, x_2 \geq 0 \}$, where $\mathbb{Q}$ is the set of rational numbers. Define $p : X \times X \to C$ as

$$p((x_1, x_2), (y_1, y_2)) = (\max\{ x_1, y_1 \}, \max\{ x_2, y_2 \}).$$

Clearly (X, C, p) is cone metric like space, $p((x_1, x_2), (x_1, x_2)) = (x_1, x_2)$, (X, C, p) is θ-complete which is not complete.

According to Example (3), Example (4) and Lemma (3.4), we conclude that every complete space is θ-complete and there are many θ-complete (cone metric like) cone pmetric like spaces which are not complete.

3.1 Applications to Coupled Fixed Point Theory

Let $(\mathbb{A}, \|.\|)$ be a normed space and C be a cone in $\mathbb{A}$. A mapping $L : \mathbb{A} \to \mathbb{A}$ is said to be V-mapping iff $L(C) \subset C$ and it preserves V-sequences in the sense that $\{L(u_n)\}_{n\in\mathbb{N}}$ is V-sequence whenever $\{u_n\}_{n\in\mathbb{N}}$ is V-sequence.

Remark 3.5. If $L : \mathbb{A} \to \mathbb{A}$ is linear bounded mapping, then it is continuous, consequently it preserves strong convergence. If it is V-mapping and $\{u_n\}_{n\in\mathbb{N}}$ is norm-convergent to $u \in C$, then $\{u_n - u\}_{n\in\mathbb{N}}$ is norm-convergent to θ, accordingly $\{u_n - u\}_{n\in\mathbb{N}}$ is V-sequence and hence $\{L(u_n) - L(u)\}_{n\in\mathbb{N}}$ is V-sequence.

We have the following theorem, cones are not necessarily normal but solid and the underlying space is the wider θ-complete cone metric like space:

Theorem 3.6. Let (X, C, p) be a θ-complete cone metric like space and $L, S : \mathbb{A} \to \mathbb{A}$ be two linear bounded V-mappings, if the spectral radius of their sum is strictly less than one, $r(L + S) < 1$ and $T : X \times X \to X$ is a mapping such that

$$q(T(x, y), T(z, w)) \preceq L(p(x, z)) + S(p(y, w)) \quad \forall\, x, y, z, w \in X, \tag{3.6}$$

then T has a unique coupled fixed point.

Proof . Select $x_0, y_0 \in X$ and set $x_1 = T(x_0, y_0)$, $y_1 = T(y_0, x_0)$, $\ldots$, $x_{n+1} = T(x_n, y_n)$, $y_{n+1} = T(y_n, x_n)$. Then by (3.6) we have

$$\begin{aligned} p(x_{n+1}, x_n) &= p(T(x_n, y_n), T(x_{n-1}, y_{n-1})) \\ &\preceq L(p(x_n, x_{n-1})) + S(p(y_n, y_{n-1})), \end{aligned}$$

and similarly,

$$\begin{aligned} p(y_{n+1}, y_n) &= p(T(y_n, x_n), T(y_{n-1}, x_{n-1})) \\ &\preceq L(p(y_n, y_{n-1})) + S(p(x_n, x_{n-1})). \end{aligned}$$

Let $v_n = p(y_n, y_{n-1}) + p(x_n, x_{n-1})$, we have

$$\begin{aligned} v_{n+1} &= p(x_{n+1}, x_n) + p(y_{n+1}, y_n) \\ &\preceq L(p(x_n, x_{n-1})) + S(p(y_n, y_{n-1})) + L(p(y_n, y_{n-1})) + S(p(x_n, x_{n-1})) \\ &= (L + S)(p(x_n, x_{n-1}) + p(y_n, y_{n-1})) = (L + S)(v_n). \end{aligned}$$

For each $n \in \mathbb{N}$, we have

$$\theta \preceq v_n \preceq (L + S)(v_{n-1}) \preceq (L + S)^2(v_{n-2}) \preceq \cdots \preceq (L + S)^{n-1}(v_1). \tag{3.7}$$

If $v_1 = \theta$, then $p(x_1, x_0) + p(y_1, y_0) = \theta$, so $p(x_1, x_0) = \theta$ and $p(y_1, y_0) = \theta$, for the cone metric like space this imply $x_1 = x_0$ and $y_1 = y_0$, consequently $T(x_0, y_0) = x_0$, $T(y_0, x_0) = y_0$ and (x_0, y_0) is coupled fixed point of T, in this case the proof will be completed. Therefore we continue by letting $\theta < v_1$. Now, let $n, m \in \mathbb{N}$, $n \leq m$. Then

$$\begin{aligned} p(x_n,x_m) &\preceq \Big[p(x_n, x_{n+1}) + p(x_{n+1}, x_m)\Big] \ldots \\ &\preceq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+3}) + \cdots + p(x_{m-1}, x_m), \end{aligned}$$

similarly,

$$p(y_n, y_m) \preceq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + p(y_{n+2}, y_{n+3}) + \cdots + p(y_{m-1}, y_m).$$

Adding, we get

$$\begin{aligned} p(x_n, x_m) + p(y_n, y_m) \preceq [p(x_n, x_{n+1}) + \ & p(y_n, y_{n+1})] + \\ & + [p(x_{n+1}, x_{n+2}) + p(y_{n+1}, y_{n+2})] + \\ & + [p(x_{n+2}, x_{n+3}) + p(y_{n+2}, y_{n+3})] + \ldots \\ & + [p(x_{m-1}, x_m) + p(y_{m-1}, y_m)]. \end{aligned}$$

This shows that

$$p(x_n, x_m) + p(y_n, y_m) \preceq v_{n+1} + v_{n+2} + v_{n+3} + \cdots + v_m. \tag{3.8}$$

Using the two inequalities (3.7) and (3.8), we have

$$\begin{aligned} p(x_n, x_m) + p(y_n, y_m) &\preceq (L + S)^n(v_1) + (L + S)^{n+2}(v_1) + \ldots \\ &\quad + (L + S)^{n+(m-n)}(v_1) \\ &= (L + S)^n \big[(L + S) + (L + S)^2 + (L + S)^3 + \cdots + \\ &\quad + (L + S)^{m-n}\big](v_1) \end{aligned} \tag{3.9}$$

Since $r(L+S)<1$, the operator $I-(L+S)$ is invertible and its inverse equals $[I-(L+S)]^{-1}=\sum_{n=0}^{\infty}(L+S)^n$,

$$\begin{aligned}
p(x_n,x_m)+p(y_n,y_m) &\preceq (L+S)^n\big[(L+S)+(L+S)^2+(L+S)^3+\cdots+\\
&+(L+S)^{m-n}\big](v_1)\\
&\preceq (L+S)^n\big[(L+S)+(L+S)^2+(L+S)^3+\cdots+\\
&+(L+S)^{m-n}+\ldots\big](v_1)\\
&=(L+S)^n\big[\sum_{n=0}^{\infty}(L+S)^n\big](v_1)\\
&=(L+S)^n[I-(L+S)]^{-1}(v_1)=(L+S)^n(\overline{v}_0),
\end{aligned}\tag{3.10}$$

where $[I-(L+S)]^{-1}(v_1)=\overline{v}_0$. On the other hand, we should have $\lim_{n\to\infty}(L+S)^n=\Theta$, where $\Theta:\mathbb{A}\to\mathbb{A}$ is the zero operator, consequently; the sequence $\{(L+S)^n(u)\}_{n\in\mathbb{N}}$ is norm convergent to θ for every $u\in\mathbb{A}$, $\lim_{n\to\infty}(L+S)^n(u)=\theta$ for every $u\in\mathbb{A}$, this imply that $\{(L+S)^n(u)\}_{n\in\mathbb{N}}$ is V-sequence in C for every $u\in C$, in particular; $\{(L+S)^n(\overline{v}_0)\}_{n\in\mathbb{N}}$ is V-sequence in C and therefore $\{p(x_n,x_m)+p(y_n,y_m)\}_{n\in\mathbb{N}}$ is V-sequence in C. Hence for every $n\le m$, we have

$$\forall\, v\in Int(C)\ \exists\, n_0\in\mathbb{N} \text{ such that } [p(x_n,x_m)+p(y_n,y_m)]\ll v\quad \forall\, n\ge n_0. \tag{3.11}$$

Since $p(x_n,x_m)\preceq p(x_n,x_m)+p(y_n,y_m)$ and $p(y_n,y_m)\preceq p(x_n,x_m)+p(y_n,y_m)$, we concluded that

$$\forall\, v\in Int(C)\ \exists\, n_0\in\mathbb{N} \text{ such that } p(x_n,x_m)\ll v\quad \forall\, n,m\ge n_0. \tag{3.12}$$

$$\forall\, v\in Int(C)\ \exists\, n_0\in\mathbb{N} \text{ such that } p(y_n,y_m)\ll v\quad \forall\, n,m\ge n_0. \tag{3.13}$$

The two inequalities (3.12) and (3.13) prove that the two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are two θ-Cauchy sequences in (X,C,p). Since (X,C,p) is θ-complete, then there are two T_p-limits of $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ say $w_0,w_1\in X$, $p(w_0,w_0)=\theta$ and $p(w_1,w_1)=\theta$, respectively. Hence

$$\forall\, v\in Int(C)\ \exists\, n_1\in\mathbb{N} \text{ such that } p(x_n,w_0)\ll \frac{1}{3}v\quad \forall\, n\ge n_1. \tag{3.14}$$

$$\forall\, v\in Int(C)\ \exists\, n_2\in\mathbb{N} \text{ such that } p(y_n,w_1)\ll \frac{1}{3}v\quad \forall\, n\ge n_2. \tag{3.15}$$

The two sequences $\{p(x_n,w_0)\}_{n\in\mathbb{N}}$ and $\{p(y_n,w_1)\}_{n\in\mathbb{N}}$ are two V- sequences in C, their images via L and S are V-sequences, in particular, we have

$$\forall\, v\in Int(C)\ \exists\, n_3\in\mathbb{N} \text{ such that } L(p(x_n,w_0))\ll \frac{1}{3}v\quad \forall\, n\ge n_3. \tag{3.16}$$

$$\forall\, v\in Int(C)\ \exists\, n_4\in\mathbb{N} \text{ such that } S(p(y_n,w_1))\ll \frac{1}{3}v\quad \forall\, n,m\ge n_4. \tag{3.17}$$

Let $N_0=\max\{n_1,n_2,n_3,n_4\}$ and $v\in Int(C)$ and use (3.14), (3.16), and (3.17). Then we have

$$\begin{aligned}
p(T(w_0,w_1),w_0) &\preceq [p(T(w_0,w_1),x_{N_0+1})+p(x_{N_0+1},w_0)]\\
&=[p(T(w_0,w_1),T(x_{N_0},y_{N_0}))+p(x_{N_0+1},w_0)]\\
&=[L(p(w_0,x_{N_0})+S(p(w_1,y_{N_0}))]+p(x_{N_0+1},w_0)\\
&\ll \frac{1}{3}v+\frac{1}{3}v+\frac{1}{3}v=v
\end{aligned}$$

Since v is an arbitrary element, we have $p(T(w_0,w_1),w_0)=\theta$, the metric like condition (3) of p insures that $T(w_0,w_1)=w_0$ and similarly $T(w_1,w_0)=w_1$, meaning that (w_0,w_1) is a coupled fixed point of T. Finally we show that such a coupled fixed point is unique. Contrarily, suppose that (w_2,w_3) is another coupled fixed point, then we have

$$\begin{aligned}
p(w_0,w_2)+p(w_1,w_3) &= p(T(w_0,w_1),T(w_2,w_3))+p(T(w_1,w_0),T(w_3,w_2))\\
&\preceq [L(p(w_0,w_2))+S(p(w_1,w_3))]+[L(p(w_1,w_3))+S(p(w_0,w_2))]\\
&\preceq (L+S)(p(w_0,w_2)+p(w_1,w_3)).
\end{aligned}$$

This gives

$$(I-(L+S))(p(w_0,w_2)+p(w_1,w_3))=\theta,$$

$$p(w_0,w_2)+p(w_1,w_3)=(I-(L+S))^{-1}(\theta)=\theta.$$

Therefore; $p(w_0,w_2)=-p(w_1,w_3)$, since both $p(w_0,w_2)$ and $p(w_1,w_3)$ are belonging to C and $C\bigcap -C=\{\theta\}$, we should have $p(w_0,w_2)=p(w_1,w_3)=\theta$, and then the metric like condition (3) of p insures that $w_0=w_2$ and $w_1=w_3$, this means that the coupled fixed point is unique and completes the proof. □

We have the following Corollaries:

Corollary 3.7. Let (X,C,p) be a θ-complete cone metric like space, $S:\mathbb{A}\to\mathbb{A}$ be a linear bounded V-mapping and the spectral radius of S is strictly less than half, $r(S)<\frac{1}{2}$. Suppose that $T:X\times X\to X$ is a mapping such that

$$q(T(x,y),T(z,w))\preceq S(p(x,z))+p(y,w)) \quad \forall\, x,y,z,w\in X. \tag{3.18}$$

Then T has a unique coupled fixed point.

Proof . Using Theorem (3.6) with $S=L$ completes the proof. □

Corollary 3.8. Let (X,C,p) be a θ-complete cone metric like space. Suppose that the mapping $T:X\times X\to X$ satisfies the following contractive condition for all $x,y,w,z\in X$:

$$p(T(x,y),T(w,z))\preceq kp(x,w)+lp(y,z),$$

where k,l are nonnegative constants with $k+l<1$. Then T has a unique coupled fixed point.

Proof . Using Theorem (3.6) with $L(u)=ku$ and $S(u)=lu$, $(L+S)(u)=(l+k)u$, $\|L+S\|=l+k$ completes the proof. □

The following corollary proves theorem (1.5) of Sabetghadam, Masiha, and Sanatpour:

Corollary 3.9. Let (X,C,p) be a complete cone metric space. Suppose that the mapping $T:X\times X\to X$ satisfies the following contractive condition for all $x,y,w,z\in X$:

$$p(T(x,y),T(w,z))\preceq kp(x,w)+lp(y,z),$$

where k,l are nonnegative constants with $k+l<1$. Then T has a unique coupled fixed point.

Proof . Since every complete cone metric space is θ-complete cone metric like space, using Corolary (3.8) with $L(u)=ku$ and $S(u)=lu$, $(L+S)(u)=(l+k)u$, $\|L+S\|=l+k$ completes the proof. □

We also have the following:

Corollary 3.10. Let (X,C,p) be a θ-complete cone metric space. Suppose that the mapping $T:X\times X\to X$ satisfies the following contractive condition for all $x,y,w,z\in X$:

$$p(T(x,y),T(w,z))\preceq t[p(x,w)+p(y,z)],$$

where t is nonnegative constant with $t<\frac{1}{2}$. Then T has a unique coupled fixed point.

Proof . Using corollary (3.8) with $l=k<\frac{1}{2}$ completes the proof. □

3.2 Counter Examples Supporting and Verifying Inclusions of Diagram (3.1)

The following gives an example of pseudo cone metric space which is neither partial cone metric space nor cone metric like, hence not cone metric space.

Example 5. Let $M_m(\mathbb{R})$ be the set of all $m \times m$ matrices of real entries, C be the cone of all matrices of non-negative real entries in $M_m(\mathbb{R})$, $X = M_m(l_p)$ be the set of all $m \times m$ matrices, where the entries of each matrix in $M_m(l_p)$ are elements of l_p,

$$A = [a^{ij}]_{1 \leq i,j \leq m} \in M_m(l_p) \Longrightarrow a^{ij} = \{a_n^{ij}\}_{n \in \mathbb{N}} \in l_p \quad \forall 1 \leq i,j \leq m,$$

with the usual linear structure of addition and scalar multiplication [note that the norm $\|A\|_\infty = \|[a^{ij}]_{1 \leq i,j \leq m}\|_\infty = \max_{i=1}^m \sum_{j=1}^m \|a^{ij}\|_p$, where $\|a^{ij}\|_p = \sqrt[p]{\sum_{n=1}^\infty |a_n^{ij}|^p}$ makes $M_m(l_p)$ Banach space], and p be the C valued function $p : M_m(l_p) \times M_m(l_p) \to C$ defined for every $A = [a^{ij}]_{1 \leq i,j \leq m}, B = [b^{ij}]_{1 \leq i,j \leq m} \in M_m(l_p)$ by

$$\begin{aligned} p(A,B) &= p([a^{ij}]_{1 \leq i,j \leq m}, [b^{ij}]_{1 \leq i,j \leq m}) \\ &:= [\max\{\|a^{ij}\|_p, \|b^{ij}\|_p\} - \min\{\|a^{ij}\|_p, \|b^{ij}\|_p\}]_{1 \leq i,j \leq m} \in C. \end{aligned}$$

Then p is pseudo cone metric on $M_m(l_p)$ which is not partial cone metric (therefore it is not cone metric). Indeed, if $A = B = [a^{ij}]_{1 \leq i,j \leq m} \in M_m(l_p)$, then

$$\begin{aligned} p(A,A) = p(A,B) &= p([a^{ij}]_{1 \leq i,j \leq m}, [a^{ij}]_{1 \leq i,j \leq m}) \\ &= [\max\{\|a^{ij}\|_p, \|a^{ij}\|_p\} - \min\{\|a^{ij}\|_p, \|a^{ij}\|_p\}]_{1 \leq i,j \leq m} \\ &= \theta = \text{matrix each one of its entry is zero} \\ &= \text{zero matrix} \in C. \end{aligned}$$

On the other side, if $p(A,B)$ equals the zero matrix, then

$$[\max\{\|a^{ij}\|_p, \|b^{ij}\|_p\} - \min\{\|a^{ij}\|_p, \|b^{ij}\|_p\}]_{1 \leq i,j \leq m} = \text{zero matrix},$$

hence, $\max\{\|a^{ij}\|_p, \|b^{ij}\|_p\} - \min\{\|a^{ij}\|_p, \|b^{ij}\|_p\} = 0$ for every $1 \leq i,j \leq m$, $\max\{\|a^{ij}\|_p, \|b^{ij}\|_p\} = \min\{\|a^{ij}\|_p, \|b^{ij}\|_p\}$ for every $1 \leq i,j \leq m$, this imply $\|a^{ij}\|_p = \|b^{ij}\|_p$ for every $1 \leq i,j \leq m$, one can construct infinitely many different matrices with this condition. Actually, for example take A and B having the same entries except

$$a^{11} = \{\frac{1}{2^{\frac{n}{p}}}\}_{n \geq 1} \text{ and } b^{11} = \{\frac{2}{3^{\frac{n}{p}}}\}_{n \geq 1},$$

we have $\|a^{11}\|_p = \sqrt[p]{1} = 1$ and $\|b^{11}\|_p = 2\sqrt[p]{\frac{\frac{1}{3}}{1-\frac{1}{3}}} = 2 \times \frac{1}{2} = 1$, hence $A \neq B$ while $p(A,B) = p(A,A) = p(B,B) = 0$.

The following is an example of partial cone metric which is not cone metric space:

Example 6. Let $M_m(\mathbb{R})$ be the set of all $m \times m$ matrices of real entries, C be the cone of all matrices of non-negative real entries, $Y = \{[a,b] : a,b \in \mathbb{R}^+, a \leq b\}$, the set of all closed bounded intervals of real numbers, $X = M_m(Y)$ be the set of all $m \times m$ matrices, the entries of each matrix in $X = M_m(Y)$ are elements of Y,

$$A = [a^{ij}]_{1 \leq i,j \leq m} \in X \Longrightarrow a^{ij} = [a_*^{ij}, a_{**}^{ij}] \in Y, \quad \forall 1 \leq i,j \leq m.$$

Define the C valued function $p : M_m(Y) \times M_m(Y) \to C$ for every $A = [a^{ij}]_{1 \leq i,j \leq m}, B = [b^{ij}]_{1 \leq i,j \leq m} \in M_m(Y)$ as follows:

$$\begin{aligned} p(A,B) &= p([a^{ij}]_{1 \leq i,j \leq m}, [b^{ij}]_{1 \leq i,j \leq m}) \\ &:= [\max\{a_{**}^{ij}, b_{**}^{ij}\} - \min\{a_*^{ij}, b_*^{ij}\}]_{1 \leq i,j \leq m}. \end{aligned}$$

Then p is a partial cone metric which is not a pseudo cone metric on $M_m(X)$ (hence, it is not cone metric space). We have

$$\begin{aligned} p(A,A) &= p([a^{ij}]_{1 \leq i,j \leq m}, [a^{ij}]_{1 \leq i,j \leq m}) \\ &= [\max\{a_{**}^{ij}, a_{**}^{ij}\} - \min\{a_*^{ij}, a_*^{ij}\}]_{1 \leq i,j \leq m} \\ &= [a_{**}^{ij} - a_*^{ij}]_{1 \leq i,j \leq m} \\ &= [length(a^{ij})]_{1 \leq i,j \leq m} \neq \text{the zero matrix}. \end{aligned}$$

The following is an example of cone pmetric like which is not partial cone metric.

Example 7. Let C be a given cone in a normed space $\mathbb{A}$, v be a non-zero element in C, $\theta \neq v \in C$, x, y be two distinct real numbers, and $X = \{x, y\}$, and $p : X \times X \to C$ defined by

$$p(x,x) = p(y,y) = p(x,y) = p(y,x) = v.$$

Then p is cone pmetric like on X which is not partial cone metric.

The following examples are examples of cone metric like spaces which are not cone pmetric like (not partial cone metric).

Example 8. Let C be the cone $C = \{(a,b) : (a,b) \in \mathbb{R}^2 : a, b \in \mathbb{R}^+\}$ in $\mathbb{R}^2$, $X = \{x, y\}$, and $p : X \times X \to C$ defined by

$$p(x,x) = (2,0),\ p(y,y) = p(x,y) = p(y,x) = (1,0).$$

Then p is cone metric like which is not partial cone metric because $p(x,x)$ is not precedes $p(x,y)$, $p(x,y) - p(x,x) = (-1,0) \notin C$, note also that $p(x,x)$ is not precedes $p(x,y) + p(y,x) - p(y,y)$. Hence, it is also not pmetric like.

Example 9. Let $M_2(\mathbb{R})$, C be the cone of all 2×2 matrices of positive entries, $X = \{a, b, c\}$, and $p : X \times X \to C$ defined by

$$p(a,a) = \begin{pmatrix} \frac{5}{4} & 0 \\ 0 & 1 \end{pmatrix}, p(b,b) = p(a,b) = p(b,a) = p(c,c) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$p(a,c) = p(c,a) = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & 1 \end{pmatrix}, p(b,c) = p(c,b) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Then p is cone metric like which is not cone pmetric (not partial cone metric) because

$$[p(a,c) + p(c,a)] - p(c,c), [p(c,b) + p(b,c)] - p(c,c), [p(a,c) + p(c,a)] - p(a,a) \in C,$$

$$[p(a,b) + p(b,a)] - p(a,a), [p(b,c) + p(c,b)] - p(b,b), [p(a,b) + p(b,a)] - p(b,b) \in C,$$

$$[p(a,c) + p(c,b)] - p(a,b) = \begin{pmatrix} (\frac{3}{4} + \frac{1}{2}) - 1 & 0 \\ 0 & (1 + \frac{1}{2}) - 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in C,$$

$$[p(b,a) + p(a,c)] - p(b,c) = \begin{pmatrix} (1 + \frac{3}{4}) - \frac{1}{2} & 0 \\ 0 & (1 + 1) - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \in C,$$

$$[p(a,b) + p(b,c)] - p(a,c) = \begin{pmatrix} (1 + \frac{1}{2}) - \frac{3}{4} & 0 \\ 0 & [1 + \frac{1}{2}] - 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in C,$$

while for example

$$[p(a,b) + p(b,c) - p(b,b)] - p(a,c) = p(b,c) - p(a,c) = \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \notin C.$$

We have the following simple example with calculations:

Example 10. Let $M_2(\mathbb{R})$, $A = [a^{ij}]_{1 \le i,j \le 2} \in M_2(\mathbb{R})$ with the norm $\|A\| = \max_{i=1,2}\{|a^{i1}| + |a^{i2}|\}$, C be the cone of all 2×2 matrices of non-negative real entries, $X = \{a, b\}$, and $p : X \times X \to C$ defined by

$$p(a,a) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, p(b,b) = p(a,b) = p(b,a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then p is cone metric like which is not cone partial metric (not cone pmetric like) because $p(a,b) - p(a,a) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \notin C$, that is; $p(a,a)$ does not precedes $p(a,b)$. On the other hand $Range(p) = \{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$ with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \preceq \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. For the open p-balls $U_p(a,v) = \{y : y \in X, p(a,y) \ll v\}$, let $v = v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, v = v_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $v \in C$ be comparable with any of v_1 or v_2, that is; one of the following hold $v = v_3 < v_1$, $v_1 < v = v_4 < v_2$, and $v_2 < v = v_5$, and $v = v_6$ be not comparable with each of v_1 and v_2. Then

$$\begin{aligned}
&U_p(a, v_1) = U_p(a, v_3) = U_p(a, v_6) = \emptyset,\\
&U_p(a, v_2) = U_p(a, v_4) = \{b\},\\
&U_p(a, v_5) = X,\\
&U_p(b, v_1) = U_p(b, v_3) = U_p(b, v_6) = \emptyset,\\
&U_p(b, v_2) = U_p(b, v_4) = U_p(b, v_5) = X.
\end{aligned}$$

Now; at least one entry of v_3, v_4, v_5, and v_6 is strictly greater than zero because they are lying in C,

$$0 \leq v_3^{11} < v_1^{11},\ 0 \leq v_3^{12} < v_1^{12},\ 0 \leq v_3^{21} < v_1^{11},\ \text{and } 0 \leq v_3^{22} < v_1^{22},$$

$$v_1^{11} < v_4^{11} < v_2^{11},\ v_1^{12} < v_4^{12} < v_2^{12},\ v_1^{21} < v_4^{21} < v_2^{11},\ \text{and } v_1^{22} < v_4^{22} < v_2^{22},$$

and

$$v_2^{11} < v_6^{11},\ v_2^{12} < v_6^{12},\ v_2^{21} < v_6^{11},\ \text{and } v_2^{22} < v_6^{22},$$

$$\begin{aligned}
&B_p(a, v_1) = B_p(a, v_2) = B_p(a, v_3) = B_p(a, v_4) = B_p(a, v_5) = B_p(a, v_6) = X,\\
&B_p(b, v_1) = \{b\},\\
&B_p(b, v_2) = B_p(b, v_3) = B_p(b, v_4) = B_p(b, v_6) = X.
\end{aligned}$$

Hence; $\mathbb{U} = \{\emptyset, X, \{b\}\}$ and $\mathbb{B} = \{X, \{b\}\}$.

We also have the following example:

Example 11. Let $\mathbb{R}$ be the Banach space of real numbers with the usual absolute value metric, $C = \mathbb{R}^+$ be the cone of non-negative real numbers, $X = \{\ [a, b] : a, b \in \mathbb{R}, a \leq b\ \}$, and define $p : X \times X \to \mathbb{R}^+$ by

$$p(x, y) = p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\} \quad \forall x = [a, b], y = [c, d] \in X.$$

Then (X, p) is a partial cone metric space. We have the following:

1. For any $x = [a, b] \in X$, $a < b$, we have $p(x, x) = p([a, b], [a, b]) = b - a$, this means that $p(x, x)$ equals the length of the interval $x = [a, b]$ for every $x \in X$, $p(x, x)$ does not equal to zero.
2. Let $x = [a, b] \in X$ and $0 < \epsilon < b - a$, we see that

$$[a, b] \notin B_p([a, b], \epsilon) = \{[c, d] \in X : \max\{b, d\} - \min\{a, c\} < \epsilon\}.$$

   But the basis elements $B_p([a, b], b - a + \epsilon)$ contains $[a, b]$.
3. A limit of a sequence in a partial metric space need not be unique, a convergent sequence may have infinitely many limits. Indeed, let $x_n := [\frac{n}{(n+1)}, \frac{2n}{(n+1)}]$ and $x = [1, 2]$. Then for the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, we have

$$\begin{aligned}
p(x_n, x) &= p([\frac{n}{(n+1)}, \frac{2n}{(n+1)}], [1, 2])\\
&= \max\{\frac{2n}{(n+1)}, 2\} - \min\{\frac{n}{(n+1)}, 1\} = 2 - \frac{n}{(n+1)}
\end{aligned}$$

$$\begin{aligned}
|p(x_n, x) - p(x, x)| &= |p([\frac{n}{(n+1)}, \frac{2n}{(n+1)}], [1, 2]) - p([1, 2], [1, 2])|\\
&= |[2 - \frac{n}{(n+1)}] - [2 - 1]| = 1 - \frac{n}{(n+1)} \longrightarrow_{n\to\infty} 0
\end{aligned}$$

   hence, $\{x_n\}_{n\in\mathbb{N}}$ is $p - |.|$-convergent to $[1, 2]$. Similarly, it is also $p - |.|$-convergent to $[1, c]$ for every c, $c \geq 2$, because

$$\lim_{n\to\infty} p([\frac{n}{(n+1)}, \frac{2n}{(n+1)}], [1, c]) = c - 1 = p([1, c], [1, c]) \quad \forall\, c \geq 2,$$

hence, we have

$$[\frac{n}{(n+1)}, \frac{2n}{(n+1)}] \longrightarrow_{n\to\infty} [1,2] - (p - |.|)$$

and

$$[\frac{n}{(n+1)}, \frac{2n}{(n+1)}] \longrightarrow_{n\to\infty} [1,c] - (p - |.|) \quad \forall\, c \geq 2,$$

That is; the generated topology is not Hausdorff.

4. Moreover, the cone valued function p is not continuous in the sense that if $x_n \to_{n\to\infty} x_0 - (p - |.|)$ and $y_n \to_{n\to\infty} y_0 - (p - |.|)$, then imply $p(x_n, y_n) \to_{n\to\infty} p(x_0, y_0)$. Indeed, for $x_n = y_n = [\frac{n}{(n+1)}, \frac{2n}{(n+1)}]$, $x_0 = [1,2]$ and $y_0 = [1,c]$ for $c > 2$, we have

$$\begin{aligned} \lim_{n\to\infty} p([\frac{n}{(n+1)}, \frac{2n}{(n+1)}], [\frac{n}{(n+1)}, \frac{2n}{(n+1)}]) &= \lim_{n\to\infty} (\frac{2n}{(n+1)} - \frac{n}{(n+1)}) \\ &= \lim_{n\to\infty} \frac{n}{(n+1)} = 1, \end{aligned}$$

while $p([1,2],[1,c]) = c - 1 \neq 1 \quad$ for $c > 2$.

3.3 Conclusion

This paper gives a study of various types of cone metric spaces and its topological characterization, and considers some generalized contraction type of mappings on θ-complete cone metric like spaces, then extends and generalizes some previous coupled fixed point theorems in this setting.

4 Declarations

Ethics Approval and Consent to Participate:. Not applicable.

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