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Stability of maximum preserving in multi-Banach lattice by fixed point method with some of homogeneity properties

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Abstract

In this research work, we demonstrate the Hyers-Ulam stability for functions that are homogeneous of degree k, in multi-Banach lattice by fixed point method.

Keywords: Hyers-Ulam stability; functional equation; fixed point technique multi- Banach lattice, minimum preserving functional equation, homogeneity of degree k, quadratic and cubic functional equations
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1 Introduction

The stability problem of functional equations originated from a question of Ulam [22] in 1940. Hyers is the first mathematician who answered the question of Ulam [12] in 1941. The Hyers stability theorem was developed by other mathematicians Hyers theorem was generalized by Aoki [8] for additive mappings and by Rassias [19] for linear mappings by considering an unbounded Cauchy difference. The paper [19] of Rassias has significantly influenced the development of what we now call the Hyers-Ulam Rassias stability of functional equations. During the past decades,Several stability problems of functional equations have been extensively investigated by a number of authors(see [9, 13, 14, 21, 15])

H. G. Dales and M. E. Polyakov [10] introduced the concept of multi-normed space in their article. Multi normed space has a relation with ordered vector spaces and operator spaces. Furthermore, this concept is somewhat similar to that of the operator sequence space. We have collected some properties of multi-normed spaces which will be used in this article. We refer readers to [11, 16, 17, 10] for more details.

Agbeko has studied the stability of maximum preserving functional equations motivated by the optimal average (see [1, 2, 3, 5, 7, 6, 4]). He has replaced addition operation with the maximum operation on a given Banach lattice. This new approach can be extended to other branches of mathematics for example see [18].

In [23] the maximum preserving functional equation for cauchy functional equation has been proved by replacing addition with supremum. In [20] this property has been proved for quadratic functional equation in Banach lattice. In this paper, we generalized them for homogeneous function of degree m ($m \in \mathbb{N}$) in multi-Banach lattice by fixed point method.

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Definition 1.1. A Banach lattice $(E, \|\cdot\|)$ is a partially ordered real Banach space for which
(i) If $x, y \in E$ such that $x \leq y$ then $x+z \leq y+z$ for all $z \in E$,
(ii) If $x, y \in E$ such that $x \leq y$ then $\alpha x \leq \alpha y$ for all $\alpha \geq 0$,
(iii) the least upper bound $x \vee y$ and the greatest lower bound $x \wedge y$ exist for every $x, y \in E$,
(iv) $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ (where $|x| = x \vee (-x)$).

Definition 1.2. Let X be a set. A function $d : X^2 \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

(M1) $d(x, y) = 0$ if and only if $x = y$;

(M2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We remark that the only difference between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We now introduce one of the fundamental results of the fixed point theory.

Theorem 1.3. Let (X,d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer n_0 such that $d(\Lambda^{n_0+1}x, \Lambda^{n_0}x) < \infty$ for some $x \in X$, then the following statements are true:

(i) The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;

(ii) x^* is the unique fixed point of Λ in $X^* = \{y \in X | d(\Lambda^{n_0}x, y) < \infty\}$;

(iii) If $y \in X^*$, then
$$d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$$

Now, recalling the notion of a multi-normed space from [11, 10]. In this paper, $(E, \|\cdot\|)$ denotes a complex normed space and let $k \in \mathbb{N}$. We denote by E^k the linear space $E \oplus \cdots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$ the linear operations E^k are defined coordinatewise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \ldots, k\}$ and by $\mathfrak{S}_k$ the group of permutations on k symbols.

Definition 1.4. A multi-norm on $\{E^k : k \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_k) = (\|\cdot\|) : k \in \mathbb{N})$ such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

N1 $\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\|_k = \|x_1, \ldots, x_k)\|_k \qquad (\sigma \in \mathfrak{S}_k; x_1, \ldots, x_k \in E)$;

N2 $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|)\|(x_1, \ldots, x_k)\|_k \qquad (\alpha_1, \ldots, \alpha_k \in \mathbb{C}; x_1, \ldots x_k \in E)$;

N3 $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1} \qquad (x_1, \ldots, x_{k-1} \in E)$;

N4 $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1} \qquad (x_1, \ldots, x_{k-1} \in E)$.

In this case, we say that $((E^k, \|\cdot\|) : k \in \mathbb{N})$ is a *multi-normed space*. The motivation for the study of multi-normed spaces (and multi-normed algebras) and many examples are detailed in the earlier investigation [11]. Suppose that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space, and take $k \in \mathbb{N}$. The following properties are almost immediate consequences of the axioms.
(a) $\|(x, \ldots, x)\|_k = \|x\| \qquad (x \in E)$;
(b) $\max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, \ldots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\| \quad (x_1, \ldots, x_k \in E)$.

It follows from the item (b) above that, if $(E, \|\cdot\|)$ is a Banach space, then $(E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$; in this case, $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Example 1.5. Let $(E, \|\cdot\|)$ be Banach lattice and define
$$\|(x_1, \ldots, x_k)\|_k := \||x_1| \vee \cdots \vee |x_k|\| \qquad (x_1, \ldots, x_k) \in E$$

Then $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space(see [11]). We say it *multi-Banach lattice*.

2 Main Results

Throughout this section, let $(E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice and $p : [0,\infty) \to [0,\infty)$ be continuous function, fixed $m \geq 1$ and $\tau, \eta \in \mathbb{R}^+$. For convenience, we use the following abbreviation for a given mapping $f : E_1 \to E_2$

$$Df(x,y) = f(\tau|x| \vee \eta|y|) - \frac{(\tau^m p(\tau)f(|x|) \vee \eta^m p(\eta)f(|y|))}{p(\tau) \vee p(\eta)}.$$

Let us recall some necessary definitions. If B is a Banach lattice, then B^+ stands for its positive cone, i.e.

$$B^+ = \{x \in B : x \geq 0\} = \{|x| : x \in B\}.$$

Definition 2.1. Let X and Y be Banach lattices, a mapping $f : X \to Y$ is called cone-related if $f(X^+) = \{f(|x|) : x \in B\} \subset Y^+$(see [3]).

Let X and Y be two Banach lattices and $f : X \to Y$ be a cone-related functional, with following properties:

I) Minimum Preserving Functional Equation: $f(|x| \vee |y|) = f(|x|) \vee f(|y|)$ for all members $x, y \in X$(see [3]).

II) Homogeneity of degree m: $f(\alpha|x|) = \alpha^m f(|x|)$ for all $x \in X$ and every number $\alpha \in [0,\infty)$.

We shall use the technics in [3] to prove the following two theorems.

Theorem 2.2. Let E_1 and E_2 be two Banach lattices and $((E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice . Suppose $\phi : E_1^{2k} \to [0,\infty)$ is a given function and there exist constants $m \geq 1$ and L, $0 < L < 1$, such that

$$\phi(2x_1, 2y_1, \ldots, 2x_k, 2y_k) \leq 2^m L\phi(x_1, y_1, \ldots, x_k, y_k) \tag{2.1}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E_1$. Furthermore, let $f : E_1 \to E_2$ be a cone-related function with $f(0) = 0$ which satisfies

$$\|Df(x_1, y_1), \ldots, Df(x_k, y_k)\|_k \leq \phi(\tau x_1, \eta y_1, \ldots, \tau x_k, \eta y_k) \tag{2.2}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E_1$, then there is a unique cone-related mapping $T : E_1 \to E_2$ which satisfies properties I, II and the inequality.

$$\|T(|x_1|) - f(|x_1|), \ldots, T(|x_k|) - f(|x_k|)\|_k \leq \frac{L}{1-L}\phi(x_1, x_1, \ldots, x_k, x_k) \tag{2.3}$$

Proof . If we define

$$X = \{g : E_1 \to E_2| \;\; g(0) = 0\}$$

and introduce a generalized metric on X as follows:

$$\begin{aligned} d(g,h) = \inf\{c \in [0,\infty] : & \|g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\|_k \\ & \leq c\phi(x_1, x_1, \ldots, x_k, x_k), \;\; \text{for all } x_1, \ldots, x_k \in E_1\} \end{aligned}$$

then (X, d) is complete. We define an operator $\Lambda : X \to X$ by

$$(\Lambda g)(x) = \frac{g(2x)}{2^m}$$

for all $x \in E_1$. First, we assert that Λ is strictly contractive on X. Given $g, h \in X$, let $c \in [0,\infty)$ be an arbitrary constant with $d(g,h) \leq c$, i.e.,

$$\|g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\|_k \leq c\phi(x_1, x_1, \ldots, x_k, x_k),$$

for all $x_1, \ldots, x_k \in E_1$. If we replace x in the last inequality with $2x$ and make use of (2.1), then we have

$$\begin{aligned} \|\Lambda g(x_1) - \Lambda h(x_1), \ldots, \Lambda g(x_k) - \Lambda h(x_k)\|_k = & 2^{-m} \; \|g(2x_1) - h(2x_1), \ldots, g(2x_k) - h(2x_k)\|_k \\ \leq & 2^{-m} \; c\phi(2x_1, 2x_1, \ldots, 2x_k, 2x_k) \\ \leq & Lc\phi(x_1, x_1, \ldots, x_k, x_k) \end{aligned}$$

for every $x_1, \dots, x_k \in E_1$, i.e., $d(\Lambda g, \Lambda h) \leq Lc$. Hence, we conclude that $d(\Lambda g, \Lambda h) \leq Ld(g,h)$ for any $g, h \in X$. Next, we assert that $d(\Lambda f, f) < \infty$. If we substitute x for y in (2.2) and $\tau = \eta = 2$, then (2.1) establishes

$$\|f(2|x_1|) - 2^m f(|x_1|), \dots, f(2|x_k|) - 2^m f(|x_k|)\|_k \leq \phi(2x_1, 2x_1, \dots, 2x_k, 2x_k)$$

then

$$\begin{aligned}\left\| \frac{f(2|x_1|)}{2^m} - f(|x_1|), \dots, \frac{f(2|x_k|)}{2^m} - f(|x_k|) \right\|_k &\leq 2^{-m}\ \phi(2x_1, 2x_1, \dots, 2x_k, 2x_k) \\ &\leq L\phi(x_1, x_1, \dots, x_k, x_k)\end{aligned}$$

and so,

$$\|\Lambda f(|x_1|) - f(|x_1|), \dots, \Lambda f(|x_k|) - f(|x_k|)\| \leq L\phi(x_1, x_1, \dots, x_k, x_k)$$

for any $x_1, \dots, x_k \in E_1$, i.e.,

$$d(\Lambda f, f) \leq L \leq \infty. \tag{2.4}$$

Now, it follows from Theorem 1.3 (i) that there exists a function $T : E_1 \to E_2$ with $T(0) = 0$, which is a fixed point of Λ, such that $\Lambda^n f \to T$, i.e.,

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{nm}}$$

for all $x \in E_1$. Since the integer n_0 of Theorem 1.3 is 0 then $f \in X^*$ which

$$X^* = \{y \in X : \quad d(\Lambda^{n_0} f, y) < \infty\}.$$

By Theorem 1.3 (iii) and (2.4) we obtain

$$d(f, T) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{L}{1-L}$$

i.e., the inequality (2.3) is true for all $x \in E$. Clearly, T is a cone-related operator. Let us show that T is maximum preserving. Let $\tau = \eta = 2^n$ in (2.2) we have

$$\begin{aligned}\|f(2^n(|x_1| \vee |y_1|)) - 2^{nm}(f(|x_1|) \vee f(|y_1|)), \dots, f(2^n(|x_k| \vee |y_k|)) \\ - 2^{nm}(f(|x_k|) \vee f(|y_k|))\|_k \leq \phi(2^n x_1, 2^n y_1, \dots, 2^n x_k, 2^n y_k).\end{aligned}$$

Substituting $x_1, \dots, x_k$ with $2^n x_1, \dots, 2^n x_k$ and $y_1, \dots, y_k$ with $2^n y_1, \dots, 2^n y_k$ in the last inequality:

$$\begin{aligned}\|f(2^{2n}(|x_1| \vee |y_1|)) - 2^{nm}(f(2^n|x_1|) \vee f(2^n|y_1|)), \dots, f(2^{2n}(|x_k| \vee |y_k|)) - 2^{nm}(f(2^n|x_k|) \vee f(2^n|y_k|))\|_k \\ \leq \phi(2^{2n}x_1, 2^{2n}y_1, \dots, 2^{2n}x_k, 2^{2n}y_k).\end{aligned}$$

Thus

$$\begin{aligned}\|4^{-nm} f(2^{2n}(|x_1| \vee |y_1|)) - 2^{-nm}(f(2^n|x_1|) \vee f(2^n|y_1|)), \dots, 4^{-nm} f(2^{2n}(|x_k| \vee |y_k|)) - 2^{-nm}(f(2^n|x_k|) \vee f(2^n|y_k|))\|_k \\ \leq 4^{-nm}\phi(2^{2n}x_1, 2^{2n}y_1, \dots, 2^{nm}x_k, 2^{2n}y_k).\end{aligned}$$

with use of (2.1)

$$\begin{aligned}\|4^{-nm} f(2^{2n}(|x_1| \vee |y_1|)) - 2^{-nm}(f(2^n|x_1|) \vee f(2^n|y_1|)), \dots, 4^{-nm} f(2^{2n}(|x_k| \vee |y_k|)) - 2^{-nm}(f(2^n|x_k|) \vee f(2^n|y_k|))\|_k \\ \leq 2^{-nm}L^n\phi(2^n x_1, 2^n y_1, \dots, 2^n x_k, 2^n y_k).\end{aligned}$$

By inequality (2.1) , we have

$$\lim_n 2^{-nm}\phi(2^n x_1, 2^n y_1, \dots, 2^n x_k, 2^n y_k) = 0 \tag{2.5}$$

By letting $n \to \infty$ and considering (2.2), replace $x_1, \dots, x_k$ with x and $y_1, \dots, y_k$ with y in the last inequality conclude

$$\lim_{n\to\infty} \|4^{-nm} f(4^n(|x| \vee |y|)) - 2^{-nm}(f(2^n|x|) \vee f(2^n|y|))\| = 0$$

we get for all $x, y \in X$ the equality

$$\|T(|x| \vee |y|) - T(|x|) \vee T(|y|)\| = 0$$

or equivalently

$$T(|x| \vee |y|) = T(|x|) \vee T(|y|),$$

because,

$$\lim_{n\to\infty} 4^{-nm} f(4^n|z|) = \lim_{p\to\infty} 2^{-pm} f(2^p|z|) = T(|z|), \quad z \in X$$

Now, we must show $T(r|x|) = r^m T(|x|)$ for all $x \in X$ and $r \in [0, \infty)$. Using the inequality (2.2) with $\eta = \tau$, $y_1, \dots, y_k = 0$ and substituting τ with $2^n\tau$:

$$\|f(2^n\tau(|x_1|)) - 2^{nm}\tau^m(f(|x_1|)), \dots, f(2^n\tau(|x_k|) - 2^{nm}\tau^m(f(|x_k|))\|_k \le \phi(2^n\tau x_1, 2^n\tau x_1, \dots, 2^n\tau x_k, 2^n\tau x_k).$$

If we replace $x_1, \dots, x_k$ with $2^n x_1, \dots, 2^n x_k$ respectively, then:

$$\|f(2^{2n}\tau(|x_1|)) - 2^{nm}\tau^2(f(2^n|x_1|)), \dots, f(2^{2n}\tau(|x_k|) - 2^{nm}\tau^2(f(2^n|x_k|))\|_k \le \phi(2^{2n}\tau x_1, 2^{2n}\tau x_1, \dots, 2^{2n}\tau x_k, 2^{2n}\tau x_k).$$

Divide by 4^{mn} both side of above inequality and use the inequality (2.1) :

$$\begin{aligned}\|4^{-nm} f(2^{2n}\tau(|x_1|)) - 2^{-nm}\tau^m(f(2^n|x_1|)), \dots, 4^{-nm} f(2^{2n}\tau(|x_k|) - 2^{-nm}\tau^m(f(2^n|x_k|))\|_k \\ \le 4^{-nm}\phi(2^{2n}\tau x_1, 2^{2n}\tau x_1, \dots, 2^{2n}\tau x_k, 2^{2n}\tau x_k) \\ \le 2^{-nm} L^n \phi(2^n\tau x_1, 2^n\tau x_1, \dots, 2^n\tau x_k, 2^n\tau x_k).\end{aligned}$$

By letting $n \to \infty$ and considering (2.1), replace $x_1, \dots, x_k$ with x in the last inequality conclude

$$\lim_{n\to\infty} \|4^{-nm} f(4^n(\tau|x|)) - 2^{-nm}\tau^m(f(2^n|x|))\| = 0$$

we get for all $x \in X$ the equality

$$\lim_{n\to\infty} 4^{-nm} f(4^n\tau|x|) = \tau^m \lim_{n\to\infty} 2^{-nm} f(2^n|x|) = \tau^m T(|x|),$$

by taking $z = \tau|x|$, we have

$$\tau^m T(|x|) = \lim_{n\to\infty} 4^{-nm} f(4^n\tau|x|) = \lim_{n\to\infty} 4^{-nm} f(4^n|z|) = T(|z|) = T(\tau|x|).$$

For uniqueness of T: Assume that the inequality (2.3) is also satisfied with another homogeneous function of degree two $S : E_1 \to E_2$ besides T. (As S is a homogeneous function of degree two, S satisfies that

$$S(x) = \frac{S(2x)}{2^m} = \Lambda S(x)$$

for all $x \in E_1$. That is, S is a fixed point of Λ.) In view of (2.3) and the definition of d, we know that

$$d(f, S) \le \frac{L}{1-L} < \infty$$

i.e., $S \in X^*$. (In view of (2.4), the integer n_0 of Theorem 1.3 is 0.) Thus, Theorem 1.3 (ii) implies that $S = T$. This proves the uniqueness of T. □

Theorem 2.3. Let E_1 and E_2 be two Banach lattices and $((E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice . Suppose $\phi : E_1^{2k} \to [0, \infty)$ is a given function and there exist constants $m \ge 1$ and L, $0 < L < 1$, such that

$$\phi(x_1, y_1, \dots, x_k, y_k) \le 2^{-m} L\phi(2x_1, 2y_1, \dots, 2x_k, 2y_k) \tag{2.6}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E_1$. Furthermore, let $f : E_1 \to E_2$ be a cone-related function with $f(0) = 0$ which satisfies

$$\|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \le \phi(\tau x_1, \eta y_1, \dots, \tau x_k, \eta y_k) \tag{2.7}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E_1$, then there is a unique cone-related mapping $T : E_1 \to E_2$ which satisfies properties I, II and the inequality.

$$\|T(|x_1|) - f(|x_1|), \dots, T(|x_k|) - f(|x_k|)\|_k \le \frac{L}{1-L}\phi(x_1, x_1, \dots, x_k, x_k) \tag{2.8}$$

Proof . We use the definitions for X and d, the generalized metric on X, as in the proof of Theorem 2.2. Then, (X,d) is complete. We define an operator $\Lambda : X \to X$ by

$$(\Lambda g)(x) = 2^m g(\frac{x}{2})$$

for all $x \in E_1$. We apply the same argument as in the proof of Theorem 2.2 and prove that Λ is a strictly contractive operator. Moreover, we prove that

$$d(\Lambda f, f) \leq L \tag{2.9}$$

instead of (2.4) in the proof of Theorem 2.2. According to Theorem 1.3 (i) there exists a function $T : E_1 \to E_2$ with $T(0) = 0$, which is a fixed point of Λ, such that

$$T(x) = \lim_{n\to\infty} 2^{nm} f(2^{-n}x)$$

for each $x \in E_1$. Since the integer n_0 of theorem 1.3 is 0 and $f \in X^*$ (see Theorem 2.2 for the definition of X^*), using theorem 1.3 (iii) and (2.9) yields

$$d(f,T) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{L}{1-L}.$$

By inequality (2.6) , we have

$$\lim_n 2^{nm}\phi(2^{-n}x_1, 2^{-n}y_1, \ldots, 2^{-n}x_k, 2^{-n}y_k) = 0 \tag{2.10}$$

which implies the validity of the inequality (2.8). In the last part of the proof of Theorem 2.2, if we replace $2^n x_1, \ldots, 2^n x_k$, $2^n y_1, \ldots, 2^n y_k$, and 4^{nm} with $2^{-n}x_1, \ldots, 2^{-n}x_k$, $2^{-n}y_1, \ldots, 2^{-n}y_k$, and 4^{-nm} , respectively, then we can prove that T is a unique homogeneous function of degree two satisfying inequality (2.8) for all $x \in E_1$. □

Theorem 2.4. Let E_1 and E_2 be two Banach lattices and $((E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice and $f : E_1 \to E_2$ be a cone-related functional for which there are numbers $\theta > 0$ and $m \geq 1$ and $0 \leq r < m$ such that

$$\|Df(x_1,y_1), \ldots, Df(x_k,y_k)\|_k \leq \theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r) \tag{2.11}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E_1$; then there is a unique cone-related mapping $T : E_1 \to E_2$ such that

$$\|T(|x_1|) - f(|x_1|), \ldots, T(|x_k|) - f(|x_k|)\|_k \leq \frac{2^{m+1}\theta}{2^m - 2^r} \sum_{i=1}^{k} \|x_i\|^r \tag{2.12}$$

and satisfies properties I, II .

Proof . It follows for theorem 2.2 by putting

$$\phi(x_1, y_1, \ldots, x_k, y_k) = \theta \sum_{i=1}^{k} \left(\|x_i\|^2 + \|y_i\|^2\right),$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E_1$, $L = 2^{r-m}$. □

Corollary 2.5. Let E_1 and E_2 be two Banach lattices and $((E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice and $p : [0,\infty) \to [0,\infty)$ be a continuous function $f : E_1 \to E_2$ be a cone-related functional for which there are numbers $\theta > 0$ and $m \geq 1$ and $0 \leq r < m$ such that

$$\|f(\tau|x_1| \vee \eta|y_1|) - \tau^m f(|x_1|) \vee \eta^m f(|y_1|), \ldots, f(\tau|x_k| \vee \eta|y_k|) - \tau^m f(|x_k|) \vee \eta^m f(|y_k|)\|_k \leq \theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r)$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E_1$ and $\tau, \eta \in \mathbb{R}^+$; then there is a unique cone-related mapping $T : E_1 \to E_2$ such that

$$\|T(|x_1|) - f(|x_1|), \ldots, T(|x_k|) - f(|x_k|)\|_k \leq \frac{2\theta}{2^m - 2^r} \sum_{i=1}^{k} \|x_i\|^r$$

and satisfies properties I, II .

Proof . Enough, we put $p(t) = 1$ in above theorem for $t \in [0, \infty)$. In this case, the sense of stability in multi-Banach lattice is similarity with stability of quadratic functional equation in Banach space. □

Corollary 2.6. Let E_1 and E_2 be two Banach lattices and $((E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice and $p : [0, \infty) \to [0, \infty)$ be a continuous function $f : E_1 \to E_2$ be a cone-related functional for which there are numbers $\theta > 0$ and $m \geq 1$ and $0 \leq r < m$ such that

$$\left\| f(\tau|x_1| \vee \eta|y_1|) - \frac{\tau^{m+1} f(|x_1|) \vee \eta^{m+1} f(|y_1|)}{\tau \vee \eta}., \dots, f(\tau|x_k| \vee \eta|y_k|) - \frac{\tau^{m+1} f(|x_k|) \vee \eta^{m+1} f(|y_k|)}{\tau \vee \eta} \right\|_k \leq \theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r).$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E_1$ and $\tau, \eta \in \mathbb{R}^+$; then there is a unique cone-related mapping $T : E_1 \to E_2$ such that

$$\|T(|x_1|) - f(|x_1|), \dots, T(|x_k|) - f(|x_k|)\|_k \leq \frac{2\theta}{2^m - 2^r} \sum_{i=1}^{k} \|x_i\|^r$$

and satisfies properties I, II .

Proof . Enough, we put $P(t) = t$ in above theorem. □

Theorem 2.7. Let E_1 and E_2 be two Banach lattices and $((E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice and $f : E_1 \to E_2$ be a cone-related functional for which there are numbers $\theta > 0$ and $m \geq 1$ and $0 \leq r < m$ such that

$$\|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \leq \theta \sum_{i=1}^{k} \sum_{j=1}^{k} \|x_i y_j\|^r \tag{2.13}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E_1$; then there is a unique cone-related mapping $T : E_1 \to E_2$ such that

$$\|T(|x_1|) - f(|x_1|), \dots, T(|x_k|) - f(|x_k|)\|_k \leq \frac{4^r \theta}{2^m - 4^r} \left(\sum_{i=1}^{k} \|x_i\|^r \right)^2 \tag{2.14}$$

and satisfies properties I, II .

Proof . It follows for theorem 2.2 by putting

$$\phi(x_1, y_1, \dots, x_k, y_k) = \theta \sum_{i=1}^{k} \sum_{j=1}^{k} \|x_i y_j\|^r,$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E_1$, $L = 2^{2r-m}$. □

Corollary 2.8. Let E be Banach algebra and $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra . Suppose $r \in (0, 1)$ and $\theta \in [0, \infty)$ and $f : E \to E$ with $f(1) = 1$, such that

$$\|D_{\tau\eta} f(x_1, y_1), \dots, D_{\tau\eta} f(x_k, y_k)\|_k \leq \theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r)$$

$$\|f(x_1 y_1) - f(y_1) f(x_1), \dots, f(x_k y_k) - f(y_k) f(x_k)\|_k \leq \theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r)$$

$$\lim_m 2^{-m} f(2^m \lim_n 2^{-n} f(2^n x)) = x$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E$ and $\tau, \eta \in \mathbb{T}^1_{\frac{1}{n_0}}$, then there is a unique involution mapping $I : E \to E$ which satisfies

$$\|I(x_1) - f(x_1), \dots, I(x_k) - f(x_k)\|_k \leq \frac{2\theta}{2 - 2^r} \sum_{i=1}^{k} \|x_i\|^r$$

moreover if

$$\left| \|x_1 f(x_1), \ldots, x_k f(x_k)\|_k - \|x_1, \ldots, x_k\|_k^2 \right| \leq 2\theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \ldots, x_k \in E$, then E is a C^*-algebra with involution $x^* = I(x)$, for all $x \in E$.

Proof . We put

$$\phi(x_1, y_1, \ldots, x_k, y_k) := \theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r),$$

for all $x_1, y_1, \ldots, x_k, y_k \in E$ and $L = 2^{r-1}$ in theorem (2.2), then as a result, the sentence is obtained. □

Corollary 2.9. Let E be Banach algebra and $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra . Suppose $r \in (0, 1)$ and $\theta \in [0, \infty)$ and $f : E \to E$ with $f(1) = 1$, such that

$$\|D_{\tau\eta} f(x_1, y_1), \ldots, D_{\tau\eta} f(x_k, y_k)\|_k \leq \theta \sum_{i=1}^{k} \sum_{j=1}^{k} \|x_i y_j\|^r$$

$$\|f(x_1 y_1) - f(y_1) f(x_1), \ldots, f(x_k y_k) - f(y_k) f(x_k)\|_k \leq \theta \sum_{i=1}^{k} \sum_{j=1}^{k} \|x_i y_j\|^r$$

$$\lim_m 2^{-m} f(2^m \lim_n 2^{-n} f(2^n x)) = x$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ and $\tau, \eta \in \mathbb{T}^1_{\frac{1}{n_0}}$, then f is unique involution on E, moreover if

$$\left| \|x_1 f(x_1), \ldots, x_k f(x_k)\|_k - \|x_1, \ldots, x_k\|_k^2 \right| \leq \theta \left(\sum_{i=1}^{k} \|x_i\|^r \right)^2$$

for all $x_1, \ldots, x_k \in E$, then E is a C^*-algebra with involution $x^* = f(x)$, for all $x \in E$.

Proof . We put

$$\phi(x_1, y_1, \ldots, x_k, y_k) := \theta \sum_{i=1}^{k} \sum_{j=1}^{k} \|x_i y_j\|^r,$$

for all $x_1, y_1, \ldots, x_k, y_k \in E$ and $L = 2^{2r-1}$ in theorem (2.2), then as a result, the sentence is obtained. □

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