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Multiple solutions of a class IBVPs for one-dimensional nonlinear wave equations

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Abstract

The purpose of studying the discipline of nonlinear wave processes is to knowledge of the current state of nonlinear wave problems in mechanics and an understanding of its main problems, as well as to develop general skills for solving these problems using new methods of mathematical. We consider a class of initial boundary value problems for one-dimensional nonlinear wave processes. A new topological approach is applied to prove the existence of at least two nonnegative classical solutions. The arguments are based on a recent theoretical result.

Keywords: Partial differential equations, Existence results, Green function, Nonlinear hyperbolic processes, Positive solution, Fixed point, Cone, Sum of operators, Nonnegative classical solutions
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1 Introduction

The wave equations arise in mathematical models which describe the wave phenomena in fields like fluid dynamics and electromagnetics. Many authors such as H. Brésiz, J. Mawhin, K. C. Chang etc. have developed topological tools, index theory, and variational methods to get some existence results for the one-dimensional problem with various nonlinearities. The related results are [2], [3], [4], [5], [6], [12] and the references therein. Nonlinear wave processes are usually modeled using nonlinear partial differential equations. For nonlinear analogs of the wave equation, let f be a nonlinear function, the structure of which is determined by the geometric and (or) physical features of the problem, where non-linear ripple effects are many and varied. A very important model of nonlinear waves is the nonlinear Klein-Gordon equation [13]

$$u_{tt} - u_{xx} = \phi(u),$$

where $\phi(u)$ is some smooth or discontinuous function that describes distributed nonlinear restoring forces. In the linear approximation $\phi(u) = -\kappa u (\kappa > 0)$, we have the well-known string model *on an elastic bed*.
Restricting ourselves to considering sufficiently long waves, one can obtain additional terms of the equation of motion that depend only on the deformation u_x, but not on its derivatives. The equation of a nonlinear string or *the equation of longitudinal vibrations of a nonlinear rod*, can be reduced to the form [10]

$$u_{tt} - u_{xx} + l^2 u_{xxxx} - b u_x u_{xx} = 0,$$

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here l is a scale, considered small, b is also a small parameter characterizing the intensity of nonlinear forces.
One of a very interesting basic model is given by the so-called Korteweg-de Vries equation *the KdV equation*, which turns out to be fundamental when considering models of nonlinear waves

$$u_t - u_x + a u_{xxx} - b u u_x = 0, a, b > 0,$$

By differentiating with respect to t and by performing fundamental calculations, we obtain

$$u_{tt} - u_{xx} - 2a u_{xxxx} + 2b(u u_x)_x + ab(2u u_{xx} + \frac{1}{2} u_x^2)_{xx} - a^2 u_{xxxxxx} - b^2 (u^2 u_x)_x = 0,$$

The main aim of this paper is to investigate for multiple nontrivial nonnegative solutions of the following IBVP

$$\begin{aligned} & u_{tt} - u_{xx} = f(t, x, u, u_t, u_x), \quad t \geq 0, \quad x \in [0, 1], \\ & u = u_0(x), \quad x \in [0, 1], t = 0, \\ & u_t = u_1(x), \quad x \in [0, 1], t = 0, \\ & u = 0, \quad x = 0, t \geq 0, \\ & u_x = 0, \quad x = 1, t \geq 0, \end{aligned} \tag{1.1}$$

where

(H1) $u_0, u_1 \in \mathcal{C}^2([0, 1])$,

$$\begin{aligned} 0 & \leq \max_{x\in[0,1]} u_0(x), \max_{x\in[0,1]} |u_{0x}(x)|, \max_{x\in[0,1]} |u_{0xx}(x)| < \infty, \\ 0 & \leq \max_{x\in[0,1]} |u_1(x)|, \max_{x\in[0,1]} |u_{1x}(x)|, \max_{x\in[0,1]} |u_{1xx}(x)| < \infty, \\ r & = \max\{ \max_{x\in[0,1]} u_0(x), \max_{x\in[0,1]} |u_{0x}(x)|, \max_{x\in[0,1]} |u_{0xx}(x)|, \\ & \max_{x\in[0,1]} |u_1(x)|, \max_{x\in[0,1]} |u_{1x}(x)|, \max_{x\in[0,1]} |u_{1xx}(x)| \} > 0. \end{aligned}$$

(H2) $f \in \mathcal{C}([0, \infty) \times [0, 1] \times \mathbf{R}^3)$ satisfies the condition

$$\begin{aligned} 0 & \leq |f(t, x, w_1, w_2, w_3)| \\ & < \sum_{j=1}^{l} \left(a_j(t, x)|w_1|^{p_j} + b_j(t, x)|w_2|^{p_j} + c_j(t, x)|w_3|^{p_j} \right), \end{aligned}$$

$(t, x) \in [0, \infty) \times [0, 1]$, where $l \in \mathbf{N}$, $a_j, b_j, c_j \in \mathcal{C}([0, \infty) \times [0, 1])$ such that there exist

$$0 \leq A_j = \sup_{t\in[0,\infty), x\in[0,1]} a_j(t, x) < \infty, \quad 0 \leq B_j = \sup_{t\in[0,\infty), x\in[0,1]} b_j(t, x) < \infty,$$

$$0 \leq C_j = \sup_{t\in[0,\infty), x\in[0,1]} c_j(t, x) < \infty,$$

$$(A_1, \ldots, A_l, B_1, \ldots, B_l, C_1, \ldots, C_l) \neq (0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0),$$

$p_j \geq 0$, $j \in \{1, \ldots, l\}$, $(p_1, \ldots, p_l) \neq (0, \ldots, 0)$.

In addition, we suppose

(H3) Suppose that m, A, A_j, B_j, C_j, $j \in \{1, \dots, l\}$, r_1, L_1 and R_1 are positive constants such that

$$0 < r_1 < \frac{L_1}{20} < L_1 < R_1, \quad m \in (0,1), \quad \epsilon > 1, \quad R_1 < \epsilon \frac{L_1}{20},$$

$$A\left(2R_1 + 2r + \sum_{j=1}^{l} (A_j + B_j + C_j) R_1^{p_j}\right) < \frac{L_1}{20},$$

$$A\left(2L_1 + 2r + \sum_{j=1}^{l} (A_j + B_j + C_j) L_1^{p_j}\right) < \left(\frac{4}{5} - m\right) L_1,$$

$$0 < r < L_1, \quad 0 < r < \sum_{j=1}^{l} (A_j + B_j + C_j) L_1^{p_j},$$

$$r \geq \sup_{\substack{t \in [0,\infty), \quad x \in [0,1], \\ r \leq w_1, |w_2|, |w_3| < L_1}} |f(t, x, w_1, w_2, w_3)|.$$

(H4) There exists a nonnegative function $g \in \mathcal{C}([0,\infty) \times [0,1])$ such that

$$g(0,x) = g(t,0) = 0, \quad t \geq 0, \quad x \in [0,1], \quad g(t,x) > 0, \quad t > 0, \quad x \in (0,1],$$

$$2 \int_0^t \int_0^x (1 + t_1 + t_1^2)(1 + t - t_1 + (t - t_1)^2)(1 + |x - x_1| + (x - x_1)^2) g(t_1, x_1) dx_1 dt_1 \leq A,$$

$t \geq 0$, $x \in [0,1]$.

In the last section, we will give an example for constants m, A, A_j, B_j, C_j, $j \in \{1, \dots, l\}$, r_1, L_1, R_1 and a function g that satisfy $(H3)$ and $(H4)$. In Remark 3.1, we will give motivation for the last two conditions of $(H3)$ and using them we remove the case when

$$f(t, x, w_1, w_2, w_3) = \sum_{j=1}^{l} (a_j(t,x) w_1^{p_j} + b_j(t,x) w_2^{p_j} + c_j(t,x) w_3^{p_j}), \quad t \geq 0, \quad x, w_1, w_2, w_3 \in [0,1],$$

as well as the cases when f is a linear function or a constant. Our main result is as follows.

Theorem 1.1. Suppose $(H1)$-$(H4)$. Then the IBVP (1.1) has at least two non trivial nonnegative classical solutions.

To prove our main result we use a new topological approach. So far, for the authors they are not known investigations for existence of multiple solutions for the IBVP (1.1).

The paper is organized as follows, In the next section, we give some auxiliary results. In Section 3, we prove our main result. In Section 4, we give an example.

2 Auxiliary Results

Let X be a real Banach space.

Definition 2.1. A mapping $K : X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

The concept for k-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.2. Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Omega_X \to [0,\infty)$ is defined by

$$\alpha(Y) = \inf \left\{ \delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad \text{and} \quad \operatorname{diam}(Y_j) \leq \delta, \quad j \in \{1, \dots, m\} \right\},$$

where $\operatorname{diam}(Y_j) = \sup\{\|x - y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1, \dots, m\}$.

For the main properties of measure of noncompactness we refer the reader to [7], Chapter 7, Section 7.3.

Definition 2.3. A mapping $K : X \to X$ is said to be k-set contraction if there exists a constant $k \geq 0$ such that

$$\alpha(K(Y)) \leq k\alpha(Y)$$

for any bounded set $Y \subset X$.

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction(see [8], Chapter 5, Section 5.1).

Definition 2.4. Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

Definition 2.5. A closed, convex set $\mathcal{P}$ in X is said to be cone if

1. $\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,
2. $x, -x \in \mathcal{P}$ implies $x = 0$.

Denote $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$,

$$\mathcal{P}_{r_1} = \{u \in \mathcal{P} : \|u\| \leq r_1\},$$

$$\mathcal{P}_{r_1,r_2} = \{u \in \mathcal{P} : r_1 \leq \|u\| \leq r_2\}$$

for positive constants r_1, r_2 such that $0 < r_1 \leq r_2$. The following result will be used to prove our main result. We refer the reader to [9] for more details.

Theorem 2.6. Let Ω be a subset of $\mathcal{P}$, $0 \in \Omega$ and $0 < r < L < R$ are real constants. Let also, $T : \Omega \to E$ is an expansive operator with a constant $h > 1$, $F : \overline{\mathcal{P}_R} \to E$ is a k-set contraction with $0 \leq k < h - 1$ and $F(\overline{\mathcal{P}_R}) \subset (I - T)(\Omega)$. Assume that $\mathcal{P}_{r,L} \bigcap \Omega \neq \emptyset$, $\mathcal{P}_{L,R} \bigcap \Omega \neq \emptyset$ and there exist an $u_0 \in \mathcal{P}^*$ such that $T(x - \lambda u_0) \in \mathcal{P}$ for all $\lambda \geq 0$ and $x \in \partial\mathcal{P}_r \bigcap(\Omega + \lambda u_0)$ and the following conditions hold.

(a) $Fx \neq x - \lambda u_0$, $x \in \partial\mathcal{P}_r$, $\lambda \geq 0$,

(b) $\|Fx + T0\| \leq (h - 1)\|x\|$ and $Tx + Fx \neq x$, $x \in \partial\mathcal{P}_L \bigcap \Omega$,

(c) $Fx \neq x - \lambda u_0$, $x \in \mathcal{P}_R$, $\lambda \geq 0$.

Then $T + F$ has at least two fixed points $x_1 \in \mathcal{P}_{r,L} \bigcap \Omega$, $x_2 \in \mathcal{P}_{L,R} \bigcap \Omega$, i.e.,

$$r < \|x_1\| < L < \|x_2\| < R.$$

In [1], it is proved that the function $G(t, s) = \min\{t, s\}$, $t, s \in [0, 1]$, is the Green function for the BVP

$$\begin{aligned} y'' + g(t) &= 0, \quad t \in [0, 1], \\ y(0) = 0 &= y'(1). \end{aligned} \tag{2.1}$$

We have $0 \leq G(t, s) \leq 1$, $t, s \in [0, 1]$. Let $E = \mathcal{C}^2([0, \infty) \times [0, 1])$ be endowed with the norm

$$\|u\| = \max\left\{\|u\|_\infty, \quad \|u_t\|_\infty, \quad \|u_{tt}\|_\infty, \quad \|u_x\|_\infty, \quad \|u_{xx}\|_\infty\right\},$$

provided it exists, where

$$\|v\|_\infty = \sup_{(t,x)\in[0,\infty)\times[0,1]} |v(t, x)|.$$

For $u \in E$, define the operator

$$\begin{aligned} F_1u(t,x) &= \int_0^1 \int_0^t (t-t_1)G(x,x_1)f(t_1,x_1,u(t_1,x_1),u_t(t_1,x_1),u_x(t_1,x_1))dt_1dx_1 \\ &- \int_0^1 G(x,x_1)(u(t,x_1)-u_0(x_1)-tu_1(x_1))dx_1 \\ &- \int_0^t (t-t_1)u(t_1,x)dt_1, \quad t \geq 0, \quad x \in [0,1]. \end{aligned}$$

Lemma 2.7. Suppose $(H1)$ and $(H2)$. If $u \in E$ satisfies the integral equation

$$F_1u(t,x) = 0, \quad t \geq 0, \quad x \in [0,1], \tag{2.2}$$

then u is a solution to the IBVP (1.1).

Proof . We differentiate with respect to t the equation (2.2) and we find

$$\begin{aligned} 0 &= \int_0^1 \int_0^t G(x,x_1)f(t_1,x_1,u(t_1,x_1),u_t(t_1,x_1),u_x(t_1,x_1))dt_1dx_1 \\ &- \int_0^1 G(x,x_1)(u_t(t,x_1)-u_1(x_1))dx_1 \\ &- \int_0^t u(t_1,x)dt_1, \quad t \geq 0, \quad x \in [0,1]. \end{aligned} \tag{2.3}$$

Now, we put $t = 0$ in (2.2) and (2.3) and we get

$$\int_0^1 G(x,x_1)(u(0,x_1)-u_0(x_1))dx_1 = 0, \quad x \in [0,1], \tag{2.4}$$

and

$$\int_0^1 G(x,x_1)(u_t(0,x_1)-u_1(x_1))dx_1 = 0, \quad x \in [0,1], \tag{2.5}$$

respectively. We differentiate twice with respect to x the equation (2.4) and the equation (2.5) and we obtain

$$u(0,x) = u_0(x) \quad \text{and} \quad u_t(0,x) = u_1(x), \quad x \in [0,1],$$

respectively. We differentiate with respect to t the equation (2.3) and we arrive at

$$\begin{aligned} 0 &= \int_0^1 G(x,x_1)f(t,x_1,u(t,x_1),u_t(t,x_1),u_x(t,x_1))dx_1 \\ &- \int_0^1 G(x,x_1)u_{tt}(t,x_1)dx_1 - u(t,x), \quad t \geq 0, \quad x \in [0,1], \end{aligned}$$

or

$$u(t,x) = \int_0^1 G(x,x_1)\left(f(t,x_1,u(t,x_1),u_t(t,x_1),u_x(t,x_1)) - u_{tt}(t,x_1)\right)dx_1,$$

$t \geq 0$, $x \in [0,1]$. Using that $G(\cdot,\cdot)$ is the Green function for the BVP (2.1), we get

$$u_{xx}(t,x) + f(t,x,u(t,x),u_t(t,x),u_x(t,x)) - u_{tt}(t,x) = 0, \quad t \geq 0, \quad x \in [0,1],$$

and

$$u(t,0) = u_x(t,1) = 0, \quad t \geq 0.$$

This completes the proof. □

Lemma 2.8. Suppose $(H1)$ and $(H2)$. If $u \in E$ and $\|u\| \le b$ for some positive constant b, then

$$|F_1 u(t,x)| \le \left(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\right)t^2+(b+r)t, \quad t \ge 0, \quad x \in [0,1].$$

Proof . We have

$$\begin{aligned}
|F_1 u(t,x)| &\le \int_0^1\int_0^t (t-t_1)G(x,x_1)|f(t_1,x_1,u(t_1,x_1),u_t(t_1,x_1),u_x(t_1,x_1))|dt_1dx_1\\
&\quad +\int_0^1 G(x,x_1)(|u(t,x_1)|+u_0(x_1)+t|u_1(x_1)|)dx_1\\
&\quad +\int_0^t (t-t_1)|u(t_1,x)|dt_1\\
&\le \int_0^1\int_0^t (t-t_1)\sum_{j=1}^{l}\Big(a_j(t,x)|u(t_1,x_1)|^{p_j}+b_j(t,x)|u_t(t_1,x_1)|^{p_j}\\
&\quad +c_j(t_1,x_1)|u_x(t_1,x_1)|^{p_j}\Big)dt_1dx_1\\
&\quad +(b+r+rt)t\\
&\le \sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}t^2+(b+r)t+(b+r)t^2\\
&= \left(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\right)t^2+(b+r)t, \quad t \ge 0, \quad x \in [0,1].
\end{aligned}$$

This completes the proof. □ For $u \in E$, define the operator

$$Fu(t,x)=\int_0^t\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)F_1u(t_1,x_1)dx_1dt_1, \quad t \ge 0, \quad x \in [0,1].$$

Lemma 2.9. Suppose $(H1)$ and $(H2)$. If $u \in E$ is a solution to the integral equation

$$Fu(t,x)+\frac{L_1}{5}=0, \quad t \ge 0, \quad x \in [0,1], \tag{2.6}$$

then u is a solution to the IBVP (1.1).

Proof . We differentiate trice with respect to t and trice with respect to x the equation (2.6) and we find

$$g(t,x)F_1u(t,x)=0, \quad t \ge 0, \quad x \in [0,1],$$

whereupon

$$F_1u(t,x)=0, \quad t>0, \quad x \in (0,1].$$

Now, using that $F_1u(\cdot,\cdot)$ is a continuous function on $[0,\infty)\times[0,1]$, we get

$$\begin{aligned}
0 &= \lim_{t\to 0}F_1u(t,x)=F_1u(0,x)\\
&= \lim_{x\to 0}F_1u(t,x)=F_1u(t,0)\\
&= \lim_{t\to 0,x\to 0}F_1u(t,x)=F_1u(0,0), \quad t>0, \quad x \in (0,1].
\end{aligned}$$

Consequently

$$F_1u(t,x)=0,\quad t\geq 0,\quad x\in[0,1].$$

Then, the assertion follows from Lemma 2.7. This completes the proof. □

Lemma 2.10. Suppose $(H1)$, $(H2)$ and $(H4)$. If $u\in E$ and $\|u\|\leq b$ for some positive constant b, then

$$\|Fu\|\leq A\left(2b+2r+\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}\right).$$

Proof . Applying Lemma 2.8 and $(H3)$, we get

$$\begin{aligned}
|Fu(t,x)| &\leq \int_0^t\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)\bigg(\bigg(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\bigg)t_1^2\\
&\quad +(b+r)t_1\bigg)dx_1dt_1\\
&\leq A\left(2b+2r+\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}\right),\quad t\geq 0,\quad x\in[0,1],
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial}{\partial t}Fu(t,x)\right| &\leq 2\int_0^t\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)\bigg(\bigg(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\bigg)t_1^2\\
&\quad +(b+r)t_1\bigg)dx_1dt_1\\
&= 2\bigg(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\bigg)\int_0^t\int_0^x t_1^2(t-t_1)(x-x_1)^2 g(t_1,x_1)dx_1dt_1\\
&\quad +2(b+r)\int_0^t\int_0^x t_1(t-t_1)(x-x_1)^2 g(t_1,x_1)dx_1dt_1\\
&\leq A\left(2b+2r+\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}\right),\quad t\geq 0,\quad x\in[0,1],
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial^2}{\partial t^2}Fu(t,x)\right| &\leq 2\int_0^t\int_0^x (x-x_1)^2 g(t_1,x_1)\bigg(\bigg(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\bigg)t_1^2\\
&\quad +(b+r)t_1\bigg)dx_1dt_1\\
&= 2\bigg(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\bigg)\int_0^t\int_0^x t_1^2(x-x_1)^2 g(t_1,x_1)dx_1dt_1\\
&\quad +2(b+r)\int_0^t\int_0^x t_1(x-x_1)^2 g(t_1,x_1)dx_1dt_1\\
&\leq A\left(2b+2r+\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}\right),\quad t\geq 0,\quad x\in[0,1],
\end{aligned}$$

and

$$
\begin{aligned}
\left|\frac{\partial}{\partial x}Fu(t,x)\right| &\leq 2\int_0^t\int_0^x (t-t_1)^2(x-x_1)g(t_1,x_1)\Bigg(\Bigg(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\Bigg)t_1^2\\
&\quad +(b+r)t_1\Bigg)dx_1dt_1\\
&= 2\Bigg(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\Bigg)\int_0^t\int_0^x t_1^2(t-t_1)^2(x-x_1)g(t_1,x_1)dx_1dt_1\\
&\quad +2(b+r)\int_0^t\int_0^x t_1(t-t_1)^2(x-x_1)g(t_1,x_1)dx_1dt_1\\
&\leq A\left(2b+2r+\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}\right), \quad t\geq 0, \quad x\in[0,1],
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial x^2}Fu(t,x)\right| &\leq 2\int_0^t\int_0^x (t-t_1)^2 g(t_1,x_1)\Bigg(\Bigg(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\Bigg)t_1^2\\
&\quad +(b+r)t_1\Bigg)dx_1dt_1\\
&= 2\Bigg(\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}+(b+r)\Bigg)\int_0^t\int_0^x t_1^2(t-t_1)^2 g(t_1,x_1)dx_1dt_1\\
&\quad +2(b+r)\int_0^t\int_0^x t_1(t-t_1)^2 g(t_1,x_1)dx_1dt_1\\
&\leq A\left(2b+2r+\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}\right), \quad t\geq 0, \quad x\in[0,1].
\end{aligned}
$$

Consequently

$$
\|Fu\| \leq A\left(2b+2r+\sum_{j=1}^{l}(A_j+B_j+C_j)b^{p_j}\right).
$$

This completes the proof. □

3 Proof of the Main Result

Let

$$
\widetilde{P} = \{u\in E : u\geq 0 \quad \text{on} \quad [0,\infty)\times[0,1]\}.
$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\mathcal{P}$. For $v\in E$, define the operators

$$
\begin{aligned}
Tv(t,x) &= (1+m\epsilon)v(t,x)-\epsilon\frac{L_1}{10},\\
Sv(t,x) &= -\epsilon Fv(t,x)-m\epsilon v(t,x)-\epsilon\frac{L_1}{10},
\end{aligned}
$$

$(t,x) \in [0,\infty) \times [0,1]$. Note that any fixed point $v \in E$ of the operator $T+S$ is a solution to the IBVP (1.1). Define

$$\begin{aligned}
\mathcal{P}_{r_1} &= \{v \in \mathcal{P} : \|v\| < r_1\},\\
\mathcal{P}_{L_1} &= \{v \in \mathcal{P} : \|v\| < L_1\},\\
\mathcal{P}_{R_1} &= \{v \in \mathcal{P} : \|v\| < R_1\},\\
\mathcal{P}_{r_1,L_1} &= \{v \in \mathcal{P} : r_1 < \|v\| < L_1\},\\
\mathcal{P}_{L_1,R_1} &= \{v \in \mathcal{P} : L_1 < \|v\| < R_1\},\\
R_2 &= R_1 + \frac{A}{m}\left(2R_1 + 2r + \sum_{j=1}^{l}(A_j+B_j+C_j)R_1^{p_j}\right) + \frac{L_1}{5m},\\
\Omega &= \mathcal{P}_{R_2} = \{v \in \mathcal{P} : \|v\| < R_2\}.
\end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have
$$\|Tv_1 - Tv_2\| = (1+m\epsilon)\|v_1 - v_2\|,$$
whereupon $T : \Omega \to E$ is an expansive operator with a constant $1+m\epsilon > 1$.
2. For $v \in \overline{\mathcal{P}}_{R_1}$, we get
$$\begin{aligned}
\|Sv\| &\le \epsilon\|Fv\| + m\epsilon\|v\| + \epsilon\frac{L_1}{10}\\
&\le \epsilon\Bigg(A\left(2R_1 + 2r + \sum_{j=1}^{l}(A_j+B_j+C_j)R_1^{p_j}\right)\\
&\qquad +mR_1 + \frac{L_1}{10}\Bigg).
\end{aligned}$$
Therefore $S(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S : \overline{\mathcal{P}}_{R_1} \to E$ is continuous, we have that $S(\overline{\mathcal{P}}_{R_1})$ is equicontinuous. Consequently $S : \overline{\mathcal{P}}_{R_1} \to E$ is a 0-set contraction.
3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set
$$v_2 = v_1 + \frac{1}{m}Fv_1 + \frac{L_1}{5m}.$$
Note that by the second inequality of $(H3)$ and by Lemma 2.10, it follows that $\epsilon Fv + \epsilon\frac{L_1}{5} \ge 0$ on $[0,\infty)\times[0,1]$. We have $v_2 \ge 0$ on $[0,\infty)\times[0,1]$ and
$$\begin{aligned}
\|v_2\| &\le \|v_1\| + \frac{1}{m}\|Fv_1\| + \frac{L_1}{5m}\\
&\le R_1 + \frac{A}{m}\left(2R_1 + 2r + \sum_{j=1}^{l}(A_j+B_j+C_j)R_1^{p_j}\right) + \frac{L_1}{5m}\\
&= R_2.
\end{aligned}$$
Therefore $v_2 \in \Omega$ and
$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon F v_1 - \epsilon\frac{L_1}{10} - \epsilon\frac{L_1}{10}$$
or
$$\begin{aligned}
(I-T)v_2 &= -\epsilon m v_2 + \epsilon\frac{L_1}{10}\\
&= Sv_1.
\end{aligned}$$
Consequently $S(\overline{\mathcal{P}}_{R_1}) \subset (I-T)(\Omega)$.

4. Suppose that there exists an $v_0 \in \mathcal{P}^*$ such that $T(v-\lambda v_0) \in \mathcal{P}$, $v \in \partial\mathcal{P}_{r_1}$, $v \in \partial\mathcal{P}_{r_1} \bigcap (\Omega+\lambda u_0)$ and $Sv = v-\lambda v_0$ for some $\lambda \geq 0$. Then
$$\begin{aligned} r_1 &= \|v-\lambda v_0\| = \|Sv\| \\ &\geq -Sv(t,x) = \epsilon Fv(t,x) + \epsilon m v(t,x) + \epsilon \frac{L_1}{10} \\ &\geq \epsilon \frac{L_1}{20}, \quad (t,x) \in [0,\infty) \times [0,1], \end{aligned}$$
because by the second inequality of $(H3)$ and by Lemma 2.10, it follows that $\epsilon Fv + \epsilon \frac{L_1}{20} \geq 0$ on $[0,\infty)\times[0,1]$.
5. Let $v \in \partial\mathcal{P}_{L_1}$. Then
$$\begin{aligned} \|Sv+T0\| &= \|\epsilon Fv + m\epsilon v + \epsilon \frac{L_1}{5}\| \\ &\leq \epsilon(\|Fv\| + m\|v\| + \frac{L_1}{5}) \\ &\leq \epsilon\left(A\left(2L_1 + 2r + \sum_{j=1}^{l}(A_j+B_j+C_j)L_1^{p_j}\right)(m+\frac{1}{5})L_1\right) \\ &\leq \epsilon L_1 = \epsilon\|v\|. \end{aligned}$$
Note that in the last inequality we have used the third inequality of $(H3)$.
6. Now, assume that $v \in \partial\mathcal{P}_{L_1} \bigcap \Omega$ is such that
$$v = Tv + Sv,$$
whereupon
$$Fv + \frac{L_1}{5} \equiv 0 \quad \text{on} \quad [0,\infty)\times[0,1].$$
Since $v \in \partial\mathcal{P}_{L_1}$, we have that $v \not\equiv 0$ on $[0,\infty)\times[0,1]$ and by the second inequality of $(H3)$ and by Lemma 2.10, it follows that $Fv + \frac{L_1}{5} > Fv + \frac{L_1}{20} \geq 0$ on $[0,\infty)\times[0,1]$. This is a contradiction.
7. Suppose that there exists an $v_0 \in \mathcal{P}^*$ such that $T(v-\lambda v_0) \in \mathcal{P}$, $v \in \partial\mathcal{P}_{R_1}$, $v \in \partial\mathcal{P}_{R_1} \bigcap (\Omega+\lambda u_0)$ and $Sv = v-\lambda v_0$ for some $\lambda \geq 0$. Then
$$\begin{aligned} R_1 &= \|v-\lambda v_0\| = \|Sv\| \\ &\geq -Sv(t,x) = \epsilon Fv(t,x) + \epsilon m v(t,x) + \epsilon \frac{L_1}{10} \\ &\geq \epsilon \frac{L_1}{20}, \quad (t,x) \in [0,\infty) \times [0,1], \end{aligned}$$
which is a contradiction.

Therefore all conditions of Theorem 2.6 hold. Hence, the IBVP (1.1) has at least two solutions v_1 and v_2 so that
$$r_1 < \|v_1\| < L_1 < \|v_2\| < R_1. \tag{3.1}$$
This completes the proof.

Remark 3.1. 1. For any solution u of the IBVP (1.1), we have
$$\|u\| \geq \max\{\max_{x\in[0,1]} u_0(x), \max_{x\in[0,1]} |u_{0x}(x)|, \max_{x\in[0,1]} |u_{0xx}(x)|, \max_{x\in[0,1]} |u_1(x)|, \max_{x\in[0,1]} |u_{1x}(x)|, \max_{x\in[0,1]} |u_{1xx}(x)|\} = r.$$
Hence, using (3.1), we conclude that $0 \leq r < L_1$ and $\|v_2\| > r$. If $\|v_1\| = r$, since v_1 is a nontrivial nonnegative solution of (1.1), then $r > 0$.

2. If $r = \|v_1\| = |v_{1tt}(t_1,x_1)|$ and $v_{1xx}(t_1,x_1) = 0$ for some $t_1 \in [0,\infty)$, $x_1 \in [0,1]$, then
$$r \leq v_1(t,x), \quad |v_{1t}(t,x)|, \quad |v_{1tt}(t,x)|, \quad |v_{1x}(t,x)|, \quad |v_{1xx}(t,x)| < L_1, \tag{3.2}$$
$t \in [0,\infty)$, $x \in [0,1]$. Hence,
$$\begin{aligned} r &= |v_{1tt}(t_1,x_1)| = |v_{1tt}(t_1,x_1) - v_{1xx}(t_1,x_1)| \\ &= |f(t_1,x_1,v_1(t_1,x_1),v_{1t}(t_1,x_1),v_{1x}(t_1,x_1))|, \end{aligned}$$

which is true because the last condition of $(H3)$ holds. Moreover,

$$r = |f(t_1, x_1, v_1(t_1, x_1), v_{1t}(t_1, x_1), v_{1x}(t_1, x_1))| < \sum_{j=1}^{l} (A_j + B_j + C_j) L_1^{p_j}.$$

3. If $r = \|v_1\| = |v_{1xx}(t_2, x_2)|$ and $v_{1tt}(t_2, x_2) = 0$ for some $t_2 \in [0, \infty)$, $x_2 \in [0, 1]$, then we have (3.2). Hence,

$$\begin{aligned} r &= |v_{1xx}(t_2, x_2)| = |v_{1tt}(t_2, x_2) - v_{1xx}(t_2, x_2)| \\ &= |f(t_2, x_2, v_1(t_2, x_2), v_{1t}(t_2, x_2), v_{1x}(t_2, x_2))|, \end{aligned}$$

which is true because the last condition of $(H3)$ holds. Moreover,

$$\begin{aligned} r &= |f(t_2, x_2, v_1(t_2, x_2), v_{1t}(t_2, x_2), v_{1x}(t_2, x_2))| \\ &< \sum_{j=1}^{l} (A_j + B_j + C_j) L_1^{p_j}. \end{aligned}$$

4. If

$$f(t, x, w_1, w_2, w_3) = \sum_{j=1}^{l} \left(a_j(t) w_1^{p_j} + b_j(t) w_2^{p_j} + b_j(t) w_3^{p_j} \right),$$

$t \in [0, \infty)$, $x, w_1, w_2, w_3 \in [0, 1]$, then

$$\sup_{\substack{t \in [0,\infty), \quad x \in [0,1], \\ r \leq w_1, |w_2|, |w_3| < L_1}} |f(t, x, w_1, w_2, w_3)| = \sum_{j=1}^{l} (A_j + B_j + C_j) L_1^{p_j}.$$

Hence and by the last two conditions of $(H3)$, we get

$$\sum_{j=1}^{l} (A_j + B_j + C_j) L_1^{p_j} \leq r < \sum_{j=1}^{l} (A_j + B_j + C_j) L_1^{p_j},$$

which is impossible.

4 An Example

Let

$$\begin{aligned} l &= 2, \quad p_1 = \frac{3}{5}, \quad p_2 = 0, \qquad R_1 = \frac{3}{10^{10}}, \quad r = \frac{4}{3 \cdot 10^{10}}, \quad L_1 = \frac{2}{10^{10}}, \quad r_1 = \frac{1}{10^{12}}, \\ m &= \frac{1}{2}, \quad \epsilon = 50, \quad A = \frac{1}{10^{10}}, \quad A_1 = \frac{2}{10^{10}}, \quad A_2 = \left(\frac{4}{3 \cdot 10^{10}} \right)^{\frac{3}{5}}, \quad B_1 = B_2 = C_1 = C_2 = 0. \end{aligned}$$

Then

$$R_1 = \frac{3}{10^{10}} < \frac{5}{10^{10}} = \epsilon \frac{L_1}{20}, \quad r_1 < L_1 < R_1, \quad r_1 < \frac{L_1}{20} = \frac{1}{10^{11}}.$$

Also,

$$\begin{aligned} A \left(2R_1 + 2r + \sum_{j=1}^{l} (A_j + B_j + C_j) R_1^{p_j} \right) &= \frac{1}{10^{10}} \left(\frac{6}{10^{10}} + \frac{8}{3 \cdot 10^{10}} + \frac{2}{10^{10}} \left(\frac{3}{10^{10}} \right)^{\frac{3}{5}} + \left(\frac{4}{3 \cdot 10^{10}} \right)^{\frac{3}{5}} \right) \\ &< \frac{1}{10^{11}} = \frac{L_1}{20}. \end{aligned}$$

Next,

$$A\left(2L_1+2r+\sum_{j=1}^{l}(A_j+B_j+C_j)L_1^{p_j}\right) = \frac{1}{10^{10}}\left(\frac{4}{10^{10}}+\frac{8}{3\cdot 10^{10}}+\frac{2}{10^{10}}\left(\frac{2}{10^{10}}\right)^{\frac{3}{5}}+\left(\frac{4}{3\cdot 10^{10}}\right)^{\frac{3}{5}}\right)$$
$$< \frac{6}{10^{11}}=\left(\frac{4}{5}-m\right)L_1.$$

Consequently $(H3)$ holds. Now, we will construct the function g in $(H4)$. Let

$$h(x)=\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}},\quad l(s)=\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}},\quad s\in[0,1].$$

Then

$$h'(s) = \frac{22\sqrt{2}s^{10}(1-s^{22})}{(1-s^{11}\sqrt{2}+s^{22})(1+s^{11}\sqrt{2}+s^{22})},$$
$$l'(s) = \frac{11\sqrt{2}s^{10}(1+s^{20})}{1+s^{40}},\quad s\in[0,1].$$

Therefore

$$-\infty < \lim_{s\to\pm\infty}(1+s+s^2)h(s)<\infty,$$
$$-\infty < \lim_{s\to\pm\infty}(1+s+s^2)l(s)<\infty.$$

Hence, there exists a positive constant C_1 so that

$$(1+s+s^2)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \leq C_1,$$
$$(1+s+s^2)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \leq C_1,$$

$t\in[0,\infty)$, $s\in[0,1]$. Note that by [11](pp. 707, Integral 79), we have

$$\int\frac{dz}{1+z^4}=\frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2}+\frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s)=\frac{s^{10}}{(1+s^{44})(1+s+s^2)^2(1+s^2)^2},\quad s\in[0,1],$$

and

$$g_1(t,x)=Q(t)Q(x),\quad t\in[0,\infty),\quad x\in[0,1].$$

Then there exists a constant $C_2>0$ so that

$$C_2 \geq \int_0^t\int_0^x g_1(t_1,s_1)\left(1+|x-s_1|+(x-s_1)^2\right)\times\left(1+(t-t_1)+(t-t_1)^2\right)t_1^2s_1^2ds_1dt_1,\quad (t,x)\in[0,\infty)\times[0,1].$$

Now, we take

$$g(t,x)=\frac{1}{10^{20}C_2}g_1(t,x),\quad (t,x)\in[0,\infty)\times[0,1].$$

Then

$$A = \frac{1}{10^{10}} \geq \int_0^t \int_0^x g(t_1, s_1)\left(1 + |x_1 - s_1| + (x_1 - s_1)^2\right)$$
$$\times \left(1 + (t - t_1) + (t - t_1)^2\right)(1 + t_1 + t_1^2)(1 + |s_1| + s_1^2)ds_1 dt_1, \quad (t, x) \in [0, \infty) \times [0, 1].$$

Now, consider the IBVP

$$\begin{aligned} u_{tt} - u_{xx} &= w(t)\left(u - \tfrac{4}{3\cdot 10^{10}}\right)^{\frac{3}{5}}, \quad (t, x) \in (0, \infty) \times [0, 1] \\ u(0, x) &= \tfrac{4}{3\cdot 10^{10}}, \quad u_t(0, x) = 0, \quad x \in [0, 1], \\ u(t, 0) &= u_x(t, 1) = 0, \quad t \geq 0, \end{aligned} \tag{4.1}$$

where

$$w(t) = \begin{cases} \frac{1}{10^{10}}(9t^2 - 9t + 2), & t \in [0, 1], \\ \frac{2}{10^{10}}, & t > 1. \end{cases}$$

Next,

$$0 < r < L_1, \quad r = \frac{4}{3\cdot 10^{10}} < \frac{2}{10^{10}} \cdot \left(\frac{2}{10^{10}}\right)^{\frac{3}{5}} + \left(\frac{4}{3\cdot 10^{10}}\right)^{\frac{3}{5}} = A_1 L_1^{p_1} + A_2$$

and

$$r = \frac{4}{3\cdot 10^{10}} \geq \sup_{\substack{t \in [0, \infty) \\ \frac{4}{3\cdot 10^{10}} \leq w_1 < \frac{2}{10^{10}}}} \left| w(t)\left(w_1 - \frac{4}{3\cdot 10^{10}}\right)\right| = \frac{2}{10^{10}} \frac{2}{3\cdot 10^{10}}.$$

We have that $(H1)$-$(H4)$ hold. The IBVP (4.1) has two nontrivial nonnegative solutions.

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