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Space-time Müntz spectral collocation approach for parabolic Volterra integro-differential equations with a singular kernel

Bahareh Sadeghi, Mohammad Maleki*, Homa Almasieh

Department of Mathematics, Isfahan (Khorasgan) Branch, Islamic Azad University, Isfahan, Iran

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Abstract

We consider a type of Volterra integro-differential equations of the parabolic type that arise naturally in the study of heat flow in materials with memory. We present a simple and accurate numerical method for problems with a weakly singular kernel subject to an initial condition and given boundary conditions. In this method, both the space and time discretizations are based on the Müntz-Legendre collocation method that converts the problem to a system of algebraic equations. For numerical stability purposes, the Müntz-Legendre polynomials and their partial derivatives are stated in terms of Jacobi polynomials. Moreover, to deal with the weakly singular integral term of the problem, two efficient schemes based on the integration by parts and nonclassical Gaussian quadrature are derived. Comparisons between the two proposed schemes and other methods in the literature are made to demonstrate the efficiency, convergence and superiority of our method in the space and time directions.

Keywords: Parabolic integro-differential equation; Heat flow, Singular kernel, Müntz-Legendre collocation, Nonclassical Gauss quadrature formulas
2020 MSC: 45K05, 65D32, 65N20, 65N35, 80Axx

1 Introduction

A physical understanding of ordinary and partial integro-differential equations is given by their usefulness in modeling problems in heat flow [1, 2, 3]. Consider a rigid heat conductor in which heat flows in only one direction. Let the functions $u(x,t)$, $e(x,t)$, $q(x,t)$ and $h(x,t)$ denote the temperature, internal energy, heat flux and heat supply, respectively. Moreover, x denotes the position in the body and t denotes time. The energy balance equation is

$$\frac{\partial}{\partial t}e(x,t) = -\frac{\partial}{\partial x}q(x,t) + h(x,t). \tag{1.1}$$

In a homogeneous isotropic material, the classical linear theory for heat flow is investigated by assuming that the internal energy depends linearly on the temperature, and the heat flux is related to the temperature by Fourier's law, i.e.

$$e(x,t) = e_0 + b_0 u(x,t), \quad b_0 > 0, \tag{1.2}$$

$$q(x,t) = -c_0 \frac{\partial}{\partial x}u(x,t), \quad c_0 > 0. \tag{1.3}$$

*Corresponding author
Email address: mm.maleki2013@gmail.com (Mohammad Maleki)

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Nonetheless, the assumptions (1.2)-(1.3) are inadequate in materials of fading memory type. Indeed, in these types of materials it should be assumed that the internal energy and heat flux are functionals of the temperature and the gradient of the temperature, respectively. In the linear theory for materials with memory, a natural choice for the functionals $e(x,t)$ and $q(x,t)$ are as follows:

$$e(x,t) = e_0 + b_0 u(x,t) + \int_0^t k_2(t-s)u(x,s)\mathrm{d}s, \quad t \geqslant 0, \tag{1.4}$$

$$q(x,t) = -c_0 \frac{\partial}{\partial x} u(x,t) + \int_0^t k_1(t-s)\frac{\partial}{\partial x} u(x,s)\mathrm{d}s, \quad t \geqslant 0. \tag{1.5}$$

Note that, the history of the temperature is prescribed as zero for $t \leqslant 0$. Applying (1.4)-(1.5) to the energy balance (1.1), leads to the parabolic Volterra integro-differential equation

$$b_0 \frac{\partial}{\partial t} u(x,t) = c_0 \frac{\partial^2}{\partial x^2} u(x,t) + \int_0^t \left[k_1(t-s)\frac{\partial^2}{\partial x^2} u(x,s) - k_2'(t-s)u(x,s) \right] \mathrm{d}s - k_2(0)u(x,t) + h(x,t). \tag{1.6}$$

Equations similar to (1.6), can also be found in the modeling of phenomena associated with linear viscoelastic mechanics [4], combined conduction, convection and radiation problems [5, 6].

This paper concerns the numerical solution of (1.6) with the weakly singular kernel $k_1(t) = t^{-\frac{1}{2}}$, and $k_2(t) = h(x,t) = 0$, i.e., we consider

$$b_0 \frac{\partial}{\partial t} u(x,t) = c_0 \frac{\partial^2}{\partial x^2} u(x,t) + \int_0^t (t-s)^{-\frac{1}{2}} \frac{\partial^2}{\partial x^2} u(x,s)\mathrm{d}s, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant T, \tag{1.7}$$

with initial and boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 \leqslant t \leqslant T, \tag{1.8}$$

$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant 1, \tag{1.9}$$

where f is a given continuous function. For the questions of existence and uniqueness of the solution to this problem refer to [1, 7].

Despite the vast literature concerning ordinary integro-differential equations, in recent decade, only few numerical methods have been developed for solving partial integro-differential equations with singular kernel. In [8] a quasi wavelet based numerical method is given. Cubic B-splines collocation method is developed in [9]. Several finite difference procedures are presented and compared in [10, 11]. A space-time spectral method is proposed in [12]. Compact difference schemes are derived in [13, 14]. A combination of Crank-Nicolson and Legendre spectral collocation methods is suggested in [15]. A Crank-Nicolson-type compact difference method is constructed in [16]. In [17], the fuzzy Laplace transform method is examined for solving the fuzzy parabolic Volterra partial integro-differential equations. Authors of [18] proposed the combination of meshless method based on radial basis functions with a geometric numerical integration method. A hybrid method between homotopy analytical method and Harris hawks optimization algorithm has been developed in [19].

In this paper, we aim to introduce a Müntz-Legendre spectral collocation method for the solution of parabolic integro-differential equation (1.7)-(1.9). The Müntz-Legendre polynomials, which have the property of orthogonality, are used as trial functions for both space and time discretizations. We represent the trial functions and their partial derivatives in terms of Jacobi polynomials that makes the proposed method numerically more stable. Two different schemes are proposed to treat the singular integral term. In the first scheme, we utilize integration by parts to convert the singular integral term to a nonsingular one. Then, the classical Legendre-Gauss (LG) quadrature rule is employed for its approximation. In the second scheme, we first introduce a nonclassical weight function and then calculate the singular integral term using a nonclassical Gaussian quadrature. We compare the numerical results of our proposed methods with some available methods in the literature and show that the new method is more accurate.

This article is outlined as follows: In the next section, a brief review of orthogonal polynomials and Müntz-Legendre functions is given. In Section 3, the Müntz-Legendre pseudospectral method is derived for partial integro-differential equations with a weakly singular kernel. In Section 4, numerical results of two test examples are given. Section 5 is devoted to discussion of numerical findings and conclusions are given in Section 6.

2 Preliminaries

2.1 Nonclassical Gauss points and weights

Let $w(t)$ be a non-negative, continuous and integrable weight function on the interval $[a, b]$. The weighted inner product of two functions f and g is

$$\langle f, g\rangle_w = \int_a^b w(t)f(t)g(t)\mathrm{d}t. \tag{2.1}$$

Corresponding to each weighted inner product, there exist a set of orthogonal polynomials with the leading coefficient 1. Let $Q_n(t)$ be the n^{th}-degree nonclassical orthogonal polynomial with respect to the weight w that can be obtained from the following three-term recurrence relation

$$\begin{aligned} &Q_{-1}(t) = 0, \qquad Q_0(t) = 1, \\ &Q_{n+1}(t) = (t - \alpha_n)Q_n(t) - \beta_n Q_{n-1}(t), \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{2.2}$$

with the coefficients

$$\alpha_n = \frac{\langle tQ_n, Q_n\rangle_w}{\langle Q_n, Q_n\rangle_w}, \quad n = 0, 1, 2, \ldots \tag{2.3}$$

$$\beta_0 = \langle Q_0, Q_0\rangle_w, \quad \beta_n = \frac{\langle Q_n, Q_n\rangle_w}{\langle Q_{n-1}, Q_{n-1}\rangle_w}, \quad n = 1, 2, 3, \ldots. \tag{2.4}$$

Computing orthogonal polynomials via the three-term recurrence relation (2.2) is a quite stable scheme that can be conveniently employed. Nonetheless, special care must be taken for the computation of the coefficients α_n and β_n. One of the more practical approaches is the one introduced by Gautschi [20] referred to as the discretized Stieltjes procedure. The method involves the accurate calculation of the integrals in Eqs. (2.3)-(2.4) by subdividing the domain of interest into many subdomains and evaluating the contribution from each subdomain with a high-order quadrature. The calculation begins with calculating α_0 from $Q_0(t) = 1$. This then allows $Q_1(t)$ to be determined from Eq. (2.2), from which α_1 and β_1 can be calculated and so on.

For computing a set of nonclassical Gauss quadrature points and weights, a symmetric tridiagonal matrix called Jacobi matrix is introduced. Specially, the tridiagonal Jacobi matrix of order $n+1$ is defined by

$$J^G_{n+1} = \begin{pmatrix} \alpha_0 & \sqrt{\beta_1} & & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & \\ & \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & & \\ & \ddots & \ddots & \ddots & & \\ & & \sqrt{\beta_{n-1}} & \alpha_{n-1} & \sqrt{\beta_n} \\ & & & \sqrt{\beta_n} & \alpha_n \end{pmatrix}.$$

The Gauss points $a < t_0 < t_1 < \ldots < t_{n-1} < t_n < b$ and weights w_j for $j = 0, 1, \ldots, n$ are obtained by the method outlined by Golub [21].

Theorem 2.1. *(Golub [21])* The Gauss nodes $a < t_0 < t_1 < \ldots < t_{n-1} < t_n < b$ are the eigenvalues of J^G_{n+1} and the quadrature weights w_j are given by

$$w_j = \beta_0(v_{1j})^2, \quad j = 0, 1, \ldots, n,$$

where v_j is the normalized eigenvector of J^G_{n+1} corresponding to the eigenvalue t_j (i.e. $v_j^T v_j = 1$) and v_{1j} its first component.

An $(n+1)$–point Gaussian quadrature rule for the weight function w has the formula of the form

$$\int_a^b w(t)g(t)\mathrm{d}t = \sum_{j=0}^{n} w_j g(t_j) + R_n[f], \tag{2.5}$$

where $R_n[f]$ is the error.

2.2 Müntz polynomials

The Weierstrass theorem states that every continuous function on a compact interval can be uniformly approximated by algebraic polynomials. A generalization of this theorem to polynomials with real number exponents is stated in the following theorem:

Theorem 2.2. *(Münts–Szász Theorem [22])* Let $\{\lambda_k\}_{k=0}^{\infty}$ be a sequence of real numbers such that $\inf_k \lambda_k > -\frac{1}{2}$. Then, $\text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ is dense in $L^2(0,1)$ if and only if $\sum_{k=0}^{\infty} \frac{1}{\lambda_k + \frac{1}{2}} = +\infty$.

Utilizing the above theorem allows us to define Müntz-Legendre polynomials. Consider the set of complex numbers $\Lambda_n = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with the condition $\text{Re}(\lambda_k) > -\frac{1}{2}$, $0 \leqslant k \leqslant n$. The Müntz-Legendre polynomials on the interval [0, 1] are defined by [22, 23]

$$P_n(x) = \sum_{k=0}^{n} c_{n,k} x^{\lambda_k}, \qquad c_{n,k} = \frac{\prod_{i=0}^{n-1} (\lambda_k + \bar{\lambda}_i + 1)}{\prod_{i=0, i\neq k}^{n} (\lambda_k - \lambda_i)}. \tag{2.6}$$

Noteworthy, Müntz-Legendre functions share many of the basic properties of their algebraic polynomial cousins [23]. We have that

$$P_n(0) = 0, \quad P_n(1) = 1. \tag{2.7}$$

In addition, these functions are orthogonal on the interval $[0,1]$ with respect to the weight function $w(x) = 1$, i.e.,

$$\langle P_m, P_n \rangle = \int_0^1 P_m(x), \bar{P}_n(x) dx = \frac{\delta_{mn}}{\lambda_m + \bar{\lambda}_n + 1}, \tag{2.8}$$

where δ_{mn} is the Kronecker delta function.

In this paper, we suppose that $\lambda_k = k\alpha$ for a positive real value α. In this case, we set $P_n(x) := P_{n,\alpha}(x)$ and we recall that the functions $P_{k,\alpha}(x)$, $k = 0, 1, \ldots, n$ form an orthogonal basis for the set $\mathbb{S}_{n,\alpha} = \text{span}\{1, x^{\alpha}, \ldots, x^{n\alpha}\}$. It is noted that the shifted Müntz-Legendre polynomials on the interval $[0,T]$ are represented by the formula

$$\widetilde{P}_{n,\alpha}(t) = P_{n,\alpha}(\frac{t}{T}), \quad 0 \leqslant t \leqslant T. \tag{2.9}$$

As Milovanović states in [23], due to rapid growth in the coefficients $c_{n,k}$, evaluating $P_{n,\alpha}(x)$ using the power form (2.6) can be numerically unstable when n is large and x is close to 1. A stable method for the numerical evaluation of shifted Müntz-Legendre polynomials has been proposed in [24]. Indeed, the following representation holds true:

$$\widetilde{P}_{n,\alpha}(t) = P_n^{(0,\frac{1}{\alpha}-1)}\Big(2(\frac{t}{T})^{\alpha} - 1\Big), \tag{2.10}$$

where $P_n^{(\beta,\gamma)}$ is the classical Jacobi polynomial of degree n with parameters $\beta, \gamma > -1$. Consequently, using the recurrence relation of Jacobi polynomials [25], the shifted Müntz-Legendre polynomials can be computed by means of the three-term recurrence relation

$$\begin{aligned} &\widetilde{P}_{0,\alpha}(t) = 1, \qquad \widetilde{P}_{1,\alpha}(t) = \left(\frac{1}{\alpha} + 1\right)\left(\frac{t}{T}\right)^{\alpha} - \frac{1}{\alpha}, \\ &a_n \widetilde{P}_{n+1,\alpha}(t) = b_n(t)\widetilde{P}_{n,\alpha}(t) - c_n \widetilde{P}_{n-1,\alpha}(t), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} a_n &= 2(n+1)(n+\frac{1}{\alpha})(2n+\frac{1}{\alpha}-1), \\ b_n(t) &= (2n+\frac{1}{\alpha})\left((2n+\frac{1}{\alpha}-1)(2n+\frac{1}{\alpha}+1)\Big(2(\frac{t}{T})^{\alpha}-1\Big) - \Big(\frac{1}{\alpha}-1\Big)^2\right), \\ c_n &= 2n(2n+\frac{1}{\alpha}-1)(2n+\frac{1}{\alpha}+1). \end{aligned}$$

Another helpful formula that relates Jacobi polynomials and their derivatives is,

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\beta,\gamma)}(x)=\frac{1}{2}(n+\beta+\gamma+1)P_{n-1}^{(\beta+1,\gamma+1)}(x),\quad -1\leqslant x\leqslant 1. \tag{2.12}$$

By Eqs. (2.10) and (2.12) one can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{P}_{n,\alpha}(t)=\frac{1}{T^\alpha}(1+n\alpha)t^{\alpha-1}P_{n-1}^{(1,\frac{1}{\alpha})}\Big(2\big(\frac{t}{T}\big)^\alpha-1\Big),\quad 0<t\leqslant T. \tag{2.13}$$

Therefore, by defining a three-term recurrence relation as in (2.11) for computing $P_{n-1}^{(1,\frac{1}{\alpha})}\big(2\big(\frac{t}{T}\big)^\alpha-1\big)$ and substituting the result into (2.13), the derivatives of Müntz-Legendre polynomials can be evaluated in a stable manner.

3 Proposed method

Consider the parabolic Volterra intergro-differential equation (1.7)-(1.9). At first, the solution $u(x,t)$ is approximated by

$$u(x,t)\approx I_nu(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{n}u_{ij}P_{i,\alpha_1}(x)\widetilde{P}_{j,\alpha_2}(t),\quad 0\leqslant x\leqslant 1,\quad 0\leqslant t\leqslant T, \tag{3.1}$$

where $\alpha_1,\alpha_2>0$ and u_{ij} are unknown coefficients. Then, the partial derivatives $\frac{\partial}{\partial t}u(x,t)$ and $\frac{\partial^2}{\partial x^2}u(x,t)$ are approximated using Eq. (2.13) as

$$\begin{aligned}\frac{\partial}{\partial t}I_nu(x,t) &= \frac{t^{\alpha_2-1}}{T^{\alpha_2}}\sum_{i=0}^{n}\sum_{j=1}^{n}u_{ij}(1+j\alpha_2)P_{i,\alpha_1}(x)P_{j-1}^{(1,\frac{1}{\alpha_2})}\Big(2\big(\frac{t}{T}\big)^{\alpha_2}-1\Big)\\ &= \frac{t^{\alpha_2-1}}{T^{\alpha_2}}\sum_{i=0}^{n}\sum_{j=1}^{n}u_{ij}(1+j\alpha_2)P_i^{(0,\frac{1}{\alpha_1}-1)}(2x^{\alpha_1}-1)P_{j-1}^{(1,\frac{1}{\alpha_2})}\Big(2\big(\frac{t}{T}\big)^{\alpha_2}-1\Big),\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x^2}I_nu(x,t) &= x^{\alpha_1-2}\sum_{i=2}^{n}\sum_{j=0}^{n}u_{ij}(1+i\alpha_1)\Big[(\alpha_1-1)P_{i-1}^{(1,\frac{1}{\alpha_1})}(2x^{\alpha_1}-1)\\ &\quad+\big(1+(i-1)\alpha_1\big)xP_{i-2}^{(2,\frac{1}{\alpha_1}+1)}(2x^{\alpha_1}-1)\Big]\widetilde{P}_{j,\alpha_2}(t)\\ &= x^{\alpha_1-2}\sum_{i=2}^{n}\sum_{j=0}^{n}u_{ij}(1+i\alpha_1)\Big[(\alpha_1-1)P_{i-1}^{(1,\frac{1}{\alpha_1})}(2x^{\alpha_1}-1)\\ &\quad+\big(1+(i-1)\alpha_1\big)xP_{i-2}^{(2,\frac{1}{\alpha_1}+1)}(2x^{\alpha_1}-1)\Big]P_j^{(0,\frac{1}{\alpha_2}-1)}\Big(2\big(\frac{t}{T}\big)^{\alpha_2}-1\Big).\end{aligned} \tag{3.3}$$

The unknown coefficients u_{ij} are obtained from the fact that $I_nu(x,t)$ should satisfy the problem (1.7)-(1.9) in a suitably chosen univalent set of collocation points $(x_k,t_l),\quad 0\leqslant k,l\leqslant n$. More precisely, the collocation equations

$$b_0\frac{\partial}{\partial t}I_nu(x_k,t_l)=c_0\frac{\partial^2}{\partial x^2}I_nu(x_k,t_l)+\int_0^{t_l}(t_l-s)^{-\frac{1}{2}}\frac{\partial^2}{\partial x^2}I_nu(x_k,s)\mathrm{d}s,\quad 1\leqslant k\leqslant n-1,\quad 1\leqslant l\leqslant n, \tag{3.4}$$

and initial and boundary conditions

$$I_nu(0,t_l)=I_nu(1,t_l)=0,\qquad 1\leqslant l\leqslant n, \tag{3.5}$$

$$I_nu(x_k,0)=f(x_k),\qquad 0\leqslant k\leqslant n, \tag{3.6}$$

have to be valid. It is well known that in spectral collocation methods, an appropriate choice of collocation points is crucial for keeping the so called spectral accuracy [25]. In this study, a particularly convenient choice for the collocation points (x_k, t_l) is

$$(x_k, t_l) = (\chi_k^{\frac{1}{\alpha_1}}, \tau_l^{\frac{1}{\alpha_2}}), \qquad 0 \leqslant k, l \leqslant n,$$

where χ_k and τ_l are the standard LG quadrature points on the intervals $(0, 1)$ and $(0, T)$, respectively.

The integral on the right-hand side of (3.4) is singular. This singularity wrecks the accuracy of the numerical solution and affects on the computational stability. To remedy this deficiency, we propose two efficient schemes.

Scheme 1 (integration by parts): In the first scheme, we set $U = \frac{\partial^2}{\partial x^2} I_n u(x_k, s)$ and $\mathrm{d}V = (t_l - s)^{-\frac{1}{2}}\mathrm{d}s$ and we utilize the integration by parts formula to rewrite (3.4) as

$$b_0 \frac{\partial}{\partial t} I_n u(x_k, t_l) = c_0 \frac{\partial^2}{\partial x^2} I_n u(x_k, t_l) + 2t_l^{\frac{1}{2}} \frac{\partial^2}{\partial x^2} I_n u(x_k, 0) + 2\int_0^{t_l} (t_l - s)^{\frac{1}{2}} \frac{\partial^3}{\partial x^2 \partial t} I_n u(x_k, s)\mathrm{d}s. \tag{3.7}$$

Applying the change of variable $s = t_l v$, Eq. (3.7) is transcribed to

$$b_0 \frac{\partial}{\partial t} I_n u(x_k, t_l) = c_0 \frac{\partial^2}{\partial x^2} I_n u(x_k, t_l) + 2t_l^{\frac{1}{2}} \frac{\partial^2}{\partial x^2} I_n u(x_k, 0) + 2t_l^{\frac{3}{2}} \int_0^1 (1 - v)^{\frac{1}{2}} \frac{\partial^3}{\partial x^2 \partial t} I_n u(x_k, t_l v)\mathrm{d}v. \tag{3.8}$$

Next, we use $(n + 1)$–point LG quadrature rule associated with the interval $[0, 1]$ to approximate the integral on the right-hand side of (3.8),

$$b_0 \frac{\partial}{\partial t} I_n u(x_k, t_l) = c_0 \frac{\partial^2}{\partial x^2} I_n u(x_k, t_l) + 2t_l^{\frac{1}{2}} \frac{\partial^2}{\partial x^2} I_n u(x_k, 0) + t_l^{\frac{3}{2}} \sum_{p=0}^{n} w_p^{LG} (1 - \tau_p)^{\frac{1}{2}} \frac{\partial^3}{\partial x^2 \partial t} I_n u(x_k, t_l \tau_p), \tag{3.9}$$

where w_p^{LG} are LG quadrature weights. The collocation equations (3.9) for $1 \leqslant k \leqslant n - 1$ and $1 \leqslant l \leqslant n$, together with (3.5)–(3.6) form a linear system of $(n+1)^2$ equations for the $(n+1)^2$ unknown coefficients u_{ij} that can be solved using one of the known methods. Note that, for large mode n the dimension of the generated linear system becomes large and it is more convenient to solve it using an iterative method.

Scheme 2 (nonclassical Gaussian quadrature): Based on the approximation of $\frac{\partial^2}{\partial x^2} I_n u$ given in (3.3), let $\frac{\partial^2}{\partial x^2} I_n u(x, t) := x^{\alpha_1 - 2} v(x, t)$. In the second scheme, we first apply the change of variable

$$s = t_l x^{3-\alpha_1}, \quad x \in [0, 1],$$

to rewrite (3.4) as

$$b_0 \frac{\partial}{\partial t} I_n u(x_k, t_l) = c_0 \frac{\partial^2}{\partial x^2} I_n u(x_k, t_l) + t_l^{\frac{1}{2}} (3 - \alpha_1) \int_0^1 (1 - x^{3-\alpha_1})^{-\frac{1}{2}} x^{2-\alpha_1} \frac{\partial^2}{\partial x^2} I_n u(x_k, t_l x^{3-\alpha_1})\mathrm{d}x. \tag{3.10}$$

To calculate the integral on the right-hand side of (3.10), the $(N + 1)$–point gaussian quadrature rule (2.5) in the form

$$\int_0^1 (1 - x^{3-\alpha_1})^{-\frac{1}{2}} g(x)\mathrm{d}x = \sum_{p=0}^{N} w_p g(v_p), \qquad g \in \mathbb{P}_{2N+1} \tag{3.11}$$

is used. The weight function $w(x) = (1 - x^{3-\alpha_1})^{-\frac{1}{2}}$ is a nonclassical one and no explicit formulae are known for v_p and w_p. Nevertheless, we can use the Stieltjes procedure and Golub algorithm to calculate the nodes and weights as discussed in Section 2. It is worth mentioning that the Gaussian quadrature rule (3.11) with $N = \lceil \frac{n+1}{2} \rceil$ becomes exact for computing the integral in (3.10). Hence, after obtaining the nodes v_p and the weights w_p, Eq. (3.10) becomes

$$b_0 \frac{\partial}{\partial t} I_n u(x_k, t_l) = c_0 \frac{\partial^2}{\partial x^2} I_n u(x_k, t_l) + t_l^{\frac{1}{2}} (3 - \alpha_1) \sum_{p=0}^{\lceil \frac{n+1}{2} \rceil} w_p v_p^{2-\alpha_1} \frac{\partial^2}{\partial x^2} I_n u(x_k, t_l v_p^{3-\alpha_1}). \tag{3.12}$$

Again, the collocation equations (3.12) for $1 \leqslant k \leqslant n - 1$ and $1 \leqslant l \leqslant n$, together with (3.5)–(3.6) form an $(n+1)^2 \times (n+1)^2$ linear system of algebraic equations for the unknowns u_{ij}.

Table 1: Maximum absolute errors for Test 1.

	Scheme 1		Scheme 2		CDS [13]		BSCM [9]	
n	$\alpha=\frac{1}{2}$	$\alpha=\frac{3}{4}$	$\alpha=\frac{1}{2}$	$\alpha=\frac{3}{4}$	(J,N)	Error	(M,N)	Error
5	2.9×10^{-1}	3.5×10^{-1}	1.4×10^{-1}	4.1×10^{-1}	(10,80)	3.2×10^{-2}	(800,10)	9.7×10^{-3}
10	1.1×10^{-2}	4.6×10^{-3}	5.4×10^{-3}	3.8×10^{-3}	(10,160)	1.2×10^{-2}	(800,25)	7.3×10^{-3}
15	6.8×10^{-4}	1.2×10^{-3}	2.9×10^{-3}	3.2×10^{-3}	(10,320)	4.2×10^{-3}	(800,50)	7.1×10^{-3}
20	2.2×10^{-4}	9.1×10^{-4}	1.8×10^{-3}	2.4×10^{-3}	(10,640)	1.5×10^{-3}	(800,100)	7.0×10^{-3}
25	8.8×10^{-5}	4.9×10^{-4}	9.3×10^{-4}	1.1×10^{-3}	–	–	–	–

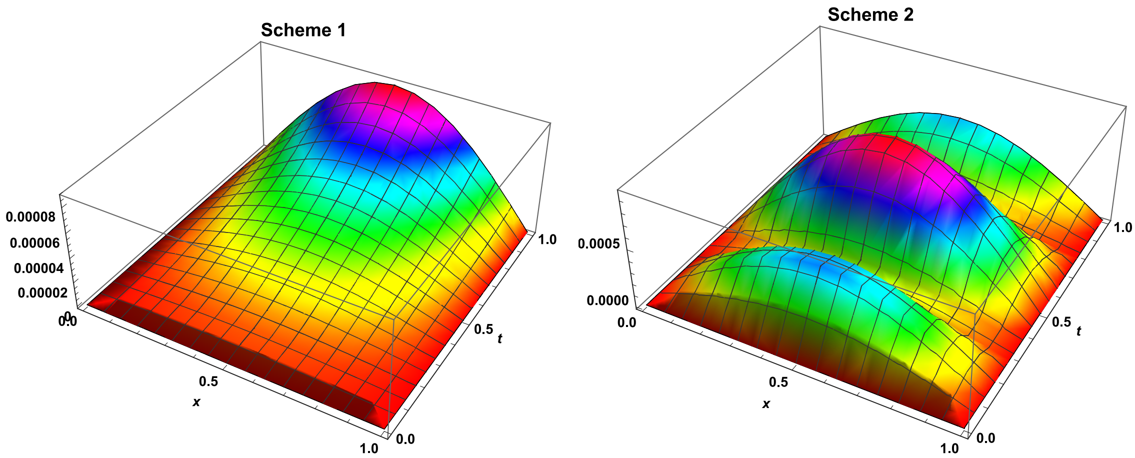

Figure 1: Absolute error functions for Test 1 with $\alpha = 0.5$ and $n = 25$.

4 Numerical tests

To test the Müntz collocation method developed here for solving the parabolic integro-differential equation with a weakly singular kernel, we give two numerical examples.

Test 1: Consider the problem (1.7)-(1.9) with $b_0 = 1$, $c_0 = 0$, $f(x) = \sin(\pi x)$ and $T = 1$. In this case the exact solution is

$$u(x,t) = \sin(\pi x)\sum_{i=0}^{\infty}(-1)^i\Gamma\left(\frac{3}{2}i+1\right)^{-1}\left(\pi^{\frac{5}{2}}t^{\frac{3}{2}}\right)^i.$$

The maximum absolute errors obtained using both schemes of the presented method with $\alpha_1 = \alpha_2 = 0.5, 0.75$ and different values of n, together with the errors obtained in [13] using a compact difference scheme (CDS) and the errors in [9] using a cubic B-splines collocation method (BSCM) are presented in Table 1. In the CDS, J is the number of grides in space and N is the number of grides in time. In the BSCM, M and N are the number for temporal and spatial divisions, respectively. Table 1 shows the convergence of both Schemes 1 and 2. However, in this test problem, Scheme 1 with $\alpha_1 = \alpha_2 = 0.5$ provides more accurate numerical results. Moreover, Table 1 demonstrates the superiority of our collocation method over the BSCM as our method is more accurate and it uses much fewer total number of collocation points. Fig. 1 depicts the absolute error functions of Schemes 1 and 2 for $\alpha = 0.5$ and $n = 25$.

Test 2: In the second test example, we consider the problem (1.7)-(1.9) with $b_0 = c_0 = 1$, $f(x) = \sin(\pi x)$ and $T = 1$. In this case no analytical solution is available. The approximated solution for $\alpha_1 = \alpha_2 = 0.5$ with $n = 20$ (441 nodes) is plotted in Fig. 2. The numerical results for this problem at $T = 1$ computed for $\alpha_1 = \alpha_2 = 0.5$ and different values of n are given in Table 2. The convergence of the proposed spectral collocation schemes is apparent from this table.

5 Discussion of results

In our numerical implementations, we observed some key features of the derived spectral collocation method with the proposed schemes for approximating the integral term, which are explained now:

Table 2: Results for $u(x,1)$ with $\alpha_1 = \alpha_2 = 0.5$ for Test 2.

	Scheme 1			Scheme 2		
x	$n = 12$	$n = 16$	$n = 20$	$n = 12$	$n = 16$	$n = 20$
0.1	−0.003786	−0.003827553	−0.003985512	−0.003926	−0.0039408200	−0.0039438554
0.2	−0.007202	−0.007280438	−0.007408356	−0.007468	−0.0074958850	−0.0075016588
0.3	−0.009914	−0.010020663	−0.010275678	−0.010279	−0.0103172006	−0.0103251475
0.4	−0.011655	−0.011779996	−0.012035748	−0.012083	−0.0121285966	−0.0121379389
0.5	−0.012254	−0.012386221	−0.012865964	−0.012705	−0.0127527612	−0.0127625842
0.6	−0.011654	−0.011779995	−0.012035748	−0.012083	−0.0121285966	−0.0121379389
0.7	−0.009914	−0.010020662	−0.010275678	−0.010279	−0.0103172005	−0.0103251475
0.8	−0.007204	−0.007280437	−0.007408356	−0.007468	−0.0074958849	−0.0075016588
0.9	−0.003788	−0.003827552	−0.003985512	−0.003926	−0.0039408199	−0.0039438554

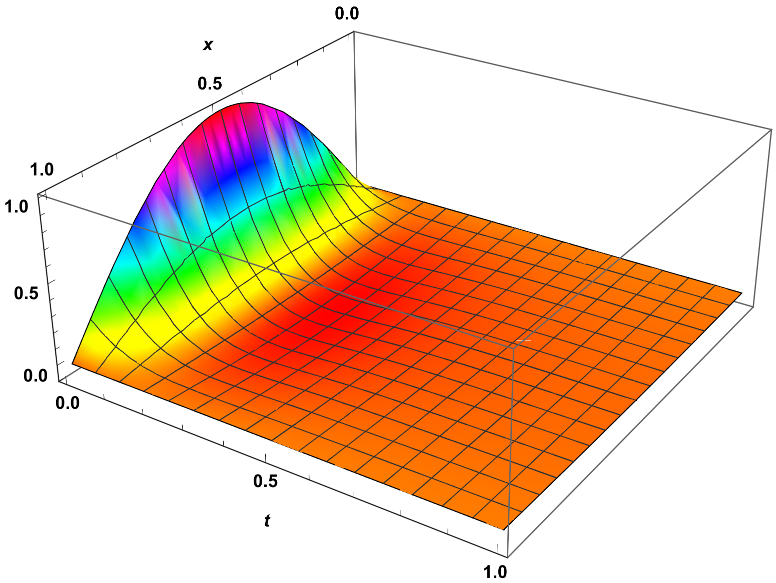

Figure 2: Numerical solution of Test problem 2 with $\alpha = 0.5$ and $n = 20$.

- The present collocation method choose a suitable mesh and basis functions and by tuning the parameters α_1 and α_2 the accuracy of the solution may be improved; however, we found out that the choice $\alpha_1 = \alpha_2 = 0.5$ provides more accurate results.
- Table 1 shows that the errors using the Scheme 1 are smaller than those for the Scheme 2 and Scheme 1 provides higher convergence rate. Nevertheless, Scheme 1 has more computational complexity than Scheme 2. A drawback of Scheme 1 is the appearance of the third order partial derivative $\frac{\partial^3}{\partial x^2 \partial t}$ in the formulation of the problem. On the other hand, the main difficulty in Scheme 2 is the generation of a set of nonclassical nodes and weights.
- Although the kernel in the considered problem is weakly singular, our proposed schemes provide satisfactory numerical results with moderate mode n. Nevertheless, very large mode n is required so that the errors achieve machine precision that is not convenient.
- In Fig. 1 it is seen that the absolute error function of Scheme 1 is increasing in time while the absolute error function of Scheme 2 is oscillatory in time.

6 Conclusions

We have considered a space-time spectral collocation method based on the Müntz-Legendre polynomials for computing the solutions of a parabolic integro-differential equation with a weakly singular kernel that appears in the study of heat flow in materials of fading memory type. A opposed to finite difference schemes for time discretization, we utilized the collocation method for discretizing both the time and space variables simultaneously that improves the accuracy. The singular integral term involve in the problem, limit the accuracy of the spectral methods. Hence, two efficient stable schemes have been introduced to approximate it and the numerical results were quite satisfactory. The new method outlined here were tested on two problems and it was seen that the results converge to the exact solution when n goes to infinity.

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