Int. J. Nonlinear Anal. Appl. 14 (2023) 2, 221–231
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.27809.3721

Fixed point theorems for generalized orthogonal F-contraction and F-expansion of Wardowski kind via the notion of $\psi-$fixed point

Sahil Arora[a,*], Monika Masta[b], Mahender Poonia[b]

[a]Department of Mathematics, K.R.M.D.A.V. College, Nakodar, Punjab, 144040, India

[b]Department of Mathematics, Om Sterling Global University, Hisar, Haryana, 125001, India

(Communicated by Ali Farajzadeh)

Abstract

In this manuscript, we introduce generalized orthogonal $(\mathfrak{f}^*, \psi)$-contraction of kind (S) and use this concept to establish ψ-fixed point theorems in the frame of O-complete orthogonal metric space. Secondly, we introduce the new notion of generalized orthogonal $(\mathfrak{f}^*, \psi)$ expansive mapping and utilize the same to prove some fixed point results for surjective mapping satisfying certain conditions. Our results extend and improve the results of [3] and [7] by omitting the continuity condition of $F \in \Im$ with the aid of ψ-fixed point. We also give an illustrative example which yields the main result. Also, many existing results in the frame of metric spaces are established.

Keywords: Generalized orthogonal $(\mathfrak{f}^*, \psi)$-contraction; Generalized orthogonal $(\mathfrak{f}^*, \psi)$-expansion, ψ-fixed point, $\perp$-preserving function, $\perp$-continuous function, Lower semi-continuous function
2020 MSC: 47H10, 54H25

1 Introduction and Preliminaries

In 1922, Banach established a useful result in fixed point theory regarding a contraction mapping, known as the Banach contraction principle. In 2012, Wardowski [12] gave a new contraction known as F-contraction and prove fixed point theorem concerning F-contractions. In this manner, Wardowski conclude the Banach contraction principle in a different way from the eminent results from the literature. Numerous generalities of F-contraction have been discussed by several prominant researchers. In 2012, Shahi et al. [9] recommend the view of (ξ, α) mapping of expansion and demonstrated a few aftereffects of fixed point for such assortment of functions in complete metric spaces. In 2013, Murthy and Prasad [6] generalize weak contraction by making combinations of $\sigma(\Omega, \mho)$. Later, Piri and Kumam [7] established Wardowski type fixed point theorems in complete metric spaces. Motivated by the perception of Dung and Hang [1], in 2016, Piri and Kumam [7] generalized the concept of generalized F-contraction and established some fixed point theorems for such kind of functions in complete metric spaces by addition of four terms $d(f^2x, x), d(f^2x, fx), d(f^2x, y), d(f^2x, fy)$. In 2017, Gornicki [3] established some results for F-expansion mapping in the context of metric and G-metric spaces.

*Corresponding author
Email addresses: drprofsahilarora@gmail.com (Sahil Arora), monikamasta93@gmail.com (Monika Masta), drmahender@osgu.ac.in (Mahender Poonia)

Received: July 2022 *Accepted:* September 2022

On the other hand one of the significant and imperative ideas, φ fixed point result, was presented by Jleli et al. [4] and conclude some results for partial metric spaces. They also established various $\varphi-$fixed point results for graphic and weak $(F,\varphi)-$contraction mappings in the edge of metric spaces. In 2018, Senapati et al. [8] established fixed point results for orthogonal metric spaces with the aid of w-distance function. In 2019, Taheri and Farajzadeh [10] established some results for α-type almost-F-contractions and F-Suzuki contractions. Afterwards, Kumar and Arora [5] proved results for generalized F-contraction in the frame of G-metric spaces. Motivated by the work of [8] very recently, Touaila and Moutawakil [11] investigated some results for orthogonal contractive mapping in the edge of O-complete orthogonal bounded metric space.
Wardowski [12] defined the F-contraction as follows.

Definition 1.1. [12] Let $(\mathcal{H},\sigma)$ be a metric space and $\mathcal{Q}:\mathcal{H}\to\mathcal{H}$ be a mapping. Then, $\mathcal{Q}$ is named an F-contraction on $(\mathcal{H},\sigma)$, if we can find $F\in\Im$ such that $\sigma(\mathcal{Q}\Omega,\mathcal{Q}\mho)>0\Rightarrow\gamma+F(\sigma(\mathcal{Q}\Omega,\mathcal{Q}\mho))\le F(\sigma(\mathcal{Q}\Omega,\mathcal{Q}\mho))$, for every $\Omega,\mho\in\mathcal{H}$, where $\Im$ be class of all mappings $F:(0,\infty)\to\mathcal{R}$ such that

(F1) F is strictly increasing function, that is, for all $\Omega,\mho\in(0,\infty)$,
if $\Omega<\mho$, then $F(\Omega)<F(\mho)$.

(F2) For each sequence $\{a_n\}$ of natural numbers, we have $\lim_{n\to\infty}a_n=0$ if and only if $\lim_{n\to\infty}F(a_n)=-\infty$.

(F3) There exists $q\in(0,1)$ such that $\lim_{a\to0^+}(a^qF(a))=0$.

Wardowski [12] gave some examples of $\Im$ as follows:

1.$F(\zeta)=\ln\zeta$.

2.$F(\zeta)=-\frac{1}{\zeta^{\frac{1}{2}}}$.

3.$F(\zeta)=\ln(\zeta)+\zeta$.

4.$F(\zeta)=\ln(\zeta^2+\zeta)$.

Remark 1.2. Let $F:\mathbb{R}_+\to\mathbb{R}$ be defined as $F(\beta)=\ln(\beta)$, then $F\in\Im$. Now, F-contraction changes to a Banach contraction. Consequently, the Banach contractions are special case of F-contractions. There are F-contractions which are not Banach contractions
(see[12]).

F-weak contraction was established by Wardowski and Dung in 2014 which is given as follows:

Definition 1.3. [13] Let $\mathcal{Q}$ be a self map in a metric space $(\mathcal{H},\sigma)$. Then, $\mathcal{Q}$ is named as F-weak contraction on $(\mathcal{H},\sigma)$ if there occur $F\in\Im$ and $\gamma>0$ with the goal that $\sigma(\mathcal{Q}\Omega,\mathcal{Q}\mho)>0$ implies that

$$\gamma+F(\sigma(\mathcal{Q}\Omega,\mathcal{Q}\mho))\le F(\max\{\sigma(\Omega,\mho),\sigma(\mathcal{Q}\Omega,\Omega),\sigma(\mho,\mathcal{Q}\mho),\frac{\sigma(\Omega,\mathcal{Q}\mho)+\sigma(\mho,\mathcal{Q}\Omega)}{2}\}),$$

for every $\Omega,\mho\in\mathcal{H}$.

Theorem 1.4. [13] Let $(\mathcal{H},\sigma)$ be a complete metric space and $\mathcal{Q}$ be an F-weak contraction mapping. If F or $\mathcal{Q}$ is continuous, then $\mathcal{Q}$ has a fixed point ${\Theta_1}^*\in\mathcal{H}$ which is unique and the sequence $\{\mathcal{Q}^n\Omega\}$ tends to ${\Theta_1}^*$, where n varies from 1 to ∞.

Hang and Dung [1] investigated the concept of generalized F-contraction and proved useful fixed point results for such kind of functions.

Definition 1.5. [1] Let $\mathcal{Q}:\mathcal{H}\to\mathcal{H}$ be a self-mapping in $(\mathcal{H},\sigma)$. Then, $\mathcal{Q}$ is named as generalized F-contraction on $(\mathcal{H},\sigma)$ if we find $F\in\Im$ and $\delta>0$ with the goal that

$$\sigma(\mathcal{Q}\Omega,\mathcal{Q}\mho)>0\Rightarrow\delta+F(\sigma(\mathcal{Q}\Omega,\mathcal{Q}\mho))\le F(\max\{\sigma(\Omega,\mho),\sigma(\Omega,\mathcal{Q}\Omega),\sigma(\mho,\mathcal{Q}\mho),$$
$$\frac{\sigma(\Omega,\mathcal{Q}\mho)+\sigma(\mho,\mathcal{Q}\Omega)}{2},\frac{\sigma(\mathcal{Q}^2\Omega,\Omega)+\sigma(\mathcal{Q}^2\Omega,\mathcal{Q}\mho)}{2},\sigma(\mathcal{Q}^2\Omega,\mathcal{Q}\Omega),\sigma(\mathcal{Q}^2\Omega,\mho),\sigma(\mathcal{Q}^2\Omega,\mathcal{Q}\mho)\}),$$

for all $\Omega,\mho\in\mathcal{H}$.

Subsequently, Kumam and Piri replace the $(F3)$ with $(F3^{'})$ in the definition of F-contraction given by Wardowski [12].

$(F3^{'})$: F is continuous on positive reals.

They gave the notation $\mathfrak{F}$ to denote the class of all maps $F : \mathbb{R}^{+} \rightarrow \mathbb{R}$ which fulfil $(F1),(F2)$ and $(F3^{'})$. Piri and Kumam also proved some useful fixed point results for $(\mathcal{H},\sigma)$. Now, the conditions $(F3)$ and $(F3^{'})$ are not associated with each other. For example, for $q \geq 1, F(\beta) = \frac{-1}{\beta^{q}}$, then F meet the conditions (F1) and (F2) but it does not fulfil(F3), while it fulfils the condition $(F^{'})$. In view of this, it is significant to observe the sequel of Wardowski [12] with the functions $F \in \Im$ rather than F $\in \mathfrak{F}$. In 2017, Gornicki introduced the notion of F-expansion as follows:

Definition 1.6. [3] Let $\mathcal{Q}$ be a self-mapping in $(\mathcal{H},\sigma)$. Then, $\mathcal{Q}$ is named an F-expansion on $(\mathcal{H},\sigma)$ if we can find $F \in \Im$ with the goal that $\sigma(\mathcal{Q}\Omega,\mathcal{Q}\mho) > 0$ implies that $F(\sigma(\mathcal{Q}\Omega,\mathcal{Q}\mho)) \geq F(\sigma(\mathcal{Q}\Omega,\mathcal{Q}\mho)) + \gamma$, for every $\Omega,\mho \in \mathcal{H}$, where $\Im$ be class of all mappings $F : (0,\infty) \rightarrow \mathcal{R}$ such that

(F1) F is strictly increasing function, that is, for all $\Omega,\mho \in (0,\infty)$,
if $\Omega < \mho$, then $F(\Omega) < F(\mho)$.

(F2) For each sequence $\{a_n\}$ of natural numbers, we have $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

(F3) There exists $q \in (0,1)$ such that $\lim_{a\to 0^{+}}(a^{q}F(a)) = 0$.

In 2014, Jleli et al. introduced the notion of ψ-fixed point as follows:

Definition 1.7. [4] Let $(\mathcal{H},\sigma)$ be a metric space, $\psi : \mathcal{H} \rightarrow [0,\infty)$ be a function and $h : \mathcal{H} \rightarrow \mathcal{H}$ be an operator. An element $z \in \mathcal{H}$ is said to be ψ-fixed point of operator h if and only if $z \in \{\Omega \in \mathcal{H} : h\Omega = \Omega\} \cap \{\Omega \in \mathcal{H} : \psi(\Omega) = 0\}$.

In 2017, Gordji et al. [2] described the notion of orthogonal set and orthogonal metric spaces. In their work, they held a generalization of Banach fixed point theorem in this interestingly-defined construction and in addition, applied their acquired results to demonstrate the existence of a solution of an ordinary differential equation.

Definition 1.8. [2] Let $\perp \subseteq \mathcal{H} \times \mathcal{H}$ be a binary relation, where $\mathcal{H}$ is a non-void set. If $\perp$ satisfies the following property:

$$\exists \Omega_0[(\forall \mho \in \Omega, \mho \perp \Omega_0) or (\forall \mho \in \Omega, \Omega_0 \perp \mho)],$$

then it is called an orthogonal set (briefly, O-set) and Ω_0 is called an orthogonal element.

Definition 1.9. [2] The triplet $(\mathcal{H},\perp,\sigma)$ is called an orthogonal metric space if $(\mathcal{H},\perp)$ is an O-set and $(\mathcal{H},\sigma)$ is a metric space.

Definition 1.10. [2] Let $(\mathcal{H},\perp)$ be an O-set. A function $h : \mathcal{H} \rightarrow \mathcal{H}$ is called $\perp$-preserving function if $h(\Omega) \perp h(\mho)$, when $\Omega \perp \mho$.

Definition 1.11. [2] Let $(\mathcal{H},\perp,\sigma)$ be an orthogonal metric space. A function $h : \mathcal{H} \rightarrow \mathcal{H}$ is called $\perp$-continuous function, if for every O-sequence $\{\Omega_n\}$ in $\mathcal{H}$ with $\Omega_n \rightarrow \Omega$, when $n \rightarrow \infty$, we have $h\Omega_n \rightarrow h\Omega$, when $n \rightarrow \infty$.

The aim of this paper is to introduce generalized orthogonal $(\mathfrak{f}^{*},\psi)$-contraction and generalized orthogonal $(\mathfrak{f}^{*},\psi)$-expansion mapping with the aid of ψ-fixed point. Also some ψ-fixed point results have been established by replacing the conditions $(F1)$, $(F3)$ and $(F3^{'})$ of [7] by a single condition (E).

Throughout this paper, we denote by $\mathcal{H}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_{+}$, $\mathbb{Z}$ and $\mathbb{Z}_{+}$ the nonempty set, the set of natural numbers, the set of real numbers, the set of positive real numbers the set of integers and the set of positive integers, respectively.

$$F_h = \{\Omega \in \mathcal{H} : h\Omega = \Omega\} \text{ and } K_{\psi} = \{\Omega \in \mathcal{H} : \psi(\Omega) = 0\}.$$

2 Main Results

Let $\mathcal{F}_{\mathcal{E}}$ be the class of all continuous functions $\mathfrak{f}^{*} : (0,\infty)\times[0,\infty)^{2} \rightarrow \mathbb{R}$, which gratify the accompanying condition:
(E) For all sequences $\{e_n\},\{f_n\}$ and $\{g_n\} \in \mathbb{R}_{+}$,

$$\lim_{n\to\infty} \mathfrak{f}^{*}(e_n,f_n,g_n) = -\infty \text{ if and only if } \lim_{n\to\infty} e_n^{2} + e_n + f_n + g_n = 0.$$

Now, we introduce generalized orthogonal $(\mathfrak{f}^{*},\psi)$-contraction of kind (S).

Remark 2.1. If $\{e_n\}, \{f_n\}$ and $\{g_n\} \in \mathbb{R}_+$, then

$$\lim_{n\to\infty} e_n^2 + e_n + f_n + g_n = 0 \Leftrightarrow \lim_{n\to\infty} e_n = 0, \lim_{n\to\infty} f_n = 0 \, and \, \lim_{n\to\infty} g_n = 0.$$

Definition 2.2. Let $\psi : \mathcal{H} \to [0, \infty)$ be a map in orthogonal metric space. A function $h : \mathcal{H} \to \mathcal{H}$ is called generalized orthogonal $(\mathfrak{f}^*, \psi)$-contraction of kind (S), if there occur $\mathfrak{f}^* \in \mathcal{F}_\mathcal{E}$ and $\chi > 0$ in order that

$$\Omega \perp \mho[\sigma(h\Omega, h\mho) > 0 \Rightarrow \chi + \mathfrak{f}^*(\sigma(h\Omega, h\mho), \psi(h\Omega), \psi(h\mho)) \leq \mathfrak{f}^*(S(\Omega, \mho), \psi(\Omega), \psi(\mho))], \tag{2.1}$$

where

$$S(\Omega, \mho) = max\{\sigma(\Omega, \mho), \frac{\sigma(\Omega, h\mho) + \sigma(\mho, h\Omega)}{2}, \frac{\sigma(h^2\Omega, \Omega) + \sigma(h^2\Omega, h\mho)}{2}, \sigma(h^2\Omega, h\Omega), \sigma(h^2\Omega, \mho), \sigma(h^2\Omega, h\mho) + \sigma(\Omega, h\Omega), \sigma(h\Omega, \mho) + \sigma(\mho, h\mho)\},$$

for all $\Omega, \mho \in \mathcal{H}$.

Lemma 2.3. Let $\psi : \mathcal{H} \to [0, \infty), F_h \subseteq K_\psi$ and $h : \mathcal{H} \to \mathcal{H}$ be $\perp$-preserving generalized orthogonal $(\mathfrak{f}^*, \psi)$-contraction in orthogonal metric space, where $\mathfrak{f}^* \in \mathcal{F}_\mathcal{E}$. If $\Omega_n \neq \Omega_{n+1}, \forall n \in \mathbb{Z}_+$, then

(i) $\lim_{n\to\infty} \sigma(\Omega_n, \mho_n) = \lim_{n\to\infty} \psi(\Omega_n)$;
(ii) Ω_n is Cauchy O-sequence.

Proof . From the definition of the orthogonality, it indicates that $\Omega_0 \perp h(\Omega_0)$ or $h(\Omega_0) \perp \Omega_0$, where $\Omega_0 \in \mathcal{H}$ be any point. Inserting $h^n\Omega_0 = \Omega_{n+1}$, for every $n \in \mathbb{N} \cup \{0\}$. If there occur $n \in \mathbb{N} \cup \{0\}$ such that $h\Omega_n = \Omega_n$ and using given assumption $F_h \subseteq K_\psi$ of Lemma 2.3, we get that Ω_n is ψ-fixed point of h. Imagine that $\sigma(\Omega_n, h\Omega_n) > 0$, for every $n \in \mathbb{N} \cup \{0\}$. Since $h : \mathcal{H} \to \mathcal{H}$ is $\perp$-preserving, which indicates that $\Omega_{n-1} \perp \Omega_n$ or $\Omega_n \perp \Omega_{n-1}$, for every $n \in \mathbb{N} \cup \{0\}$. This shows that $\{\Omega_n\}$ is an O-sequence. Since, h is generalized orthogonal $(\mathfrak{f}^*, \psi)$-contraction mapping, we have

$$\chi + \mathfrak{f}^*(\sigma(h\Omega_{n-1}, h\Omega_n), \psi(h\Omega_{n-1}), \psi(h\Omega_n)) \leq \mathfrak{f}^*(S(\Omega_{n-1}, \Omega_n), \psi(\Omega_{n-1}), \psi(\Omega_n)), \tag{2.2}$$

for all $\Omega_{n-1}, \Omega_n \in \mathcal{H}$, where

$$\begin{aligned} S(\Omega_{n-1}, \Omega_n) =& \max\{\sigma(\Omega_{n-1}, \Omega_n), \frac{\sigma(\Omega_{n-1}, h\Omega_n) + \sigma(\Omega_n, h\Omega_{n-1})}{2}, \frac{\sigma(h^2\Omega_{n-1}, \Omega_{n-1}) + \sigma(h^2\Omega_{n-1}, h\Omega_n)}{2}, \\ & \sigma(h^2\Omega_{n-1}, h\Omega_{n-1}), \sigma(h^2\Omega_{n-1}, \Omega_n), \sigma(h^2\Omega_{n-1}, h\Omega_n) + \sigma(\Omega_{n-1}, h\Omega_{n-1}), \sigma(h\Omega_{n-1}, \Omega_n) + \sigma(\Omega_n, h\Omega_n)\}, \\ =& \max\{\sigma(\Omega_{n-1}, \Omega_n), \frac{\sigma(\Omega_{n-1}, \Omega_{n+1}) + \sigma(\Omega_n, \Omega_n)}{2}, \frac{\sigma(\Omega_{n+1}, \Omega_{n-1}) + \sigma(\Omega_{n+1}, \Omega_{n+1})}{2}, \\ & \sigma(\Omega_{n+1}, \Omega_{n+1}), \sigma(\Omega_{n+1}, \Omega_n), \sigma(\Omega_{n+1}, \Omega_{n+1}) + \sigma(\Omega_{n-1}, \Omega_n), \sigma(\Omega_n, \Omega_n) + \sigma(\Omega_n, \Omega_{n+1})\}, \\ =& \max\{\sigma(\Omega_{n-1}, \Omega_n), \sigma(\Omega_n, \Omega_{n+1})\}. \end{aligned}$$

If there occur $n \in \mathbb{Z}_+$ in order that $max\{\sigma(\Omega_{n-1}, \Omega_n), \sigma(\Omega_n, \Omega_{n+1})\} = \sigma(\Omega_n, \Omega_{n+1})$, subsequently (2.2) becomes $\chi + \mathfrak{f}^*(\sigma(h\Omega_{n-1}, h\Omega_n), \psi(h\Omega_{n-1}), \psi(h\Omega_n)) \leq \mathfrak{f}^*(\sigma(\Omega_n, \Omega_{n+1}), \psi(\Omega_{n-1}), \psi(\Omega_n))$. Thus,

$$\chi + \mathfrak{f}^*(\sigma(\Omega_n, \Omega_{n+1}), \psi(\Omega_n), \psi(\Omega_{n+1})) \leq \mathfrak{f}^*(\sigma(\Omega_n, \Omega_{n+1}), \psi(\Omega_{n-1}), \psi(\Omega_n)). \tag{2.3}$$

Since $\chi > 0$, we get a counterstatement. Thus,

$$\max\{\sigma(\Omega_{n-1}, \Omega_n), \sigma(\Omega_n, \Omega_{n+1})\} = \sigma(\Omega_{n-1}, \Omega_n).$$

Now, (2.2) becomes

$$\begin{aligned} \mathfrak{f}^*(\sigma(h\Omega_{n-1}, h\Omega_n), \psi(h\Omega_{n-1}), \psi(h\Omega_n)) &\leq \mathfrak{f}^*(\sigma(\Omega_{n-1}, \Omega_n), \psi(\Omega_{n-1}), \psi(\Omega_n)) - \chi \\ &\leq \mathfrak{f}^*(\sigma(\Omega_{n-2}, \Omega_{n-1}), \psi(\Omega_{n-2}), \psi(\Omega_{n-1})) - 2\chi \\ &\leq \mathfrak{f}^*(\sigma(\Omega_{n-3}, \Omega_{n-2}), \psi(\Omega_{n-3}), \psi(\Omega_{n-2})) - 3\chi \\ &\vdots \\ &\leq \mathfrak{f}^*(\sigma(\Omega_0, \Omega_1), \psi(\Omega_0), \psi(\Omega_1)) - n\chi. \end{aligned} \tag{2.4}$$

Let $n \to \infty$ in (2.4), we acquire

$$\lim_{n\to\infty} \mathfrak{f}^*(\sigma(h\Omega_{n-1}, h\Omega_n), \psi(h\Omega_{n-1}), \psi(h\Omega_n)) = -\infty.$$

With the assistance of (E), we acquire

$$\lim_{n\to\infty} \sigma(\Omega_{n-1}, \Omega_n) = \lim_{n\to\infty} \psi(\Omega_n) = 0. \tag{2.5}$$

Imagine that $\{\Omega_n\}$ is not Cauchy O-sequence in $\mathcal{H}$. Then, there exist $\delta > 0$ and subsequences $\{\Omega_{n_e}\}$ and $\{\Omega_{r_e}\}$ of $\{\Omega_n\}$ such that $\sigma(\Omega_{n_e}, \Omega_{r_e}) \geq \delta$ and $\sigma(\Omega_{n_e}, \Omega_{r_e-1}) < \delta$, for each $r_e > n_e > e$, where $e \in \mathbb{Z}_+$. Now,

$$\begin{aligned}\delta &\leq \sigma(\Omega_{n_e}, \Omega_{r_e})\\ &\leq \sigma(\Omega_{r_e}, \Omega_{r_e-1}) + \sigma(\Omega_{r_e-1}, \Omega_{n_e})\\ &\leq \sigma(\Omega_{r_e}, \Omega_{r_e-1}) + \delta.\end{aligned}$$

With the assistance of (2.5) and making $e \to \infty$, we acquire

$$\lim_{e\to\infty} \sigma(\Omega_{n_e}, \Omega_{r_e}) = \delta. \tag{2.6}$$

Further for each $e \geq n_1$, $\exists\, n_1 \in \mathbb{Z}_+$ ensuring that

$$\sigma(\Omega_{r_e}, \Omega_{r_e+1}) < \frac{\delta}{8} \text{ and } \sigma(\Omega_{n_e}, \Omega_{n_e+1}) < \frac{\delta}{8}. \tag{2.7}$$

Now, we exhibit that $\sigma(\Omega_{r_e+1}, \Omega_{n_e+1}) > 0$, for every $e \geq n_1$. Let us imagine $\exists\ g \geq n_1$ ensuring that

$$\sigma(\Omega_{r_g+1}, \Omega_{n_g+1}) = 0. \tag{2.8}$$

With the aid of (2.6), (2.7) and (2.8), we acquire

$$\begin{aligned}\delta &\leq \sigma(\Omega_{r_g}, \Omega_{n_g})\\ &\leq \sigma(\Omega_{r_g}, \Omega_{r_g+1}) + \sigma(\Omega_{r_g+1}, \Omega_{n_g})\\ &\leq \sigma(\Omega_{r_g}, \Omega_{r_g+1}) + \sigma(\Omega_{r_g+1}, \Omega_{n_g+1}) + \sigma(\Omega_{n_g+1}, \Omega_{n_g})\\ &< \frac{\delta}{8} + 0 + \frac{\delta}{8} = \frac{\delta}{4},\end{aligned}$$

which is a counterstatement. Consequently,

$$\sigma(\Omega_{r_e+1}, \Omega_{n_e+1}) > 0 \tag{2.9}$$

for every $e \geq n_1$. Further inserting $\Omega = \Omega_{r_e}$ and $\mho = \Omega_{n_e}$ in (2.1), we acquire

$$\chi + \mathfrak{f}^*(\sigma(h\Omega_{r_e}, h\Omega_{n_e}), \psi(h\Omega_{r_e}), \psi(h\Omega_{n_e})) \leq \mathfrak{f}^*(S(\Omega_{r_e}, \Omega_{n_e}), \psi(\Omega_{r_e}), \psi(\Omega_{n_e})),$$

where

$$\begin{aligned}S(\Omega_{r_e}, \Omega_{n_e}) &= \max\{\sigma(\Omega_{r_e}, \Omega_{n_e}), \frac{\sigma(\Omega_{r_e}, h\Omega_{n_e}) + \sigma(\Omega_{n_e}, h\Omega_{r_e})}{2}, \frac{\sigma(h^2\Omega_{r_e}, \Omega_{r_e}) + \sigma(h^2\Omega_{r_e}, h\Omega_{n_e})}{2},\\ &\sigma(h^2\Omega_{r_e}, h\Omega_{r_e}), \sigma(h^2\Omega_{r_e}, \Omega_{n_e}), \sigma(h^2\Omega_{r_e}, h\Omega_{n_e}) + \sigma(\Omega_{r_e}, h\Omega_{r_e}), \sigma(h\Omega_{r_e}, \Omega_{n_e}) + \sigma(\Omega_{n_e}, h\Omega_{n_e})\}\\ &= \max\{\sigma(\Omega_{r_e}, \Omega_{n_e}), \frac{\sigma(\Omega_{r_e}, \Omega_{n_e+1}) + \sigma(\Omega_{n_e}, \Omega_{r_e+1})}{2}, \frac{\sigma(\Omega_{r_e+2}, \Omega_{r_e}) + \sigma(\Omega_{r_e+2}, \Omega_{n_e+1})}{2},\\ &\sigma(\Omega_{r_e+2}, \Omega_{r_e+1}), \sigma(\Omega_{r_e+2}, \Omega_{n_e}), \sigma(\Omega_{r_e+2}, \Omega_{n_e+1}) + \sigma(\Omega_{r_e}, \Omega_{r_e+1}), \sigma(\Omega_{r_e+1}, \Omega_{n_e}) + \sigma(\Omega_{n_e}, \Omega_{n_e+1})\}.\end{aligned}$$

With the aid of (2.5), (2.6) and continuity property of $\mathfrak{f}^*$, we acquire

$$\chi + \mathfrak{f}^*(\delta, 0, 0) \leq \mathfrak{f}^*(\delta, 0, 0),$$

a counterstatement, which indicates that $\{\Omega_n\}$ is a Cauchy O-sequence in $\mathcal{H}$. $\square$

Theorem 2.4. Let $h : \mathcal{H} \to \mathcal{H}$ be $\perp$-preserving and $\perp$-continuous generalized orthogonal $(\mathfrak{f}^*, \psi)$-contraction of kind (S) and $\psi : \mathcal{H} \to [0, \infty)$ be lower semi-continuous mapping in O-complete orthogonal metric space $(\mathcal{H}, \perp, \sigma)$. Then, h possess a unique ψ-fixed point.

Proof . With the assistance of Lemma 2.3, we get that $\{\Omega_n\}$ is a Cauchy O-sequence. But $(\mathcal{H}, \perp, \sigma)$ is O-complete, which indicates that there occur $\Omega \in \mathcal{H}$ such that

$$\lim_{n\to\infty} \Omega_n = \Omega. \tag{2.10}$$

With the assistance of Lemma 2.3(i) and lower semi-continuity property of ψ, we acquire

$$0 \leq \psi(\Omega) \leq \lim_{n\to\infty} inf \psi(\Omega_n) = 0,$$

which yields that

$$\psi(\Omega) = 0. \tag{2.11}$$

Eventually, we show that $h\Omega = \Omega$. Since, $h : \mathcal{H} \to \mathcal{H}$ be $\perp$-continuous, we get

$$\sigma(\Omega, h\Omega) = \lim_{n\to\infty} \sigma(\Omega_n, h\Omega_n) = \lim_{n\to\infty} \sigma(\Omega_n, \Omega_{n+1}) = 0. \tag{2.12}$$

Consequently,

$$\sigma(\Omega, h\Omega) = 0. \tag{2.13}$$

Equations (2.11) and (2.13) yields that Ω is ψ-fixed point of h. Now, we exhibit that ψ-fixed point of h is unique. Let us imagine that Ω_1, Ω_2 be distinct ψ-fixed points of h. Thus, $\sigma(h\Omega_1, h\Omega_2) = \sigma(\Omega_1, \Omega_2) > 0$.

If $\Omega_n \to \Omega_2$, when $n \to \infty$, we have $\Omega_1 = \Omega_2$. If $\Omega_n \nrightarrow \Omega_2$, when $n \to \infty$, then there is a subsequence $\{\Omega_{n_k}\}$ such that $h\Omega_{n_k} \neq \Omega_2$, for all $k \in \mathbb{N}$. By the choice of Ω_0 in the starting portion of proof of Lemma 2.3, we have

$$(\Omega_0 \perp \Omega_2) or (\Omega_2 \perp \Omega_0).$$

Since h is $\perp$-preserving and $h^n\Omega_2 = \Omega_2, \forall n \in \mathbb{N}$, we have

$$(h^n\Omega_0 \perp \Omega_2) or (\Omega_2 \perp h^n\Omega_0),$$

for all n $\in \mathbb{N}$. Consequently, $\Omega_1 \perp \Omega_2$. Now, inserting $\Omega = \Omega_1$ and $\mho = \Omega_2$ in (2.1), we acquire

$$\chi + \mathfrak{f}^*(\sigma(h\Omega_1, h\Omega_2), \psi(h\Omega_1), \psi(h\Omega_2)) \leq \mathfrak{f}^*(S(\Omega_1, \Omega_2), \psi(\Omega_1), \psi(\Omega_2)), \tag{2.14}$$

where

$$\begin{aligned} S(\Omega_1, \Omega_2) =& \max\{\sigma(\Omega_1, \Omega_2), \frac{\sigma(\Omega_1, h\Omega_2) + \sigma(\Omega_2, h\Omega_1)}{2}, \frac{\sigma(h^2\Omega_1, \Omega_1) + \sigma(h^2\Omega_1, h\Omega_2)}{2}, \\ & \sigma(h^2\Omega_1, h\Omega_1), \sigma(h^2\Omega_1, \Omega_2), \sigma(h^2\Omega_1, h\Omega_2) + \sigma(\Omega_1, h\Omega_1), \sigma(h\Omega_1, \Omega_2) + \sigma(\Omega_2, h\Omega_2)\} \\ =& \max\{\sigma(\Omega_1, \Omega_2), \frac{\sigma(\Omega_1, \Omega_2) + \sigma(\Omega_2, \Omega_1)}{2}, \frac{\sigma(\Omega_1, \Omega_1) + \sigma(\Omega_1, \Omega_2)}{2}, \\ & \sigma(\Omega_1, \Omega_1), \sigma(\Omega_1, \Omega_2), \sigma(\Omega_1, \Omega_2) + \sigma(\Omega_1, \Omega_1), \sigma(\Omega_1, \Omega_2) + \sigma(\Omega_2, \Omega_2)\} \\ =& \sigma(\Omega_1, \Omega_2). \end{aligned}$$

With the aid of (2.14), we acquire

$$\chi + \mathfrak{f}^*(\sigma(\Omega_1, \Omega_2), 0, 0) \leq \mathfrak{f}^*(\sigma(\Omega_1, \Omega_2), 0, 0),$$

which is a counterstatement. Thus, $\Omega_1 = \Omega_2$, which indicates that h possess a unique ψ-fixed point in $\mathcal{H}$. $\square$

Example 2.5. Consider $\mathcal{H} = [0, 6]$ associated with the usual metric σ. We formulize

$$\Omega \perp \mho \, if \, \Omega\mho \leq max\left\{\frac{\Omega}{2}, \frac{\mho}{2}\right\}.$$

Let $\Omega \perp \mho$ and $\Omega\mho \leq \frac{\Omega}{2}$. If $\{\Omega_k\}$ is an arbitrary Cauchy O-sequence in $\mathcal{H}$, then there exists a subsequence $\{\Omega_{k_n}\}$ of $\{\Omega_k\}$ for which $\Omega_{k_n} = 0$, for all n or there exists a subsequence $\{\Omega_{k_n}\}$ of $\{\Omega_k\}$ for which $\Omega_{k_n} = \frac{1}{2} \Rightarrow \{\Omega_{k_n}\}$ converges to a Ω. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent, which indicates that $\{\Omega_k\}$ is convergent. Now, it is clear that $(\mathcal{H}, \perp, \sigma)$ is O-complete. We define $h : \mathcal{H} \to \mathcal{H}$ by

$$h(\Omega) = \begin{cases} 0, & \text{if } 0 \leq \Omega < 5.5, \\ s\ln(\frac{\Omega}{9}), & 5.5 \leq \Omega < 6, \end{cases}$$

for all $\Omega \in \mathcal{H}$ and $s < 1$. Now, it is clear that h is not continuous at $\Omega = 5.5$. Three cases arise:

Case 1: If $\Omega, \mho \in [0, 5.5)$, then $h\Omega = 0$ and $h\mho = 0$. Thus,

$$h\Omega h\mho \leq \frac{h\Omega}{2},$$

holds trivially.

Case 2: If $\Omega, \mho \in [5.5, 6]$, then $h\Omega = s\ln(\frac{\Omega}{9})$ and $h\mho = s\ln(\frac{\mho}{9})$. Since $s < 1$, we have,

$$h\Omega h\mho \leq \frac{h\Omega}{2}.$$

Case 3: If $\Omega \in [5.5, 6]$ and $\mho \in [0, 5.5)$, then $h\Omega = s\ln(\frac{\Omega}{9})$ and $h\mho = 0$, which indicates that

$$h\Omega h\mho \leq \frac{h\Omega}{2}.$$

These cases implies that h is $\perp$-preserving. Let $\mathfrak{f}^* : (0, \infty) \times [0, \infty)^2 \to \mathbb{R}$ be defined as $\mathfrak{f}^*(e, f, g) = \ln(e + e^2 + f + g)$, for all $e, f, g \in [0, \infty)$ and $g \neq 0$. Let $\psi : \mathcal{H} \to [0, \infty)$ be defined as $\psi(\Omega) = \Omega$, for every $\Omega \in \mathcal{H}$. It is clear that ψ is lower semi-continuous and $\mathfrak{f}^* \in \mathcal{F}_{\mathcal{E}}$. Now, we assert that

$$\sigma(h\Omega, h\mho) + \psi(h\Omega) + \psi(h\mho) \leq e^{-\chi}(S(\Omega, \mho) + \psi(\Omega) + \psi(\mho)), \tag{2.15}$$

where

$$S(\Omega, \mho) = \max\{\sigma(\Omega, \mho), \frac{\sigma(\Omega, h\mho) + \sigma(\mho, h\Omega)}{2}, \frac{\sigma(h^2\Omega, \Omega) + \sigma(h^2\Omega, h\mho)}{2}, \sigma(h^2\Omega, h\Omega), \sigma(h^2\Omega, \mho), \sigma(h^2\Omega, h\mho) + \sigma(\Omega, h\Omega), \sigma(h\Omega, \mho) + \sigma(\mho, h\mho)\},$$

for all $\Omega, \mho \in \mathcal{H}$ and $h\mho \neq h\Omega$. Three cases arise:

Case 1: If $\Omega, \mho \in [0, 5.5)$, then (2.15) holds trivially.

Case 2: If $\Omega, \mho \in [5.5, 6]$, then

$$\begin{aligned} \sigma(h\Omega, h\mho) + \psi(h\Omega) + \psi(h\mho) &= 2\max\{h\Omega, h\mho\} \\ &\leq 2\max\{s\Omega, s\mho\} \\ &= s(2\max\{\Omega, \mho\}). \end{aligned} \tag{2.16}$$

Now,

$$\begin{aligned} S(\Omega, \mho) =& \max\{\sigma(\Omega, \mho), \frac{\sigma(\Omega, h\mho) + \sigma(\mho, h\Omega)}{2}, \frac{\sigma(h^2\Omega, \Omega) + \sigma(h^2\Omega, h\mho)}{2}, \sigma(h^2\Omega, h\Omega), \sigma(h^2\Omega, \mho), \sigma(h^2\Omega, h\mho) \\ &+ \sigma(\Omega, h\Omega), \sigma(h\Omega, \mho) + \sigma(\mho, h\mho)\} \\ =& 2\max\{\Omega, \mho\}. \end{aligned} \tag{2.17}$$

With the aid of (2.16) and (2.17), we acquire

$$\sigma(h\Omega, h\mho) + \psi(h\Omega) + \psi(h\mho) \leq s(\sigma(\Omega, \mho) + \psi(\Omega) + \psi(\mho)).$$

Case 3: If $\Omega \in [5.5, 6]$ and $\mho \in [0, 5.5)$, then

$$\begin{aligned}\sigma(h\Omega, h\mho) + \psi(h\Omega) + \psi(h\mho) &= 2\max\{h\Omega, h\mho\}\\ &= 2\max\{h\Omega, 0\}\\ &\leq 2\max\{s\Omega, 0\}\\ &= s(2\max\{\Omega, \mho\}).\end{aligned} \tag{2.18}$$

Now,

$$\begin{aligned}S(\Omega, \mho) =& \max\{\sigma(\Omega, \mho), \frac{\sigma(\Omega, 0) + \sigma(\mho, h\Omega)}{2}, \frac{\sigma(h^2\Omega, \Omega) + \sigma(h^2\Omega, 0)}{2}, \sigma(h^2\Omega, h\Omega), \sigma(h^2\Omega, \mho), \sigma(h^2\Omega, 0)\\ &+ \sigma(\Omega, h\Omega), \sigma(h\Omega, \mho) + \sigma(\mho, 0)\}\\ =&2\max\{\Omega, \mho\}.\end{aligned} \tag{2.19}$$

With the aid of (2.18) and (2.19), we acquire

$$\sigma(h\Omega, h\mho) + \psi(h\Omega) + \psi(h\mho) \leq s(\sigma(\Omega, \mho) + \psi(\Omega) + \psi(\mho)).$$

Since, h is generalized orthogonal $(\mathfrak{f}^*, \psi)$-contraction of kind (S), we have

$$|h(\Omega_n) - h(\Omega)| \leq \frac{1}{2}|\Omega_n - \Omega|. \tag{2.20}$$

Letting $n \to \infty$ in (2.20), we acquire h is $\perp$-continuous. But it can be easily seen that h is not continuous. Thus, h and ψ gratify all the conditions of Theorem 2.4 with $\chi = -\ln s > 0$. Consequently, h possess a unique ψ-fixed point, which is zero.

Corollary 2.6. Let $(\mathcal{H}, \sigma)$ be a complete metric space and $h : \mathcal{H} \to \mathcal{H}$ be a map gratifying

$$\sigma(h\Omega, h\mho) \leq \lambda \max\{\sigma(\Omega, \mho), \frac{\sigma(\Omega, h\mho) + \sigma(\mho, h\Omega)}{2}, \frac{\sigma(\Omega, h\Omega) + \sigma(\mho, h\mho)}{2}\},$$

for some $\lambda \in (0, 1)$ and $\Omega, \mho \in \mathcal{H}$. Then, h possess a unique fixed point.

Proof . If $\mathfrak{f}^*(e, f, g) = \ln(e + f + g)$ and $\psi(\Omega) = 0$, for all $\Omega \in \mathcal{H}$ in Theorem 2.4, we get the result. □

Corollary 2.7. (Banach Contraction Principle) Let $(\mathcal{H}, \sigma)$ be a complete metric space and $h : \mathcal{H} \to \mathcal{H}$ be a map gratifying

$$\sigma(h\Omega, h\mho) \leq \lambda\sigma(\Omega, \mho),$$

for some $\lambda \in (0, 1)$ and $\Omega, \mho \in \mathcal{H}$. Then, h possess a unique fixed point.

Proof . If $\mathfrak{f}^*(e, f, g) = \ln(e + f + g)$ and $\psi(\Omega) = 0 \,\forall\, \Omega \in \mathcal{H}$ in Theorem 2.4, we can deduce the result. □ Next, we define new notion of generalized $(\mathfrak{f}^*, \psi)$-expansion of kind (S) and our second main result.

Definition 2.8. Let $\psi : \mathcal{H} \to [0, \infty)$ be a map in orthogonal metric space $(\mathcal{H}, \perp, \sigma)$. A function $h : \mathcal{H} \to \mathcal{H}$ is called generalized orthogonal $(\mathfrak{f}^*, \psi)$-expansion of kind (S) if there occur $\mathfrak{f}^* \in \mathcal{F}_{\mathcal{E}}$ and $\chi > 0$ in order that

$$\Omega \perp \mho[\sigma(h\Omega, h\mho) > 0 \Rightarrow \mathfrak{f}^*(\sigma(h\Omega, h\mho), \psi(h\Omega), \psi(h\mho)) \geq \mathfrak{f}^*(S(\Omega, \mho), \psi(\Omega), \psi(\mho)) + \chi], \tag{2.21}$$

where

$$\begin{aligned}S(\Omega, \mho) = \max\{&\sigma(\Omega, \mho), \frac{\sigma(\Omega, h\mho) + \sigma(\mho, h\Omega)}{2}, \frac{\sigma(h^2\Omega, \Omega) + \sigma(h^2\Omega, h\mho)}{2},\\ &\sigma(h^2\Omega, h\Omega), \sigma(h^2\Omega, \mho), \sigma(h^2\Omega, h\mho) + \sigma(\Omega, h\Omega), \sigma(h\Omega, \mho) + \sigma(\mho, h\mho)\},\end{aligned}$$

for all $\Omega, \mho \in \mathcal{H}$.

Theorem 2.9. Let $h : \mathcal{H} \to \mathcal{H}$ be $\perp$-preserving and $\perp$-continuous onto generalized orthogonal $(\mathfrak{f}^*, \psi)$-expansion of kind (S) and $\psi : \mathcal{H} \to [0, \infty)$ be lower semi-continuous mapping having $F_{h^*} \subseteq K_\psi$ in complete $(\mathcal{H}, \perp, \sigma)$, where $h^* : \mathcal{H} \to \mathcal{H}$ in order that $h^* \circ h = I$, the identity function on $\mathcal{H}$. Then, h possess a unique ψ-fixed point.

Proof . Since h is surjective, there exists a function $h^* : \mathcal{H} \to \mathcal{H}$ in order that $h^* \circ h = I$, the identity function on $\mathcal{H}$. Let $\Omega \neq \mho$, $h^*\Omega = \lambda$ and $h^*\mho = \mu$. Now, it is clear that $\lambda \neq \mu$, otherwise $h\lambda = h\mu$, which indicates that $\sigma(\Omega, \mho) = 0$, which is a counterstatement. Thus, $\sigma(h\Omega, h\mho) > 0$. Due to $\perp$-preserving property of h, we acquire $\Omega \perp \mho$. Since, h is generalized $(\mathfrak{f}^*, \psi)$-expansion of kind (S), we acquire

$$\mathfrak{f}^*(\sigma(h\lambda, h\mu), \psi(h\lambda), \psi(h\mu)) \geq \mathfrak{f}^*(S(\lambda, \mu), \psi(\lambda), \psi(\mu)) + \chi, \tag{2.22}$$

where

$$\begin{aligned} S(\lambda, \mu) = \max\{&\sigma(\lambda, \mu), \frac{\sigma(\lambda, h\mu) + \sigma(\mu, h\lambda)}{2}, \frac{\sigma(h^2\lambda, \lambda) + \sigma(h^2\lambda, h\mu)}{2}, \\ &\sigma(h^2\lambda, h\lambda), \sigma(h^2\lambda, \mu), \sigma(h^2\lambda, h\mu) + \sigma(\lambda, h\lambda), \sigma(h\lambda, \mu) + \sigma(\mu, h\mu)\}, \end{aligned}$$

for all $\lambda, \mu \in \mathcal{H}$. Since, $h\lambda = \Omega$ and $h\mu = \mho$, the above inequality yields that

$$\chi + \mathfrak{f}^*(\sigma(h^*\Omega, h^*\mho), \psi(h^*\Omega), \psi(h^*\mho)) \leq \mathfrak{f}^*(S(\Omega, \mho), \psi(\Omega), \psi(\mho)),$$

where

$$\begin{aligned} S(\Omega, \mho) = \max\{&\sigma(\Omega, \mho), \frac{\sigma(\Omega, h\mho) + \sigma(\mho, h\Omega)}{2}, \frac{\sigma(h^2\Omega, \Omega) + \sigma(h^2\Omega, h\mho)}{2}, \\ &\sigma(h^2\Omega, h\Omega), \sigma(h^2\Omega, \mho), \sigma(h^2\Omega, h\mho) + \sigma(\Omega, h\Omega), \sigma(h\Omega, \mho) + \sigma(\mho, h\mho)\} \end{aligned}$$

which indicates that h^* is $(\mathfrak{f}^*, \psi)$-contraction. Therefore, by mimicking the steps of Theorem 2.4, we get that $\psi-$fixed point of h^* exists and unique. Let Ω_1 be unique fixed point of h^*. Thus, $h^*\Omega_1 = \Omega_1$ and $\psi(\Omega_1) = 0$. Also, $h(\Omega_1) = h(h^*\Omega_1) = \Omega_1$ and $\psi(\Omega_1) = 0$, consequently Ω_1 is also $\psi-$fixed point of h.
Now, we assert that h possess a unique $\psi-$fixed point.

From the definition of the orthogonality, it indicates that $\Omega_0 \perp h(\Omega_0)$ or $h(\Omega_0) \perp \Omega_0$, where $\Omega_0 \in \mathcal{H}$ be any point. Inserting

$$h^n\Omega_0 = \Omega_{n+1}, \tag{2.23}$$

for every $n \in \mathbb{N} \cup \{0\}$. Let us imagine that Ω_1, Ω_2 be distinct ψ-fixed points of h. Thus, $\sigma(h\Omega_1, h\Omega_2) = \sigma(\Omega_1, \Omega_2) > 0$. If $\Omega_n \to \Omega_2$, when $n \to \infty$, we have $\Omega_1 = \Omega_2$.

If $\Omega_n \nrightarrow \Omega_2$, when $n \to \infty$, then there is a subsequence $\{\Omega_{n_k}\}$ such that $h\Omega_{n_k} \neq \Omega_2, \forall\, k \in \mathbb{N}$. By the choice of Ω_0 in (2.23), we have

$$(\Omega_0 \perp \Omega_2) or (\Omega_2 \perp \Omega_0).$$

Since h is $\perp$-preserving and $h^n\Omega_2 = \Omega_2, \forall n \in \mathbb{N}$, we have

$$(h^n\Omega_0 \perp \Omega_2) or (\Omega_2 \perp h^n\Omega_0),$$

for all n $\in \mathbb{N}$. Consequently, $\Omega_1 \perp \Omega_2$. Since, h is $(\mathfrak{f}^*, \psi)$-expansive mapping, we acquire

$$\mathfrak{f}^*(\sigma(h\Omega_1, h\Omega_2), \psi(h\Omega_1), \psi(h\Omega_2)) \geq \mathfrak{f}^*(S(\Omega_1, \Omega_2), \psi(\Omega_1), \psi(\Omega_2)) + \chi, \tag{2.24}$$

where

$$\begin{aligned} S(\Omega_1, \Omega_2) = \max\{&\sigma(\Omega_1, \Omega_2), \frac{\sigma(\Omega_1, h\Omega_2) + \sigma(\Omega_2, h\Omega_1)}{2}, \frac{\sigma(h^2\Omega_1, \Omega_1) + \sigma(h^2\Omega_1, h\Omega_2)}{2}, \\ &\sigma(h^2\Omega_1, h\Omega_1), \sigma(h^2\Omega_1, \Omega_2), \sigma(h^2\Omega_1, h\Omega_2) + \sigma(\Omega_1, h\Omega_1), \sigma(h\Omega_1, \Omega_2) + \sigma(\Omega_2, h\Omega_2)\} = \sigma(\Omega_1, \Omega_2). \end{aligned}$$

With the aid of (2.24), we acquire

$$\mathfrak{f}^*(\sigma(\Omega_1, \Omega_2), \psi(\Omega_1), \psi(\Omega_2)) = \mathfrak{f}^*(\sigma(h\Omega_1, h\Omega_2), \psi(h\Omega_1), \psi(h\Omega_2)) \geq \mathfrak{f}^*(\sigma(\Omega_1, \Omega_2), \psi(\Omega_1), \psi(\Omega_2)) + \chi,$$

which is a counterstatement. Consequently, h possess a unique $\psi-$fixed point. □

Corollary 2.10. Let $h : \mathcal{H} \to \mathcal{H}$ be onto map and $\psi : \mathcal{H} \to [0, \infty)$ be lower semi-continuous mapping having $F_{h^*} \subseteq K_\psi$ in complete $(\mathcal{H}, \sigma)$, where $h^* : \mathcal{H} \to \mathcal{H}$ in order that $h^* \circ h = I$, the identity function on $\mathcal{H}$. If there occur $\mathfrak{f}^* \in \mathcal{F}_\mathcal{E}$ and $\chi > 0$ in order that:

$$h\Omega \neq h\mho \Rightarrow \mathfrak{f}^*(\sigma(h\Omega, h\mho), \psi(h\Omega), \psi(h\mho)) \geq \mathfrak{f}^*(\sigma(\Omega, \mho), \psi(\Omega), \psi(\mho)) + \chi, \tag{2.25}$$

for all $\Omega, \mho \in \mathcal{H}$. Then, h possess a unique ψ-fixed point.

Corollary 2.11. Let $h : \mathcal{H} \to \mathcal{H}$ be an onto mapping defined on a complete metric space $(\mathcal{H}, \sigma)$ gratifying the condition $\sigma(ha, hb) \geq c\sigma(a, b)$, for all $a, b \in \mathcal{H}$, where $c \geq 1$. Then, h possess a unique fixed point in $\mathcal{H}$.

Proof . Inserting $\mathfrak{f}^*(e, f, g) = \ln(e + f + g)$, for all $e, f, g \in [0, \infty)$ and $\psi(\Omega) = 0$, for all $\Omega \in \mathcal{H}$ in Theorem 2.9, we get the following result. □

Corollary 2.12. [3] Let $h : \mathcal{H} \to \mathcal{H}$ be an onto mapping, $\mathfrak{f}^* \in \Im$ and $\chi > 0$ in a complete metric space $(\mathcal{H}, \sigma)$ gratifying the condition

$$h\Omega \neq h\mho \Rightarrow \mathfrak{f}^*(\sigma(h\Omega, h\mho)) \geq \mathfrak{f}^*(S(\Omega, \mho)) + \chi,$$

where $\Im$ be family of all functions $F : (0, \infty) \to \mathcal{R}$ such that
(F1) F is strictly increasing, that is, for all $a, b \in (0, \infty)$, if $a < b$,then $F(a) < F(b)$.
(F2) For each sequence a_n of positive numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.
(F3) There exists $k \in (0, 1)$ such that $\lim_{a\to 0^+}(a^k F(a)) = 0$. Then, h possess a unique fixed point in $\mathcal{H}$.

Proof . Inserting $\psi(\Omega) = 0$ in Theorem 2.9, we can deduce the result. □

3 Conclusion

In this paper, ψ-fixed point results are investigated with the aid of generalized orthogonal $(\mathfrak{f}^*, \psi)$-contractive and expansive functions of kind (S) in the context of complete metric space. In this way, the relationship of orthogonal contractive and expansive functions of Wardowski kind with previous concept ψ-fixed point is investigated through indispensable theorems. Moreover, we established the results by substituting the continuity condition of $\mathfrak{f}^*$ by lower semi-continuity of ψ, for detail please see ([3], [5] and [7]). Additionally, an illustrative example and corollaries are provided to demonstrate the main results. Our results can be utilized to find solution of fractional non-linear differential and integral equations (see [2] and references therein).

Acknowledgment

The author is extremely grateful to the anonymous editor and reviewers for their insightful reading the manuscript and their valuable suggestions, which substantially improved the standard of the paper.

References

[1] N.V. Dung and V.L. Hang, *A fixed point theorem for generalized F-contractions on complete metric spaces*, Vietnam J. Math **43** (2015), 743–753.

[2] M.E. Gordji, M. Rameani, M. Sen and Y.J. Cho, *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory **18** (2017), 569–578.

[3] J. Gornicki, *Fixed point theorems for F-expanding mappings*, Fixed Point Theory Appl. **9** (2017), 1–10.

[4] M. Jleli, B. Samet and C. Vetro, *Fixed point theory in partial metric spaces via φ-fixed point concept in metric spaces*, J. Inequal. Appl. **426** (2014), no. 1, 1–9.

[5] M. Kumar and S. Arora, *Fixed point theorems for modified generalized F-contraction in G-metric spaces*, Bol. Soc. Paran. Mat (2019) 1-8 doi: 10.5269/bspm.45061.

[6] P.P. Murthy and K.V. Prasad, *Weak contraction condition involving cubic terms of $d(x, y)$ under the fixed point consideration*, J. Math. **2013** (2013), Article ID 967045, 1–5.

[7] H. Piri and P. Kumam, *Wardowski type fixed point theorems in complete metric spaces*, Fixed Point Theory Appl. **45** (2016), 1–12.

[8] T. Senapati, L.K. Dey, B. Damjanovic and A. Chanda, *New fixed point results in orthogonal metric spaces with an application*, Kragujevac J. Math. **42** (2018), no. 4, 505–516.

[9] P. Shahi, J. Kaur, and S.S. Bhatia, *Fixed point theorems for* (ξ, α)*-expansive mappings in complete metric spaces*, Fixed Point Theory Appl. **157** (2012) 1-12.

[10] A. Taheri and A.P. Farajzadeh, *A new generalization of* α*-type almost-F-contractions and* α *-type F-Suzuki contractions in metric spaces and their fixed point theorems*, Carpathian Math. Publ. **11** (2019), 475–492.

[11] Y. Touaila and D. Moutawakila, *Fixed point theorems on orthogonal complete metric spaces with an application*, Int. J. Nonlinear Anal. Appl. **12** (2021), 1801–1809.

[12] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **94** (2012), 1–11.

[13] D. Wardowski and N.V. Dung, *Fixed points of F-weak contractions on complete metric spaces*, Demonst. Math. **XLVII** (2014), 146–155.