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A new reproducing kernel method for solving the second order partial differential equation

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Abstract

In this study, a reproducing kernel Hilbert space method with the Chebyshev function is proposed for approximating solutions of a second-order linear partial differential equation under nonhomogeneous initial conditions. Based on reproducing kernel theory, reproducing kernel functions with a polynomial form will be erected in the reproducing kernel spaces spanned by the shifted Chebyshev polynomials. The exact solution is given by reproducing kernel functions in a series expansion form, the approximation solution is expressed by an n-term summation of reproducing kernel functions. This approximation converges to the exact solution of the partial differential equation when a sufficient number of terms are included. Convergence analysis of the proposed technique is theoretically investigated. This approach is successfully used for solving partial differential equations with nonhomogeneous boundary conditions.

Keywords: Reproducing kernel Hilbert space method, shifted Chebyshev polynomials, Convergence analysis, Second order linear partial differential equation
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1 Introduction

A reproducing kernel Hilbert space is a powerful tool for constructing approximate solutions of partial differential equations. Many analytical and numerical methods have been proposed for solving linear and nonlinear partial differential equations, but we did not find a method that use reproducing kernels for solving two-dimensional initial-boundary problems with orthogonal functions.

In recent years, there has been a growing interest to investigate scientific models, such as linear and nonlinear boundary value problem various [4, 3, 5] integro-differential equations [2], delay problem [11, 17, 13], linear operator equations [7, 12], fuzzy differential equations and others [9, 10]. Reproducing kernel methods has ability to solve different problems effectively and has relatively simple implementation. Since the reproducing kernel space is a Hilbert space, this paper will apply the theory of orthogonal function with two variables for linear partial differential equation with initial-boundary conditions the reproducing kernel space and derive same useful conclusions.

Boundary value problems play on important role in the study of problems in fluid mechanics, flow in porons media, heat conduction in solids, diffusive transport of chemicals in porons media and biological [13, 18]. The study of such

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problems has attracted much attention. As a result, it is of essential importance to develop on effective numerical algorithms for solving partial differential equation with initial-boundary conditions. So far, the numerical treatment of such problems has attracted much attention.

The aim of this paper is to introduce a numerical technique based on reproduction kernel Hilbert space methods with polynomial form in order to solve the partial differential equation. More precisely, we provide a numerical approximate solution for second order partial differential equation in the following form [15]:

$$\alpha\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^2 u}{\partial t\partial x}+\gamma\frac{\partial^2 u}{\partial t^2}+\delta\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial t}+\theta u=G(x,t),\ (x,t)\in[0,1]\times[0,1], \tag{1.1}$$

subject to the initial conditions for variable t:

$$\begin{cases} u(x,0)=f(x), & x\in[0,1],\\ \frac{\partial u(x,0)}{\partial t}=m(x), & x\in[0,1], \end{cases} \tag{1.2}$$

together with the initial conditions for variable x:

$$\begin{cases} u(0,t)=h(t), & t\in(0,1],\\ \frac{\partial u(0,t)}{\partial x}=k(t), & t\in(0,1], \end{cases} \tag{1.3}$$

where $G(x,t), f(x), m(x), h(x)$ and $k(x)$ are conditions functions, and $\alpha,\beta,\gamma,\delta,\eta$ and θ are real numbers. Although the focus is on homogeneous mixed boundary conditions by the homogenization methods. In this paper, employing the reproducing property of the kernel, we give on efficient method for solving (1.1).

The rest of this paper is organized as follows. In Section 2, an overview of two dimensional shifted Chebyshev polynomials and their relevant properties required henceforward are presented. In Section 3, we will recall a brief review of the reproducing kernel spaces and establish an orthogonal basis in the two dimensional shifted Chebyshev reproducing kernel space. In Section 4, our method to approximate the solution of second order partial differential equation with shifted Chebyshev reproducing kernel basis function is considered. The convergence analysis and error estimation are presented in this section. In Section 5, some numerical results are provided to demonstrate the efficiency and accuracy of using the reproducing kernel Hilbert space method in comparison with of the results presented in [1, 15, 19].

2 Properties of Chebyshev polynomials

In this section, some preliminaries and notations of Chebyshev polynomials which are necessary for later are recalled. Let $T_n(x), x\in[-1,1]$ be the standard Chebyshev polynomial of degree n. For positive weight function $w(t)=\frac{1}{\pi\sqrt{1-(2t-1)^2}}$, we define the shifted Chebyshev polynomials $T_n^*(t)$ by

$$T_n^*(t)=T_n(2t-1),\ t\in[0,1], n=0,1,2,\dots . \tag{2.1}$$

In particular,

$$T_{n+1}^*(t)=2(2t-1)T_n^*(t)-T_{n-1}^*(t),\ n\geq 1, \tag{2.2}$$

where $T_0^*(t)=1$ and $T_1^*(t)=2t-1$. The set of $T_n^*(t)$ forms a complete $L_w^2(0,1)$ orthogonal system, where $c_0=2$, $c_n=1$ for $n\geq 1$ and $\delta_{n,m}$ is the Kroncker symbol.

$2D$ shifted Chebyshev polynomials are defined on $\Omega=[0,1]\times[0,1]$ as follows:

$$C_{i,j}(x,y)=T_i^*(x)T_j^*(y),\ i,j=0,1,2,\dots .$$

We consider the space $L_w^2(\Omega)$ equipped with the following inner product and norms

$$\langle f(x,y),g(x,y)\rangle=\int_0^1\int_0^1 f(x,y)g(x,y)W(x,y)dxdy,$$

$$\|f(x,y)\|=\langle f(x,y),f(x,y)\rangle^{\frac{1}{2}}=(\int_0^1\int_0^1|f(x,y)|^2dxdy)^{\frac{1}{2}},$$

where $w(x,y) = w(x)w(y)$. The set of two dimensional shifted Chebyshev polynomials forms a complete $L^2_w(\Omega)-$orthogonal system such that the orthogonality conditions

$$\int_0^1 \int_0^1 C_{i,j}(x,y)C_{k,l}(x,y)w(x,y)dxdy = \begin{cases} 1, & i=j=k=l=0, \\ \frac{1}{4}, & i=k\neq 0, j=l\neq 0, \\ \frac{1}{2}, & i=k=0, j=l\neq 0, \\ 0, & \text{else} \quad i,j,k,l. \end{cases} \tag{2.3}$$

3 Construction of reproducing kernel

In the section, we discuss reproducing kernel on two set of nodes in two dimensions and we obtain reproducing kernel space by re-defining the inner product of Chebyshev polynomials. We now present some necessary definitions on Theorems in the theory of reproducing kernel spaces.

Definition 3.1. A Hilbert space H of functions defined on $\Omega \subseteq \mathcal{R}^2$, is called a reproducing kernel Hilbert space if there exists a reproducing kernel K of H such that verifies the following conditions

(i) $K(.,z) \in H$ for each fixed $z \in \Omega$.

(ii) $\langle \varphi, K(.,z)\rangle = \varphi(z)$ for all $z \in \Omega$ and all $\varphi \in H$.

It is known that in the Hilbert space H are stated the following results.

Theorem 3.2. Let H be $n-$dimensional Hilbert space, $\{w_i\}_{i=1}^n$ is an orthogonal basis of H, then the reproducing kernel of H as:

$$K_n(x,y) = \sum_{j=0}^{n} w_j(x)w_j(y), \;\; x,y \in [0,1]. \tag{3.1}$$

Theorem 3.3. ([14] Theorem 1.24) For the orthogonal system $\{w_n\}_{n=1}^{\infty}$, formula (3.1) yields the Christoffel-Darboux formula:

$$K_n(x,y) = \frac{k_n(w_{n+1}(x)w_n(y) - w_n(x)w_{n+1}(y))}{k_{n+1}(x-y)}. \tag{3.2}$$

Here, $k_n > 0$ is the coefficient of x^n in $w_n(x)$. we also have

$$K_n(x,x) = \frac{k_n}{k_{n+1}}(w'_{n+1}(x)w_n(x) - w'_n(x)w_{n+1}(x)). \tag{3.3}$$

To derive on explicit formula for the reproducing kernel formula, we will use orthogonal polynomials and follow the strategy in [4, 5]. Now, we construct similarity reproducing kernels of equations (3.2) and (3.3) on two set of nodes in two dimensions. Let P^2_n denote the space of Chebyshev polynomials of degree at most n with respect to weight function $w(x,t)$ in two variables on $\Omega = [0,1]\times[0,1]$, that is

$$P^2_n(\Omega) = Span\{p^n_k(x,y) = \widehat{T_{n-k}}(x)\widehat{T_k}(y), 0 \leq k \leq n\}, \tag{3.4}$$

where $\widehat{T_0}(x) = 1, \widehat{T_k}(x) = \sqrt{2}T^*_k(x)$ for $k \geq 1$. we denote by $\boldsymbol{\mathcal{P}}_n$ the set of this basis and we also regard $\boldsymbol{\mathcal{P}}_n$ as a column vector

$$\boldsymbol{\mathcal{P}}_n = [p^n_0, p^n_1, ..., p^n_n]^t, \tag{3.5}$$

where the superscript t denotes the transposes. The reproducing kernel of the space P^2_n in $L^2_w([0,1]^2)$ is defined by [6],

$$K_n(x,y) = \sum_{k=0}^{n} \boldsymbol{\mathcal{P}}_k(x)[\boldsymbol{\mathcal{P}}_k(y)]^t = \sum_{k=0}^{n}\sum_{j=0}^{k} p^k_j(x)p^k_j(y). \tag{3.6}$$

Theorem 3.4. There is a Christoffel-Darboux formula (cf. [16]) which states that

$$K_n(x,y) = \frac{[A_{n,i}\boldsymbol{\mathcal{P}}_{n+1}(x)]^t\boldsymbol{\mathcal{P}}_n(y) - A_{n,i}\boldsymbol{\mathcal{P}}_{n+1}(y)]^t\boldsymbol{\mathcal{P}}_n(x)}{x_i - y_i},\ i = 1,2, \tag{3.7}$$

for $x \neq y$ and

$$K_n(x,x) = \boldsymbol{\mathcal{P}}_n^T(x)A_{n,i}\partial_i\boldsymbol{\mathcal{P}}_{n+1}(x) - [A_{n,i}\boldsymbol{\mathcal{P}}_n(x)]^T\boldsymbol{\mathcal{P}}_{n+1}(x),\ i = 1,2, \tag{3.8}$$

where $x = (x_1, x_2), y = (y_1, y_2)$, and $A_{n,i}$ are matrices defined by

$$\mathbf{A_{n,1}} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \sqrt{2} & 0 \end{bmatrix}, \qquad \mathbf{A_{n,2}} = \frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \dots & 1 \end{bmatrix}.$$

The proof follows just as in the case of on variable. Recall that the right hand side depends on i, but the left hand side is independent of i. It is straightforward to check that the kernel $K_n(x,y)$ has the reproducing property

$$\langle p_k^n, K_n(.;x,y)\rangle = p_k^n(x,y),$$

for all polynomials $p_n^k \in P_n^2(\Omega)$.

4 Description of the method

In this section, we give the solution (1.1)-(1.3) in the reproducing kernel space $P_n^2(\Omega)$. we define the linear operator $L : P_n^2([0,1]) \longrightarrow L_w^2([0,1]) \otimes L_w^2([0,1])$ as

$$Lv = \alpha\frac{\partial^2 v}{\partial x^2}(x,t) + \beta\frac{\partial^2 v}{\partial t\partial x}(x,t) + \gamma\frac{\partial^2 v}{\partial t^2}(x,t) + \delta\frac{\partial v}{\partial x}(x,t) + \eta\frac{\partial v}{\partial t}(x,t) + \theta v(x,t),\ v \in P_n^2(\Omega).$$

In order to put initial boundary value conditions of equations (1.2) on (1.3) into the reproducing kernel space $P_n^2(\Omega)$ constructed in the following part, it is must to homogenize the initial conditions. Put

$$v(x,t) = u(x,t) - \mathcal{B}(x,t)f(x) - \mathcal{C}(x,t)m(x) - \mathcal{B}(t,x)h(t) - \mathcal{C}(t,x)m(x)k(t),$$

where

$$\mathcal{B}(x,t) = \begin{cases} e^{-\frac{t^2}{x}}, & 0 < x < 1, \\ 0, & x = 0,1, \end{cases}$$

and

$$\mathcal{C}(x,t) = \begin{cases} te^{-\frac{t^2}{x}}, & 0 < x < 1, \\ 0, & x = 0,1. \end{cases}$$

Denote $\mathcal{B}(x,t)f(x)+\mathcal{C}(x,t)m(x)+\mathcal{B}(t,x)h(t)+\mathcal{C}(t,x)m(x)k(t)$ by $A(x,t)$, then we can obtain homogeneous initial conditions of equations (1.1)-(1.3). Immediately, we get

$$\begin{cases} Lv(x,t) = F(x,t), & (x,t) \in \Omega = [0,1] \times [0,1], \\ v(x,0) = \frac{\partial v}{\partial t}(x,0) = 0, & x \in [0,1], \\ v(0,t) = \frac{\partial v}{\partial x}(0,t) = 0, & t \in (0,1], \end{cases} \tag{4.1}$$

where

$$F(x,t) = G(x,t) + \alpha\frac{\partial^2 A}{\partial x^2}(x,t) + \beta\frac{\partial^2 A}{\partial t\partial x}(x,t) + \gamma\frac{\partial^2 A}{\partial t^2}(x,t) + \delta\frac{\partial A}{\partial x}(x,t) + \eta\frac{\partial A}{\partial t}(x,t) + \theta A(x,t).$$

Theorem 4.1. the operator $L : P_n^2([0,1]) \longrightarrow L_w^2([0,1]) \otimes L_w^2([0,1])$ is a bounded operator.

Proof . Note that

$$\begin{aligned}\|(Lv)(x,t)\|^2 &= \|\alpha v_{xx} + \beta v_{xt} + \gamma v_{tt} + \delta v_x + \eta v_t + \theta v\|^2 \\ &\leq \alpha^2\|v_{xx}\|^2 + \beta^2\|v_{xt}\|^2 + \gamma^2\|v_{tt}\|^2 + \delta^2\|v_x\|^2 + \eta^2\|v_t\|^2 + \theta^2\|v\|^2.\end{aligned}$$

Since

$$v(x,t) = \langle v(y,s), K_{n,(x,t)}(y,s)\rangle_{P_n^2},$$

for $i = 0, 1,$

$$\begin{aligned}\frac{\partial^i}{\partial x^i}v(x,t) &= \langle v(y,s), \frac{\partial^i}{\partial x^i}K_{n,(x,t)}(y,s)\rangle_{P_n^2}, \\ \frac{\partial^i}{\partial t^i}v(x,t) &= \langle v(y,s), \frac{\partial^i}{\partial t^i}K_{n,(x,t)}(y,s)\rangle_{P_n^2}, \\ \frac{\partial}{\partial t}\frac{\partial}{\partial x}v(x,t) &= \langle v(y,s), \frac{\partial}{\partial t}\frac{\partial}{\partial x}K_{n,(x,t)}(y,s)\rangle_{P_n^2}.\end{aligned}$$

Also, note that

$$\|K_{n,(x,t)}(y,s)\| = \sqrt{\langle K_{n,(x,t)}(y,s), K_{n,(x,t)}(y,s)\rangle} = \sqrt{K_{n,(x,t)}(x,t)},$$

is continuous function on the interval $[0,1]$; that is, it holds that $\|K_{n,(x,t)}(y,s)\| \leq M_0$. Meanwhile, setting

$$\begin{aligned}&\|\frac{\partial^i}{\partial x^i}K_{n,(x,t)}(y,s)\| \leq M_i, i = 1,2, \\ &\|\frac{\partial^j}{\partial t^j}K_{n,(x,t)}(y,s)\| \leq N_j, j = 1,2, \\ &\|\frac{\partial}{\partial t}\frac{\partial}{\partial x}K_{n,(x,t)}(y,s)\| \leq M_3,\end{aligned}$$

we have

$$\begin{aligned}|\frac{\partial^i}{\partial x^i}v(x,t)| &\leq \|v\|\|\frac{\partial^i}{\partial x^i}K_{n,(x,t)}(y,s)\| \leq M_i\|v\|, i = 0,1,2, \\ |\frac{\partial^i}{\partial t^i}v(x,t)| &\leq \|v\|\|\frac{\partial^i}{\partial t^i}K_{n,(x,t)}(y,s)\| \leq N_i\|v\|, i = 0,1,2, \\ |\frac{\partial}{\partial t}\frac{\partial}{\partial x}v(x,t)| &\leq \|v\|\|\frac{\partial}{\partial t}\frac{\partial}{\partial x}K_{n,(x,t)}(y,s)\| \leq M_3\|v\|.\end{aligned}$$

Hence

$$\|(Lv)(x,t)\|^2 \leq (\alpha^2 M_2^2 + \beta^2 M_3^2 + \gamma^2 N_2^2 + \delta^2 M_1^2 + \eta^2 N_1^2 + \theta^2)M_0^2.$$

The proof is complete. □

Now, choose a countable dense subset $\{(x_1,t_1),(x_2,t_2),...\}$ in Ω and define

$$\psi_i(x,t) = L_{(y,s)}K_n(x,t,y,s)|_{(y,s)=(x_i,t_i)},$$

where the subscript (y,s) in the operator L indicates that the operator L applies to the functions y, s. considering the boundary conditions, we skillfully construct $\varphi_{1i}(x,t)$, $\varphi_{2i}(x,t)$, $\varphi_{3i}(x,t)$ and $\varphi_{4i}(x,t)$, $i = 1,2,...$ as follows

$$\begin{aligned}&\varphi_{1i}(x,t) = K_n(x,t,x_i,0), \varphi_{2i}(x,t) = \frac{\partial}{\partial s}K_n(x,t,x_i,s)|_{s=0}, \quad i = 1,2,..., \\ &\varphi_{3i}(x,t) = K_n(x,t,0,t_i), \varphi_{4i}(x,t) = \frac{\partial}{\partial y}K_n(x,t,y,t_i)|_{y=0}, \quad i = 1,2,... .\end{aligned} \tag{4.2}$$

Theorem 4.2. For each fixed n, $\left\{\{\psi_i\}_{i=1}^n, \{\varphi_{ji}\}_{(j,i)=(1,1)}^{(4,n)}\right\}$ are linearly independent in $P_n^2(\Omega)$.

Proof . Assume $\sum_{i=1}^n \lambda_i\psi_i + \sum_{j=1}^4\sum_{i=1}^n r_{ji}\varphi_{ji} = 0$. For arbitrary $l \in \mathcal{N}$, let Lagrange polynomials be defined by

$$f_l(x) = \frac{\prod_{i=1,i\neq l}^n (x-x_i)}{\prod_{i=1,i\neq l}^n (x_l-x_i)},$$

and let $f_l(x,t) = f_l(x).f_l(t)$, then there exists $V_l \in P_n^2(\Omega)$, $\quad (l=1,2,...,n)$, such that

$$\left\{\begin{array}{ll} LV_l(x,t) = f_l(x,t), & (x,t)\in\Omega=[0,1]\times[0,1], \\ V_l(x,0) = \frac{\partial}{\partial t}V_l(x,0) = 0, & x\in[0,1], \\ V_l(0,t) = \frac{\partial}{\partial x}V_l(0,t) = 0, & t\in(0,1], \end{array}\right. \tag{4.3}$$

then we have

$$\begin{aligned}
0 &= \langle V_l(x,t), \sum_{i=1}^n \lambda_i\psi_i + \sum_{j=1}^4\sum_{i=1}^n r_{ji}\varphi_{ji}\rangle_{P_n^2} \\
&= \sum_{i=1}^n \lambda_i\langle V_l,\psi_i\rangle_{P_n^2} + \sum_{i=1}^n r_{1i}\langle V_l,\varphi_{1i}\rangle_{P_n^2} + \sum_{i=1}^n r_{2i}\langle V_l,\varphi_{2i}\rangle_{P_n^2} + \sum_{i=1}^n r_{3i}\langle V_l,\varphi_{3i}\rangle_{P_n^2} \\
&+ \sum_{i=1}^n r_{4i}\langle V_l,\varphi_{4i}\rangle_{P_n^2} \\
&= \sum_{i=1}^n \lambda_i\langle V_l, L_{(y,s)}K_n(x,t,y,s)|_{(y,s)=(x_i,t_i)}\rangle_{P_n^2} + \sum_{i=1}^n r_{1i}\langle V_l, K_n(x,t,x_i,0)\rangle_{P_n^2} \\
&+ \sum_{i=1}^n r_{2i}\langle V_l, \frac{\partial}{\partial s}K_n(x,t,x_i,s)|_{s=0}\rangle_{P_n^2} + \sum_{i=1}^n r_{3i}\langle V_l, K_n(x,t,0,t_i)\rangle_{P_n^2} \\
&+ \sum_{i=1}^n r_{4i}\langle V_l, \frac{\partial}{\partial y}K_n(x,t,y,t_i)|_{y=0}\rangle_{P_n^2} \\
&= \sum_{i=1}^n \lambda_i L_{(y,s)}\langle V_l, K_n(x,t,y,s)|_{(y,s)=(x_i,t_i)}\rangle_{P_n^2} + \sum_{i=1}^n r_{1i}V_l(x_i,0) \\
&+ \sum_{i=1}^n r_{2i}\frac{\partial}{\partial s}\langle V_l, K_n(x,t,x_i,s)|_{s=0}\rangle_{P_n^2} + \sum_{i=1}^n r_{3i}V_l(0,t_i) \\
&+ \sum_{i=1}^n r_{4i}\frac{\partial}{\partial y}\langle V_l, K_n(x,t,y,t_i)|_{y=0}\rangle_{P_n^2} \\
&= \sum_{i=1}^n \lambda_i L_{(y,s)}V_l(y,s)|_{(y,s)=(x_i,t_i)} + \sum_{i=1}^n r_{1i}V_l(x_i,0) + \sum_{i=1}^n r_{2i}\frac{\partial}{\partial s}V_l(x_i,0) \\
&+ \sum_{i=1}^n r_{3i}V_l(0,t_i) + \sum_{i=1}^n r_{4i}\frac{\partial}{\partial y}V_l(0,t_i)|_{y=0} \\
&= \sum_{i=1}^n \lambda_i f_l(x_i,t_i) + 0 \\
&= \lambda_l.
\end{aligned}$$

In the same manner, there exist functions $W_{1l}(x,t), W_{2l}(x,t), W_{3l}(x,t), W_{4l}(x,t) \in P_n^2(\Omega)$. and Lagrange polynomials $f_{1l}(x), f_{2l}(x), f_{3l}(t), f_{4l}(t)$, satisfying

$$\left\{\begin{array}{l} LW_{1l}(x,t) = 0, \\ W_{1l}(x,0) = f_{1l}(x), \frac{\partial}{\partial t}W_{1l}(x,0) = 0, \\ W_{1l}(0,t) = 0, \frac{\partial}{\partial x}W_{1l}(0,t) = 0, \end{array}\right. \qquad \left\{\begin{array}{l} LW_{2l}(x,t) = 0, \\ W_{2l}(x,0) = 0, \frac{\partial}{\partial t}W_{2l}(x,0) = f_{2l}(x), \\ W_{2l}(0,t) = 0, \frac{\partial}{\partial x}W_{2l}(0,t) = 0, \end{array}\right.$$

$$\begin{cases} LW_{3l}(x,t)=0, \\ W_{3l}(x,0)=0, \frac{\partial}{\partial t}W_{3l}(x,0)=0, \\ W_{3l}(0,t)=f_{3l}(t), \frac{\partial}{\partial x}W_{3l}(0,t)=0, \end{cases}$$

and

$$\begin{cases} LW_{4l}(x,t)=0, \\ W_{3l}(x,0)=0, \frac{\partial}{\partial t}W_{4l}(x,0)=0, \\ W_{4l}(0,t)=0, \frac{\partial}{\partial x}W_{3l}(0,t)=f_{4l}(t), \end{cases}$$

where

$$\begin{aligned} 0 &= \langle W_{1l}(x,t), \sum_{i=1}^{n}\lambda_i\psi_i + \sum_{j=1}^{4}\sum_{i=1}^{n} r_{ji}\varphi_{ji}\rangle_{P_n^2} \\ &= \sum_{i=1}^{n}\lambda_i L_{(y,s)}W_{1l}(y,s)|_{(y,s)=(x_i,t_i)} + \sum_{i=1}^{n} r_{1i}W_{1l}(x_i,0) + \sum_{i=1}^{n} r_{2i}\frac{\partial}{\partial s}W_{1l}(x_i,0) + \sum_{i=1}^{n} r_{3i}W_{1l}(0,t_i) + \sum_{i=1}^{n} r_{4i}\frac{\partial}{\partial y}W_{1l}(0,t_i) \\ &= 0 + \sum_{i=1}^{n} r_{1i}f_{1l}(x_i) \\ &= r_{1l}. \end{aligned}$$

Similarly, we have $r_{2l}=r_{3l}=r_{4l}=0$. Namely,

$$\lambda_l = r_{1l} = r_{2l} = r_{3l} = r_{4l} = 0, \quad l = 1,2,...,n.$$

This ends the proof. □

Let $\mathcal{S}_{4n} = span\left\{\{\psi_i\}_{i=1}^n, \{\varphi_{ji}\}_{(j,i)=(1,1)}^{(4,n)}\right\}$. next, we are going to look for an approximate solution of (4.1) in the subspace $\mathcal{S}_{4n}$. let $\mathcal{R}_{4n}$ denote the orthogonal projection from $P_n^2(\Omega)$ onto $\mathcal{S}_{4n}$, *i.e.* for any $v \in P_n^2(\Omega)$, we have

$$\langle v - \mathcal{R}_{4n}v\rangle_{P_n^2} = 0, \forall w \in \mathcal{S}_{4n}.$$

Now, in the following, we investigate the property of approximate solution v_n to equation (4.1).

Theorem 4.3. If $v \in P_n^2(\Omega)$ is the solution (4.1), then $v_n = \mathcal{R}_{4n}v$ satisfies

$$\begin{cases} \langle v_n, \psi\rangle = F(x_i,t_i), \\ \langle v_n, \varphi_{1i}\rangle = \langle v_n, \varphi_{2i}\rangle = 0, \qquad i = 1,2,\dots. \\ \langle v_n, \varphi_{3i}\rangle = \langle v_n, \varphi_{4i}\rangle = 0. \end{cases} \tag{4.4}$$

Proof . In virtue of the self-conjugation of the operator $\mathcal{R}_n$ and the properties of the reproducing kernel, it can be obtained that

$$\begin{aligned} \langle \mathcal{R}_{4n}v, \psi_i\rangle_{P_n^2} &= \langle v, \mathcal{R}_{4n}\psi_i\rangle_{P_n^2} = \langle v, \psi_i\rangle_{P_n^2} \\ &= \langle v, L_{(y,s)}K_n(x,t,y,s)|_{(y,s)=(x_i,t_i)}\rangle_{P_n^2} \\ &= L_{(y,s)}\langle v, K_n(x,t,y,s)\rangle_{P_n^2}|_{(y,s)=(x_i,t_i)} \\ &= L_{(y,s)}v(y,s)|_{(y,s)=(x_i,t_i)} \\ &= F(x_i,t_i), \qquad i=1,2,...,n, \\ \langle \mathcal{R}_{4n}v, \varphi_{1i}\rangle_{P_n^2} &= \langle v, \mathcal{R}_{4n}\varphi_{1i}\rangle_{P_n^2} = \langle v, \varphi_{1i}\rangle_{P_n^2} \\ &= \langle v, K_n(x,t,x_i,0)\rangle_{P_n^2} \\ &= v(x_i,0) = 0, \\ \langle \mathcal{R}_{4n}v, \varphi_{2i}\rangle_{P_n^2} &= \langle v, \mathcal{R}_{4n}\varphi_{2i}\rangle_{P_n^2} = \langle v, \varphi_{2i}\rangle_{P_n^2} \\ &= \langle v, \frac{\partial}{\partial s}K_n(x,t,x_i,s)|_{s=0}\rangle_{P_n^2} \\ &= \frac{\partial}{\partial s}\langle v, K_n(x,t,x_i,s)\rangle_{P_n^2}|_{s=0} \\ &= \frac{\partial}{\partial s}v(x_i,0) = 0, \end{aligned}$$

Similarly, we have

$$\langle \mathcal{R}_{4n}v, \varphi_{3i}\rangle_{P_n^2} = 0, \langle \mathcal{R}_{4n}v, \varphi_{4i}\rangle_{P_n^2} = 0, i = 1, 2, ..., n.$$

It can be shown that $v_n = \mathcal{R}_{4n}v$ is an approximate solution of v. □ Furthermore, we can prove the uniform convergence.

Theorem 4.4. $v_n(x,t)$ is the approximate solution of equation (4.1), and $v_n(x,t)$ converges to $v(x)$ on Ω uniformly. Moreover, $\frac{\partial^2}{\partial t\partial x}v_n(x,t)$, $\frac{\partial^i}{\partial x_i}v_n(x,t)$, $\frac{\partial^i}{\partial t_i}v_n(x,t)$ uniformly convergence to $\frac{\partial^2}{\partial t\partial x}v(x,t)$, $\frac{\partial^i}{\partial x_i}v(x,t)$, $\frac{\partial^i}{\partial t_i}v(x,t)$ on Ω for $i = 0, 1, 2$, respectively.

Proof . obviously, $\|v_n - v\|_{P_n^2} \longrightarrow 0$ holds as $n \longrightarrow \infty$. that is, $v_n(x,t)$ is the approximate solution of equation (4.1). Besides, from inequality

$$\begin{aligned} |\frac{\partial^i}{\partial x_i}v_n(x,t) - \frac{\partial^i}{\partial x_i}v(x,t)| &= |\frac{\partial^i}{\partial x_i}\langle v_n(.,.) - v(.,.), K_{n,(x,t)}(.,.)\rangle| \\ &= |\langle v_n(.,.) - v(.,.), \frac{\partial^i}{\partial x_i}K_{n,(x,t)}(.,.)\rangle| \\ &\le \|v_n - v\|_{P_n^2}\|\frac{\partial^i}{\partial x_i}K_{n,(x,t)}\|_{P_n^2}, \end{aligned}$$

Since $\|\frac{\partial^i}{\partial x_i}K_{n,(x,t)}\|$ is bounded on $[0,1]$, we have

$$|\frac{\partial^i}{\partial x_i}v_n(x,t) - \frac{\partial^i}{\partial x_i}v(x,t)| \le M\|v_n - v\|_{P_n^2} \longrightarrow 0,$$

where M is a positive real number. it follows the $v_n(x,t)$ converges uniformly to $v(x,t)$ on $[0,1]$. similarly, one can prove $\frac{\partial^i}{\partial t_i}v_n(x,t)$ and $\frac{\partial^2}{\partial t\partial x}v_n(x,t)$ uniformly convergence to $\frac{\partial^i}{\partial t_i}v(x,t)$ and $\frac{\partial^2}{\partial t\partial x}v(x,t)$ on $[0,1]$, $\quad i = 1, 2$. The prove is completed. □

Hence, v_n is a good approximate solution of (4.1). since $v_n \in \mathcal{S}_{4n}$, we get

$$v_n = \sum_{j=1}^{n}\alpha_j\psi_j + \sum_{k=1}^{4}\sum_{l=1}^{n}\beta_{kl}\varphi_{kl}. \tag{4.5}$$

As v_n is the solution of equation (4.4), we have

$$\begin{cases} \sum_{j=1}^n \alpha_j\langle\psi_j,\psi_i\rangle + \sum_{k=1}^4\sum_{l=1}^n \beta_{kl}\langle\varphi_{kl},\psi_i\rangle = F(x_i,t_i), \\ \sum_{j=1}^n \alpha_j\langle\psi_j,\varphi_{1i}\rangle + \sum_{k=1}^4\sum_{l=1}^n \beta_{kl}\langle\varphi_{kl},\varphi_{1i}\rangle = 0, \\ \sum_{j=1}^n \alpha_j\langle\psi_j,\varphi_{2i}\rangle + \sum_{k=1}^4\sum_{l=1}^n \beta_{kl}\langle\varphi_{kl},\varphi_{2i}\rangle = 0, \qquad i = 1, 2, ..., n, \\ \sum_{j=1}^n \alpha_j\langle\psi_j,\varphi_{3i}\rangle + \sum_{k=1}^4\sum_{l=1}^n \beta_{kl}\langle\varphi_{kl},\varphi_{3i}\rangle = 0, \\ \sum_{j=1}^n \alpha_j\langle\psi_j,\varphi_{4i}\rangle + \sum_{k=1}^4\sum_{l=1}^n \beta_{kl}\langle\varphi_{kl},\varphi_{4i}\rangle = 0. \end{cases} \tag{4.6}$$

To obtain the approximate solution v_n, we only need to obtain the coefficients of each ψ_j and φ_{kl}. Let

$$\mathbf{G} = \begin{bmatrix} \langle\psi_j,\psi_i\rangle_{n\times n} & \langle\varphi_{1j},\psi_i\rangle_{n\times n} & \langle\varphi_{2j},\psi_i\rangle_{n\times n} & \langle\varphi_{3j},\psi_i\rangle_{n\times n} & \langle\varphi_{4j},\psi_i\rangle_{n\times n} \\ \langle\psi_j,\varphi_{1j}\rangle_{n\times n} & \langle\varphi_{1j},\varphi_{1j}\rangle_{n\times n} & \langle\varphi_{2j},\varphi_{1j}\rangle_{n\times n} & \langle\varphi_{3j},\varphi_{1j}\rangle_{n\times n} & \langle\varphi_{4j},\varphi_{1j}\rangle_{n\times n} \\ \langle\psi_j,\varphi_{2j}\rangle_{n\times n} & \langle\varphi_{1j},\varphi_{2j}\rangle_{n\times n} & \langle\varphi_{2j},\varphi_{2j}\rangle_{n\times n} & \langle\varphi_{3j},\varphi_{2j}\rangle_{n\times n} & \langle\varphi_{4j},\varphi_{2j}\rangle_{n\times n} \\ \langle\psi_j,\varphi_{3j}\rangle_{n\times n} & \langle\varphi_{1j},\varphi_{3j}\rangle_{n\times n} & \langle\varphi_{2j},\varphi_{3j}\rangle_{n\times n} & \langle\varphi_{3j},\varphi_{3j}\rangle_{n\times n} & \langle\varphi_{4j},\varphi_{3j}\rangle_{n\times n} \\ \langle\psi_j,\varphi_{4j}\rangle_{n\times n} & \langle\varphi_{1j},\varphi_{4j}\rangle_{n\times n} & \langle\varphi_{2j},\varphi_{4j}\rangle_{n\times n} & \langle\varphi_{3j},\varphi_{4j}\rangle_{n\times n} & \langle\varphi_{4j},\varphi_{4j}\rangle_{n\times n} \end{bmatrix},$$

$$\begin{aligned} X &= (\alpha_1, ..., \alpha_n, \beta_{11}, \beta_{12}, ..., \beta_{1n}, \beta_{21}, \beta_{22}, ..., \beta_{2n}, ..., \beta_{41}, \beta_{42}, ..., \beta_{4n})^\top, \\ F &= (F(x_1,t_1), F(x_2,t_2), ..., F(x_n,t_n), 0, 0, ..., 0)^\top_{1\times 5n}, \end{aligned}$$

then, we overwrite the linear equations (4.6) into matrix form: $\mathbf{G}X = F$. Note that $\mathbf{G}$ is Gram matrix which is symmetric and positive definite, so the scheme (4.4) is uniquely solvable.

Theorem 4.5. $|v(x,t) - v_n(x,t)| = O(\frac{1}{n})$.

Proof . Let $S = \{(x_1, t_1), (x_2, t_2), ...\}$ be a dense subset of $[0, 1] \times [0, 1]$, for any $(x_j, t_j) \in S, j \leq n$, in virtue of the operator $\mathcal{R}_{4n}$ and the properties of the reproducing kernel, we have

$$\begin{aligned} Lv_n(x_j, t_j) &= \langle v_n(y, s), L_{(y,s)}K_n(x_j, t_j, y, s)\rangle_{P_n^2} = \langle v_n(y, s), \psi_j(y, s)\rangle_{P_n^2} \\ &= \langle \mathcal{R}_{4n}v_n(y, s), \psi_j(y, s)\rangle_{P_n^2} = \langle v_n(y, s), \mathcal{R}_{4n}\psi_j(y, s)\rangle_{P_n^2} \\ &= \langle v_n(y, s), \psi_j(y, s)\rangle_{P_n^2} = \langle v(y, s), L_{(y,s)}K_n(x_j, t_j, y, s)\rangle_{P_n^2} \\ &= L_{(y,s)}\langle v(y, s), K_n(x_j, t_j, y, s)\rangle_{P_n^2} \\ &= Lv(x_j, t_j). \end{aligned}$$

Thus for any $n \in \mathcal{N}$ and $(x, t) \in [0, 1] \times [0, 1]$, take $(x_j, t_j) \in S$, such that $|x - x_j| < \frac{1}{n}$ and $|t - t_j| < \frac{1}{n}$, we get

$$\begin{aligned} Lv_n(x, t) - Lv(x, t) &= (Lv_n(x, t) - Lv_n(x_j, t_j)) - (Lv(x, t) - Lv_n(x_j, t_j)) \\ &= \langle v_n(y, s), L_{(y,s)}K_n(x, t, y, s) - L_{(y,s)}K_n(x_j, t_j, y, s)\rangle_{P_n^2} \\ &- \langle v(y, s), L_{(y,s)}K_n(x, t, y, s) - L_{(y,s)}K_n(x_j, t_j, y, s)\rangle_{P_n^2} \\ &= \langle v_n(y, s) - v(y, s), L_{(y,s)}K_n(x, t, y, s) - L_{(y,s)}K_n(x_j, t_j, y, s)\rangle_{P_n^2}. \end{aligned}$$

By the mean value theorem, we have

$$L_{(y,s)}K_n(x, t, y, s) - L_{(y,s)}K_n(x_j, t_j, y, s) = (x - x_j)\frac{\partial}{\partial \eta}L_{(y,s)}K_n(\eta, t, y, s) - (t - t_j)\frac{\partial}{\partial \zeta}L_{(y,s)}K_n(x, \zeta, y, s).$$

□

Finally, the following conclusion follows from the above

$$\begin{aligned} |v_n(x, t) - v(x, t)| &= \langle v_n - v, L^{-1}(LK_n(x, t, ., .) - LK_n(x_j, t_j, ., .))\rangle_{P_n^2} \\ &\leq \|L^{-1}\|_{P_n^2}\|v_n - v\|_{P_n^2}\|LK_n(x, t, ., .) - LK_n(x_j, t_j, ., .)\|_{P_n^2} \\ &\leq \|L^{-1}\|_{P_n^2}\|v_n - v\|_{P_n^2}(|x - x_j|\|\frac{\partial}{\partial \eta}K_n(\eta, t, ., .)\|_{P_n^2} + |t - t_j|\|\frac{\partial}{\partial \zeta}K_n(x, \zeta, ., .)\|_{P_n^2}). \end{aligned}$$

Thus according to $\|v_n(x, t) - v(x, t)\|_{P_n^2} \longrightarrow 0, |x - x_j| < \frac{1}{n}, |t - t_j| < \frac{1}{n}$ and the boundedness of $\frac{\partial}{\partial \eta}K_n(\eta, t, ., .)\|_{P_n^2}$ and $\|\frac{\partial}{\partial \zeta}K_n(x, \zeta, ., .)\|_{P_n^2}$, we get $|v_n(x, t) - v(x, t)| = O(\frac{1}{n})$.

5 Numerical experiments

In this section, some numerical examples with exact solution are considered to illustrate the performance and accuracy of the Chebyshev reproducing kernel method. The results obtained by the method are compared with the analytical solution and are found to be in good agreement with each other. To show the efficiency of the presented method as well as the accuracy of approximate solution u_n, the maximum absolute errors are reported. Throughout this work, all computations are implemented by using Maple 16 software package. To show the rate of convergence of the present method, the values of the order of convergence of the method with respect to the norm infinity with the following formula have been reported

$$r_n = \frac{\ln\left(e_n/e_{2n}\right)}{\ln 2},$$

where

$$e_n = \|e_n(.)\|_\infty = \max_{x,t\in[0,1]} |u(x, t) - u_n(x, t)|.$$

Table 3 shows the order of convergence for different values of n. The results are reported in this table confirm the results of Theorem 4.5.

Example 5.1. [15] As our first example, we consider the following second order linear equation

$$\begin{cases} u_{xx} - 3u_{xt} + u_{tt} = 3\exp(-t)\cos(x), \\ u(x, 0) = \sin(x), \quad u_t(x, 0) = -\sin(x), \quad x \in [0, 1], \\ u(0, t) = 0, \quad u_x(0, t) = \exp(-t), \quad t \in (0, 1]. \end{cases} \tag{5.1}$$

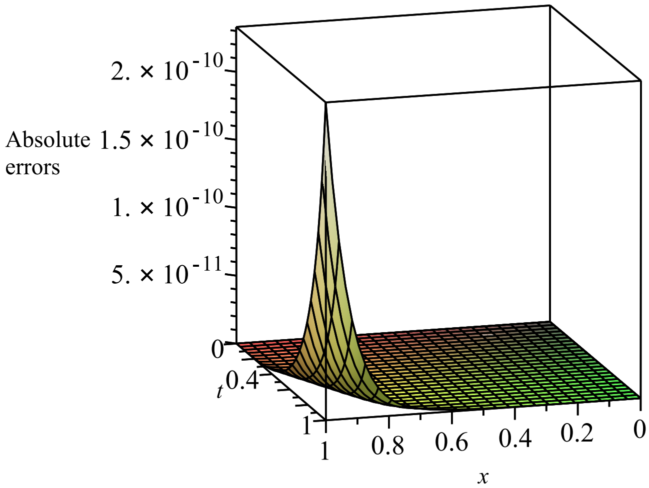

Figure 1: Absolute values of the error $|u(x,t) - u_n(x,t)|$ with $n = 10$ at the selected points of numerical example 5.1.

x	t	**Scheme in [15] with** $n = 10$	**Proposed scheme with** $n = 10$	**Proposed scheme with** $n = 14$
0.1	0.1	2.192e-009	3.055804e-014	4.645834e-016
0.2	0.2	2.556e-009	2.652734e-014	1.772636e-016
0.3	0.3	9.950e-010	7.214098e-014	6.772352e-016
0.4	0.4	1.469e-009	8.953055e-013	1.546248e-015
0.5	0.5	3.178e-009	6.223302e-012	1.016362e-014
0.6	0.6	2.959e-009	1.160306e-012	6.511884e-014
0.7	0.7	9.250e-010	5.100531e-012	3.222006e-013
0.8	0.8	1.591e-009	6.081960e-011	8.532298e-013
0.9	0.9	2.947e-009	3.508839e-010	4.680387e-013
1.0	1.0	2.331e-009	4.014137e-010	8.751284e-012

Table 1: Absolute values of the error $|u(x,t) - u_n(x,t)|$ at the selected points of numerical example 5.1.

The exact solution in $[0,1] \times [0,1]$ is given by $u(x,t) = exp(-t)sin(x)$. After homogenizing the initial conditions and using our method, we obtain the results presented in Tables and Figures. We apply the reproducing kernel Hilbert space method on this problem with $x_i = t_i = \frac{1}{2}cos(\frac{(i+1)\pi}{n}) + \frac{1}{2},\ i = 0,1,2,...,n-1$ for $n = 10$ and $n = 14$. The absolute values of the error is calculated and compared in Table 1 with those available in the literature. It can be noted from Table 1 and Figure 1 that the results of the proposed method is better than the Bernoulli matrix method presented in [15].

Example 5.2. [19] As our second example, we consider the following second-order linear telegraph equation in one-space variable given by

$$\begin{cases} u_{tt} + 20u_t + 25u - u_{xx} = -12\exp(-2t)\sinh(x), \\ u(x,0) = \sinh(x), \quad u_t(x,0) = -2\sinh(x), \quad x \in [0,1], \\ u(0,t) = 0, \qquad u_x(0,t) = \exp(-2t), \qquad t \in (0,1]. \end{cases} \tag{5.2}$$

The exact solution in $[0,1] \times [0,1]$ is given by $u(x,t) = \exp(-2t)\sinh(x)$. Figure 2 shows the absolute error graph for $n = 10$. Numerical results show that the present method is more accurate than the unconditionally stable scheme [19].

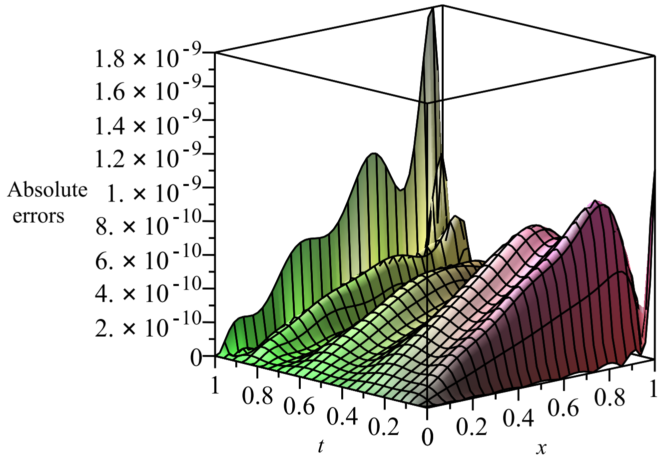

Figure 2: Absolute values of the error $|u(x,t) - u_n(x,t)|$ with $n = 10$ at the selected points of numerical example 5.2.

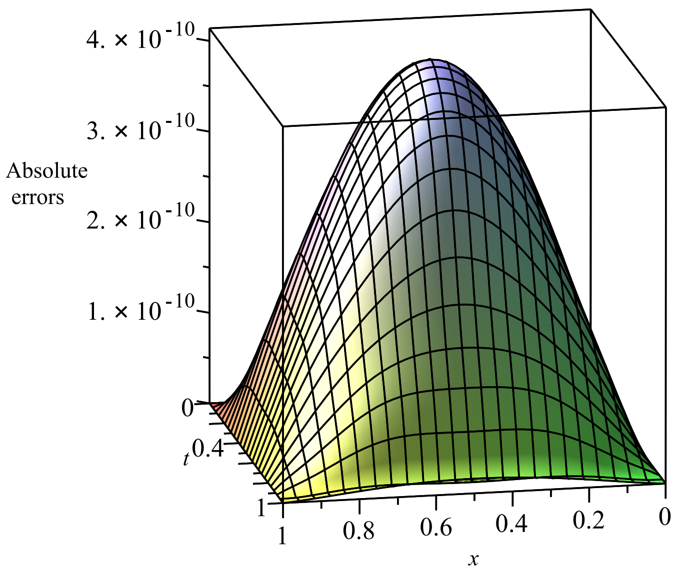

Figure 3: Absolute values of the error $|u(x,t) - u_n(x,t)|$ with $n = 10$ at the selected points of numerical example 5.3.

x	t	**Scheme in [1] with** $n = 10$	**Proposed scheme with** $n = 10$	**Proposed scheme with** $n = 14$
0.1	0.1	0.00003843767	4.603421e-012	3.459243e-016
0.2	0.2	0.00015152539	1.639043e-011	1.167257e-015
0.3	0.3	0.00028528437	2.954963e-011	2.010931e-015
0.4	0.4	0.00039472077	3.921268e-011	2.667901e-015
0.5	0.5	0.00044524883	4.020648e-011	2.761894e-015
0.6	0.6	0.00041530665	3.480960e-011	2.344389e-015
0.7	0.7	0.00032602776	2.300791e-011	1.531764e-015
0.8	0.8	0.00019090592	1.592087e-011	7.622561e-016
0.9	0.9	0.00008009819	2.542232e-011	1.676249e-016
1.0	1.0	3.40220513e-09	1.060000e-015	9.067000e-020

Table 2: Absolute values of the error $|u(x,t) - u_n(x,t)|$ at the selected points of numerical example 5.3.

r_n	**example 5.1**	**example 5.2**	**example 5.3**
r_{10}	1.83652	1.90046	1.80735
r_{14}	1.90689	1.85561	1.45066

Table 3: The rate of convergence for Examples

Example 5.3. [1] As our third example, we consider the following telegraph equation

$$\begin{cases} u_{tt} + u_t + u - u_{xx} = (2 - 2t + t^2)(x - x^2)\exp(-t) + 2t^2\exp(-t), \\ u(x,0) = 0, \quad u_t(x,0) = 0, \qquad\qquad x \in [0,1], \\ u(0,t) = 0, \quad u_x(0,t) = t\exp(-t), \quad t \in (0,1]. \end{cases} \tag{5.3}$$

The exact solution in $[0,1] \times [0,1]$ is given by $u(x,t) = (x - x^2)t\exp(-t)$. After homogenizing the initial conditions and using our method, we compare the numerical results with the result of [1]. It can be concluded that the proposed scheme has a higher efficiency and accuracy than the scheme in [1]. The results on interval $[0,1] \times [0,1]$ when $n = 10$ and $n = 14$ are shown in Table 2. It confirms that higher accuracy can be reached by increasing the number of basis functions. Figure 3 depict the absolute error functions on $[0,1] \times [0,1]$ when $n = 10$.

Conclusions

In this paper, the shifted Chebyshev reproducing kernel method is employed to compute approximate solutions of a second order linear partial differential equation under nonhomogeneous initial conditions. In this approach, a truncated series based on shifted Chebyshev reproducing kernel functions with easily computable components. Based on the orthogonal basis established in the reproducing kernel space, an efficient algorithm is provided to solve the nonlinear system of a second order linear partial differential equation on $[0,1] \times [0,1]$. The convergence analysis and error estimation of the approximate solution using the proposed method are investigated. The validity and applicability of the method is demonstrated by solving several numerical examples. The proposed method is a well-performance technique for calculating the best approximate solution of linear and nonlinear boundary value problems. The main advantage of the present method lies in the lower computational cost and high accuracy.

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