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On the stability of a two-step method for a fourth-degree family by computer designs along with applications

Mandana Moccari[a,*], Taher Lotfi[a], Vali Torkashvand[b,c]

[a] Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran

[b] Young Researchers and Elite Club, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran

[c] Farhangian University, Tehran, Iran

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Abstract

In this paper, some important features of Traub's method are studied: Analysis of the stability behavior, obtaining the 4th root of a matrix, semi-local convergence, and local convergence. The stability of Traub's method is studied by using the dynamic behavior of a family of 4th-degree polynomials. The obtained equations are very complex and do not solve with the software. Therefore, we find the results by plotting diagrams and pictures, and then we show the very stable behavior of Traub's method. Then Traub's method is extended to a matrix iterative method for calculating the 4th root of a square matrix. We also present the local and semi-local convergence of the method based on the divided differences, and therefore, the benefits of our approach are more precise error estimation in semi-local convergence and a large ball of convergence in local convergence. We confirm our theoretical results by some numerical examples such as the nonlinear integral equation of mixed Hammerstein type.

Keywords: Stability region, Parameter plane, 4th root of the matrix, Semi-local convergence, Local convergence
2020 MSC: 65F08, 37F50, 40A05

1 Introduction

In this paper, we concern with the approximation of the solution of nonlinear equation

$$F(x) = 0. \tag{1.1}$$

The study of dynamical stability, local and semi-local convergence are powerful tools for choosing the best initial values and finding the nearest approximate to the solution of (1.1). There exist many studies about these problems such as [1, 12, 18, 14, 32, 33]. In this paper, we have studied the dynamical behavior, local and semi-local convergence of Traub's scheme [37] given by

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} = y_n - F'(x_n)^{-1}F(y_n), \end{cases} \tag{1.2}$$

*Corresponding author

Email addresses: m_moccari@yahoo.com (Mandana Moccari), lotfitaher@yahoo.com (Taher Lotfi), vali_torkashvand@yahoo.com (Vali Torkashvand)

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where x_0 is given. We analyze the stability of third-order Traub's method on a family of fourth-degree polynomials $p(x) = (x^2 - 1)(x^2 + c)$, where c is an arbitrary parameter in the complex plane $\mathbb{C}$. Also, the dynamic properties of the given method such as stability regions of strange fixed points, parameter planes, and basins of attraction are shown. Dynamic analysis have been mostly done on the second and third degree polynomials, such as [13, 35] and references therein. We able to do this for a family of fourth-order polynomials, and it can be a novelty according to complexity of the software. Also, we have plotted the stability regions in a new form with respect to the other papers such as [9].

Furthermore, we use and adapt Traub's method for finding the 4th principal root of a given matrix A. Some examples are also given. Solving the equation $X^p - A = 0$, where A is a matrix, is an important problem in many areas of applications such as control theory, lattice quantum chromo-dynamics and nuclear magnetic resonance (see, for example, [25, 26, 19, 20]). Obtaining the pth root of a matrix is an important problem that many researches studied on it, for example [15, 4].

Additionally, in[3, 2], the behavior of the third-order Traub method is studied for quadratic polynomials, and it is not globally convergent, but it is quite stable. Argyros et. al. have shown in[9] that Traub's method has extremely stable behavior on cubic polynomials in the complex parameter planes except very small regions that appear as basin of attracting strange fixed points. In this paper, for a family of fourth-degree polynomials, we will show the very stable behavior of Traub's method.

Also, in this paper, local and semi-local convergence of the method (1.2) are presented based on the divided differences. The matter of the semi-local convergence is to determine the initial conditions that guarantee the convergence, while the matter of the local convergence is to compute the radius of convergence ball by using the information about solution x^* [7, 5, 11, 29, 30, 13]. Two of the important problems which are to be considered for iterative methods are enlarging the radius of convergence ball in local convergence and increasing the precision of the error estimates on the distances $||x_{n+1} - x_n||$ and $||x_n - x^*||$. Also, some advantages of our work over other studies such as[31, 8, 17, 22, 21, 6] are: we present semi-local convergence in more precise in error estimates than the others by using majorizing sequences and divided differences. In the local convergence a larger convergence ball than others is obtained. Finally, some numerical examples confirm our theoretical results.

This paper is organized as follow: In Sections 2 and 3, we analyze the dynamics of the Traub's method on a family of fourth-degree polynomials. Stability regions, parameter and dynamic planes are obtained. In Section 4, the application of Traub's method for obtaining 4th-root of a matrix A is given. We illustrate our results by some numerical examples. In Sections 5 and 6, the semi-local and local convergence of Traub's scheme are presented. Finally, some numerical examples such as nonlinear integral equation of mixed Hammerstein type are also given in Section 7.

2 Dynamic concepts

In this section, we will study the general convergence the method (1.2) over forth-degree polynomials. First, we recall some dynamical concepts of the complex dynamics. One can see more details in[16]. We define the function f on Riemann sphere $\hat{\mathbb{C}}$ that is $\mathbb{C} \cup \{\infty\}$. We have an iteration map ϕ that acts on the arbitrary function f and R is the rational operator associated with ϕ. In this paper, we suppose the function f is a polynomial "p" of degree four. The sequence

$$O^+(z_0) = \{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}$$

is called the *otbit of* $z_0 \in \hat{\mathbb{C}}$. Now, we study the phase plane of the rational map R by the behavior of the points in the orbits. If we have $R^n(z_0) = z_0$, for some natural number n, then z_0 is called a *periodic point* of *period* n. Also, if $n = 1$, z_0 is a *fixed point* of the map R. Moreover, a fixed point z_0 is called

attractor if $|R'(z_0)| < 1$,

superattractor if $|R'(z_0)| = 0$,

parabolic if $|R'(z_0)| = 1$, and

repulsive if $|R'(z_0)| > 1$.

The fixed point z_0 of the rational map R that is not the root of the polynomial p is called a *strange fixed point.* The roots of the equation $R'(z) = 0$ are *critical points.* For a iterative method of order of convergence greater than one, it is obvious that the roots of p are critical points, too. If the critical point z_0 is not one of the roots of the polynomial p, it is called a *free critical point.* All of the points that the orbits of them converge to an attractor α are the *basin of attraction* α:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to \alpha, n \to \infty\}.$$

$$\begin{tikzcd} \hat{C} \arrow[r, "R"] \arrow[d, "h"] & \hat{C} \arrow[d, "h"] \\ \hat{C} \arrow[r, "S"] & \hat{C} \end{tikzcd}$$

Figure 1: Diagram for conjugacy of functions

Now, we shall define Fatou and Julia set. For this, we need some definitions:

• Let X be a metric space with metric d. An *equicontinuous family* of functions $\{f_i : X \to X\}$ has the following property:

$$\forall \epsilon > 0, \exists \delta > 0, \quad s.t. \quad d(x, y) < \delta \Rightarrow d(f_i(x), f_i(y)) < \epsilon \quad \text{for all}\, i.$$

• A meromorphic function is a rational function where its numerator and denominator are entire functions, and the denominator is not zero.

• *A normal family* is an equicontinuous family $\{f_i : U \to \hat{\mathbb{C}}\}$ on every compact subset U that each f_i is a meromorphic function.

Thus, the *Fatou set,* $F(R)$*,* is the all of the points z_0 where there exists a neighborhood $U(z_0) \subseteq \hat{\mathbb{C}}$ such that the family of iterates $\{R^n : U(z_0) \to \hat{\mathbb{C}}\}$ is a normal family. The *Julia set* $\mathcal{J}(R)$ is the complement of the Fatou set.

In other word, when z_0 is a (super)attracting period point, it is contained in Fatou set, and when it is repulsive point, it is contained in Julia set.

Blanchard shows in[16] that the Julia set is nonempty, but the Fatou set can be void. For example, the Julia set of the map $f(z) = \frac{(z^2+1)^2}{4z(z^2-1)}$ is the entire Riemann Sphere.

In the following, we explain some key facts that are used in the interpretation of parameter planes.

Theorem 2.1. (Fatou-Julia) Let R be a rational function. There exists, at least, one critical point in the connected component of the basin of attraction of an attracting fixed periodic point[23, 27].

One of the useful objects that is used in the theory of dynamical systems is the equivalence relation *conjugation.* Two rational functions R and S mapping Riemann Sphere into itself are *analytically conjugate* if for a diffeomorphism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the diagram in Figure 1 commutes, i.e. $Roh = hoS$ .Thus, R and S are holomorphically the same dynamical system.

Theorem 2.2. (Scaling Theorem[2]) Let $T(z) = \alpha z + \beta$, with $\alpha \neq 0$, be an affine map. Let $g(z) = (foT)(z)$ such that $f(z)$ be a polynomial. Then the fixed point operators of Traub's method on f and g, R_f and R_g, respectively, are affinely conjugated by T, that is, $(ToR_g)(z) = (R_f oT)(z)$ for all z.

Our aim is to analyze the stability of Traub's method on a family of the polynomials of the form $p(z) = (z^2-1)(z^2+c)$ where the parameter c is an arbitrary complex number. Hence, all of the polynomials that can be parametrized by means of an affine map to $p(z)$ are considered in this paper.

3 Dynamical behavior of Traub's method

The fixed point operator associated to Traub's method (1.2) on polynomial $p(z, c) = (z^2 - 1)(z^2 + c)$ is applied. So, we get the rational operator $R_p(z, c) = \frac{h_p(z,c)}{g_p(z,c)}$ depending on $z \in \hat{\mathbb{C}}$ and $c \in \hat{\mathbb{C}}$ where

$$\begin{aligned} h_p(z, c) = &- c^4 + (4c^2 - 8c^3 + 8c^4 - 4c^5)z^2 + (24c - 118c^2 + 176c^3 - 118c^4 + 24c^5)z^4 \\ &+ (-12 - 136c + 520c^2 - 520c^3 + 136c^4 + 12c^5)z^6 \\ &+ (135 + 64c - 452c^2 + 64c^3 + 135c^4)z^8 + (-608 + 1004c - 1004c^2 + 608c^3)z^{10} \\ &+ (1370 - 2336c + 1370c^2)z^{12} + (-1540 + 1540c)z^{14} + 687z^{16} \end{aligned} \tag{3.1}$$

and

$$g_p(z,c) = 32z^5(-1+c+2z^2)^5 \tag{3.2}$$

It is clear that $R_p(z,c)$ is equal to infinity when $z=\infty$. Hence, infinity is a fixed point. For other fixed points we have:

Strange fixed points

Fixed points are the roots of equation $R_p(z,c)=z$. Strange fixed points are the roots of equation $s(z,c)=0$, where $s(z,c)=\frac{h_p(z,c)-zg_p(z,c)}{p(z,c)}$.

$$\begin{aligned} s(z,c) =& c^3+(-1+c)c(4+c(-3+4c))z^2+(-20+c(107+c(-161+(107-20c)c)))z^4 \\ &-5(-1+c)(33+c(-80+33c))z^6+(-507+(1081-507c)c)z^8-683(-1+c)z^{10}-337z^{12}. \end{aligned} \tag{3.3}$$

$s(z,c)$ has twelve distinct strange fixed points $s_1(c), s_2(c), \ldots, s_{12}(c)$. The equation $s(z,c)$ has only even powers of z, so the relations $|s_i(c)| = |s_{i+1}(c)|$, for $i=1,3,5,7,9,11$ are satisfied. Due to complexity, they have not any explicit form. Hence, we plot the graphs of $s_i(c)$ for $1 \le i \le 12$ when the parameter $c \in [-7,7]$ changes with a step 0.1.
In Figure 3, the Figures of Figure 2 are combined. So, Figure (c) in Figure 3 shows all of the points of complex plane

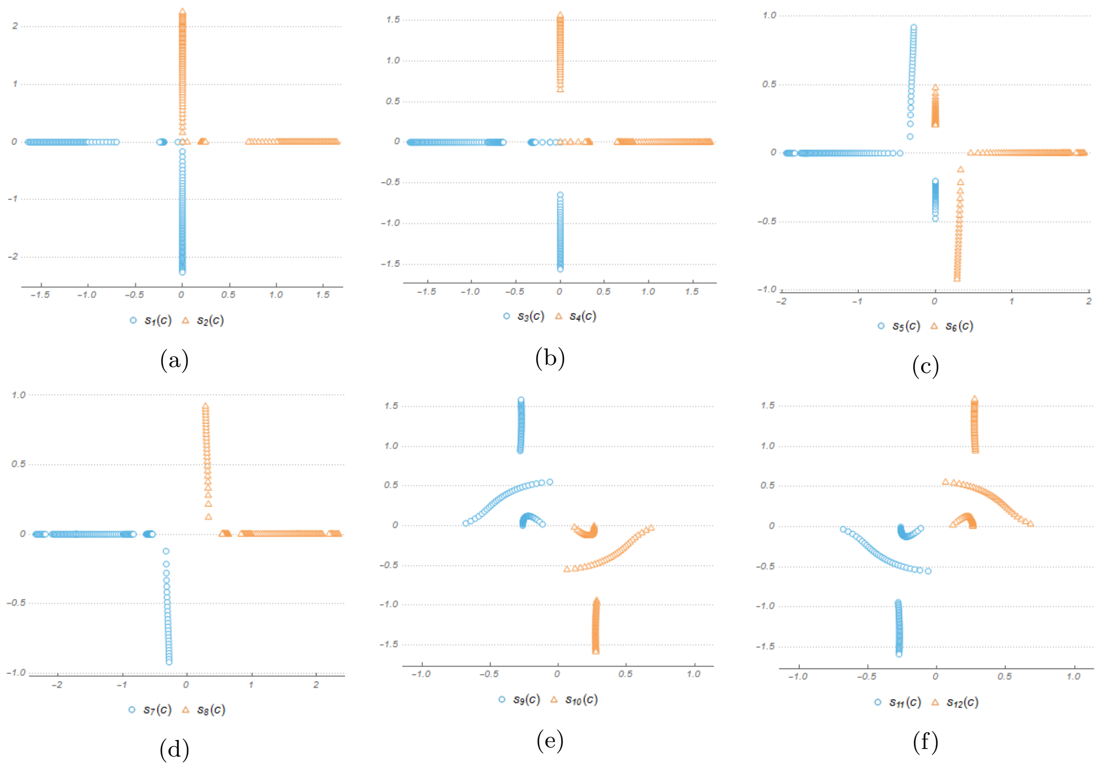


Figure 2: Strange fixed points for different values of c

where they can be strange fixed point.

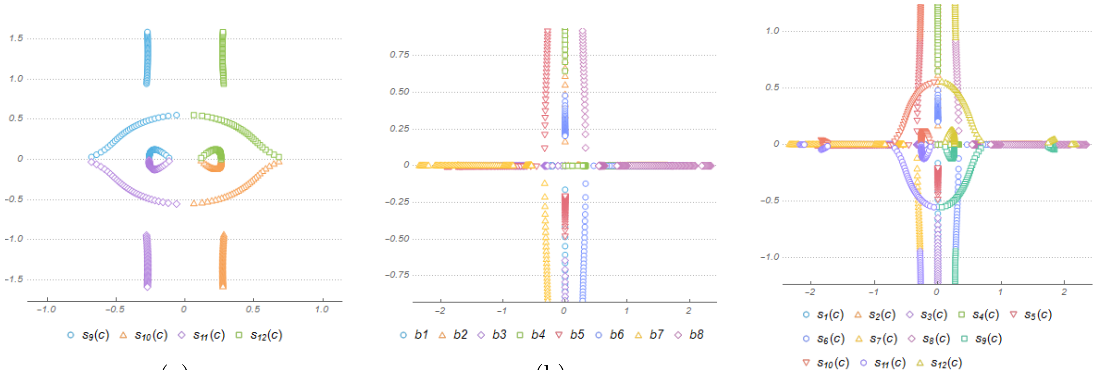
To determine the identity of the strange fixed points, we must get the derivative of the rational function $R_p(z,c)$:

$$R'_p(z,c) = (-1+z^2)^2(c+z^2)^2(-1+c+6z^2)\mathcal{A},$$

where

$$\mathcal{A} = \frac{(5c^2+2(-1+c)(6-c+6c^2)z^2+3(31-44c+31c^2)z^4+250(-1+c)z^6+229z^8)}{32z^6(-1+c+2z^2)^6}.$$

Then, $|R'_p(s_i(c),c)|$, $i=1,2,\ldots,12$, must be computed. Also, $R'_p(z,c)$ has even powers of z, so, the behavior of $|R'_p(s_i(c),c)|$ on $s_i(c)$ is similar to $s_{i+1}(c)$ for $i=1,3,5,7,9,11$. Hence, the identity of those $s_i(c)$ are studied that the



(a) (b) (c)

Figure 3: Combination of the figures in Figure 2

index of them is odd. In Figure 4, the stability regions of complex plane (in new perspective) where some of these strange fixed points become attractive are plotted. In Figures (a)-(g) in the Figure 4, the red areas are attracting, the blue areas are repulsive, the yellow areas, between red and blue regions, are parabolic points, and the green point in the middle of the cardioid is supperattracting, but in the Figures (h) and (i), the pink and orange areas are attracting fixed points related to complementary stability regions of 3th and 5th strange fixed points. The strange fixed points $s_9(c), s_{11}(c)$ are repulsive for all $c \in \mathbb{C}$. Moreover,

$s_1(c)$ is superattracting for c=–5.50779 and c=–0.181561,
$s_3(c)$ is superattracting for c=0.352618,
$s_5(c)$ is superattracting for c=2.83593, c = 0.053876 – 0.350882 I, c = 0.053876 + 0.350882 I, c = 0.427516 – 2.78432 I, and c = 0.427516 + 2.78432 I
$s_7(c)$ is superattracting for c=–6 and c=–0.166667.

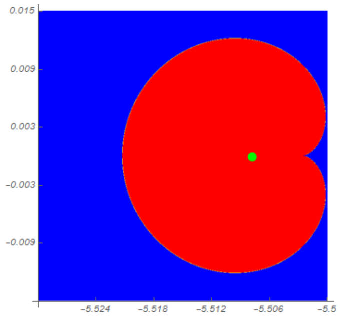
Parameter and Dynamic planes

In order to find more kinds of the behavior of the iterative method, the critical points must be calculated. They are roots of the derivative of $R[z, c]$. But the roots of $p[z, c]$ are also critical points and they are supperattracting. We need critical points different from the roots of $p[z, c]$ that they denoted by free critical points. Free critical points are roots of the equation $\frac{R'_p(z,c)}{p^2(z,c)} = 0$.

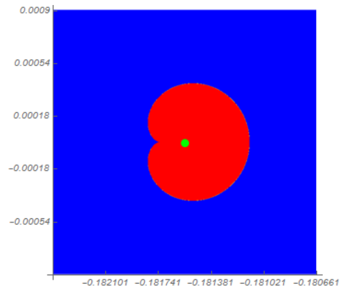
$$\begin{aligned}\frac{R'_p(z,c)}{p^2(z,c)} = & \frac{1}{(32z^6(-1+c+2z^2)^6)}(12c^4z^2 + c^3(5 - 26z^2 + 165z^4) + c^2(-5 + 58z^2 - 309z^4 + 808z^6) \\ & + cz^2(-26 + 309z^2 - 1292z^4 + 1729z^6) + z^2(12 - 165z^2 + 808z^4 - 1729z^6 + 1374z^8)).\end{aligned} \tag{3.4}$$

This equation is also function of the powers of z^2, so their roots are symmetric. The equation (3.4) has ten roots $r_i(c)$, $i = 1, \ldots, 10$, where $|r_i(c)| = |r_{i+1}(c)|$ for $i = 1, 3, 5, 7, 9$. So, five of them are independent. For the independent free critical points, we can obtain the parameter planes. We have five parameter planes P_1, P_2, P_3, P_4, P_5 associated to five free critical points $r_1(c)$, $r_3(c)$, $r_5(c)$, $r_7(c)$ and $r_9(c)$, respectively. We obtain a parameter plane by applying $R_p(z, c)$ on an independent free critical point, so, this point will be an initial estimation for iterative method. If the iterative scheme converges to any roots of $p(z, c)$, the point is colored by red, and it is colored by black if the iterative method diverges. In all of them, red points are very stable points. We have used the codes in with a mesh 1000×1000 points, a maximum number of iteration of 100 and a tolerance of 10^{-3}. In Figures 5, 6, 7, 8 and 9 contain the Parameter planes P_1, P_2, P_3, P_4 and P_5 in the left, respectively and another Figure in right is a detail.

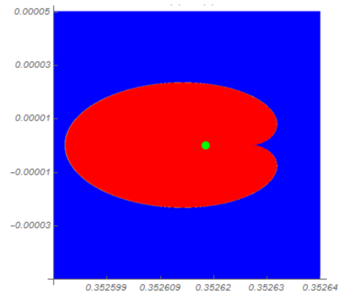
Also, we have shown Fatou and Julia sets by dynamic planes for different values of "c" according to parameter planes in Figure 10. Because the polynomial p has four distinct roots, there exist four colors, red, yellow, blue, and green, in the dynamic planes. They are the basins of attraction of roots. The roots of polynomials are always plotted with black points. The strange fixed points of the rational map R associated with the iteration method are plotted with white points. Strange fixed points stand in the range of roots perfectly symmetrical. Some basins of attraction are immersed in another one such as Figures 10c, 10e, 10a, and 10g.



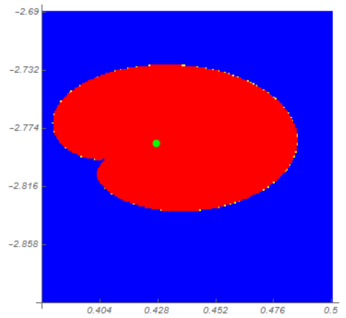
(a) $s_1(c)$



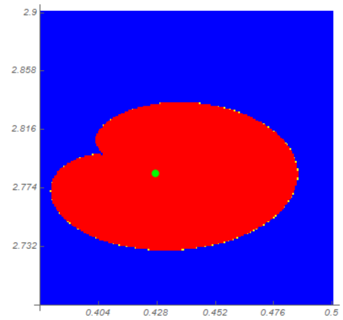
(b) $s_1(c)$



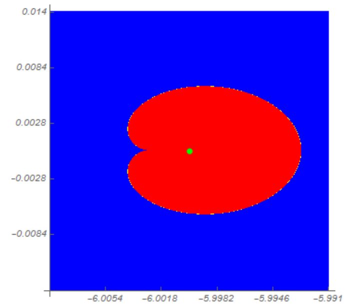
(c) $s_3(c)$



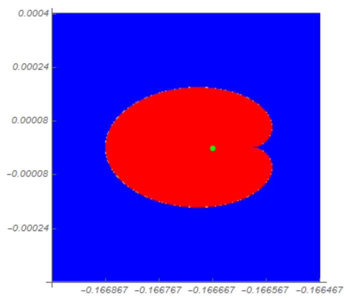
(d) $s_5(c)$



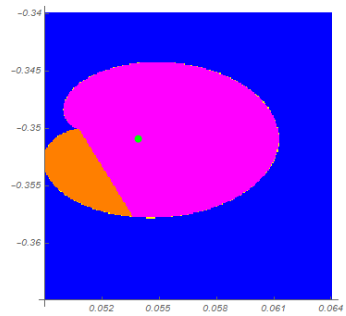
(e) $s_5(c)$



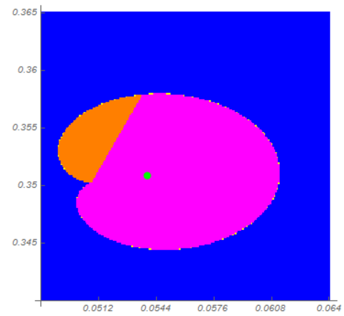
(f) $s_7(c)$



(g) $s_7(c)$



(h) $s_3(c)$ and $s_5(c)$



(i) $s_3(c)$ and $s_5(c)$

Figure 4: Stability regions for $s_i(c)$ where strange fixed points become attractive in different areas

As we see these basins of attraction and parameter planes, we conclude that the Traub method is stable.That's why, black regions in parameter planes are small areas. Moreover, when we choose any point on the Fauto set, the method converges to a superattracting point, because there is no basin of attraction for infinity. Also, the basin of attraction for attracting strange fixed points are very small regions that are not in the immediate basins of attraction of the roots of the polynomial p. For example, in Figures 11 and 12, values of c are chosen that there exist for them attracting strange fixed points. We must zoom at them until we can see the very small basins of attraction that they have lightblue color. All of these reasons show us the Traub method is very stable method.

Remark 1. In this paper, for numerical results and plotting the pictures, Mathematica 12 is used. The computer specifications are Intel(R) Xeon(R), CPU E7-4870 2.40 GHz (2 processors), with 16 GB of RAM.

Remark 2. For plotting the pictures, we have used the codes that base of them presented in [36] and some changes that we must add them. For example, while we have no explicit relation to find the strange fixed points, we apply from this routine:

```
p[x_] = (x^2 - 1) (x^2 + c);
y[x_] = x - p[x]/p'[x];
R[x_, c_] = y[x] - p[y[x]]/p'[x] // Together;
s[x_, c_] =  Numerator[R[x, c]] - x Denominator[R[x, c]] // Simplify;
fixedpoints[c_]= x /. NSolve[s[x, c]/((-1 + x^2) (c + x^2) ) == 0, x];
```

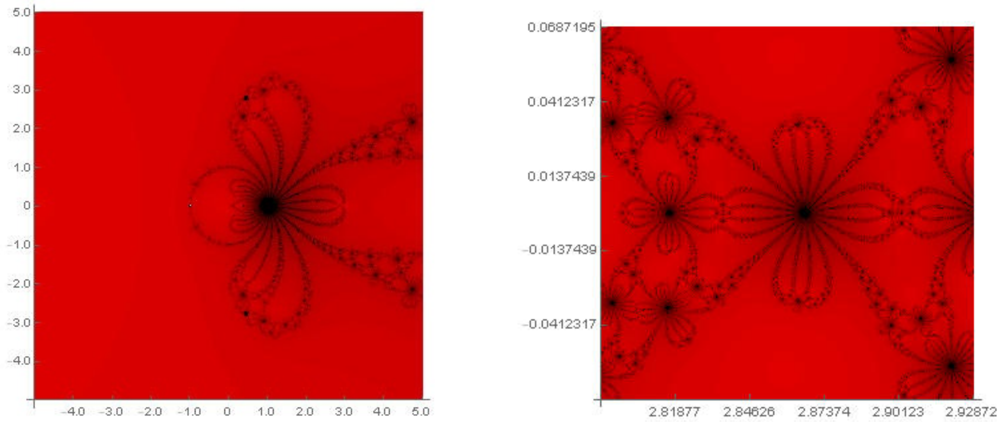


Figure 5: Parameter plane P_1 for the critical point $r_1(c)$ and detail

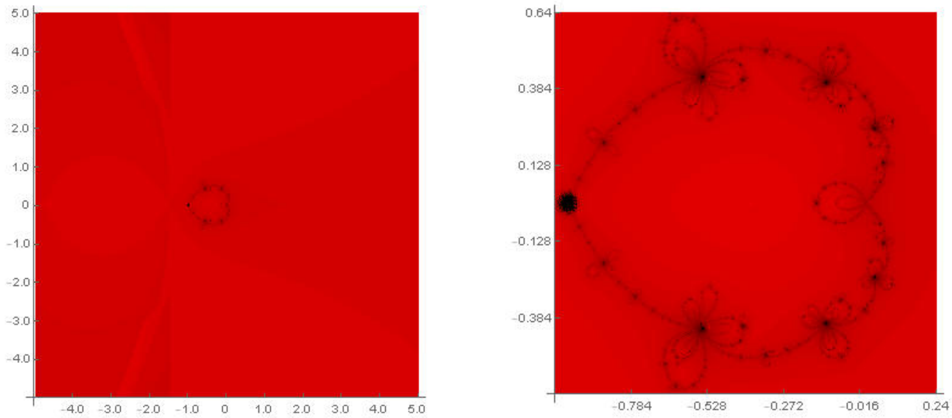


Figure 6: Parameter plane P_2 for the critical point $r_3(c)$ and detail

4 Application on matrix function

We also want to show the applicability of Traub's method for matrix functions. There are many papers about the pth root of matrix such as [24, 4, 15, 25]. Let A be a non-singular complex matrix that belongs to $\mathbb{C}^{n\times n}$ and let $L : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ be the Fréchet-derivative of the matrix function F where $F(X) = X^4 - A$. The zeroes of the function F are the $4th$ roots of the matrix A. In this Section, we want to find the principal 4th root of the matrix A. We need some concepts that we mention them in the following:

Theorem 4.1. [25](principal pth root) A principal pth root of $A \in \mathbb{C}^{n\times n}$, X, is the unique pth root of A, has no eigenvalues on $\mathbb{R}^-$ and all of its eigenvalues lie in the segment $\{z : -\frac{\pi}{p} < \arg(z) < \frac{\pi}{p}\}$. So, we write $X = A^{\frac{1}{p}}$.

Definition 4.2. [25] The Fréchet derivative of a matrix function $F : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ at a point $X \in \mathbb{C}^{n\times n}$ is a linear mapping $L : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ such that

$$F(X + E) - F(X) - L(X, E) = o(||E||),$$

for all $E \in \mathbb{C}^{n\times n}$.

Theorem 4.3. [34, 25] (existence of pth root) $A \in \mathbb{C}^{n\times n}$ has a pth root if and only if the "ascent sequence" of integers $d_1, d_2, \ldots$ defined by

$$d_i = \dim(null(A^i)) - \dim(null(A^{i-1}))$$

has the property that for every integer $k \geq 0$ no more than one element of the sequence lies strictly between pk and $p(k + 1)$.

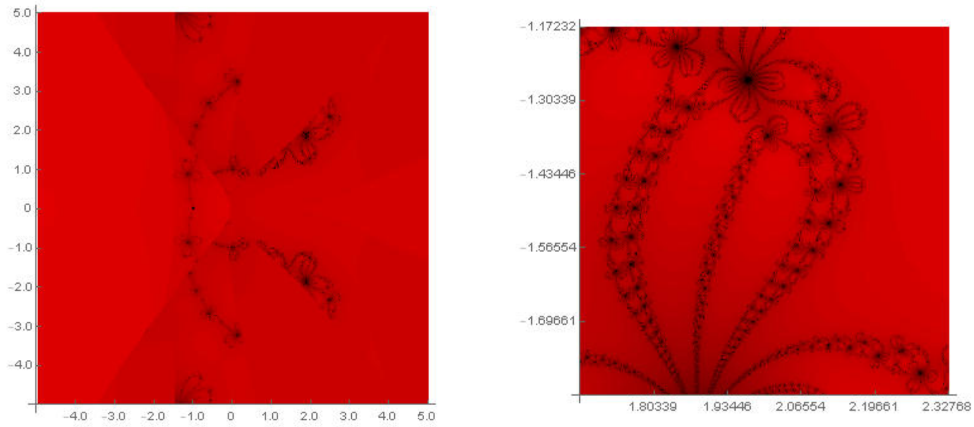


Figure 7: Parameter plane P_3 for the critical point $r_5(c)$ and detail

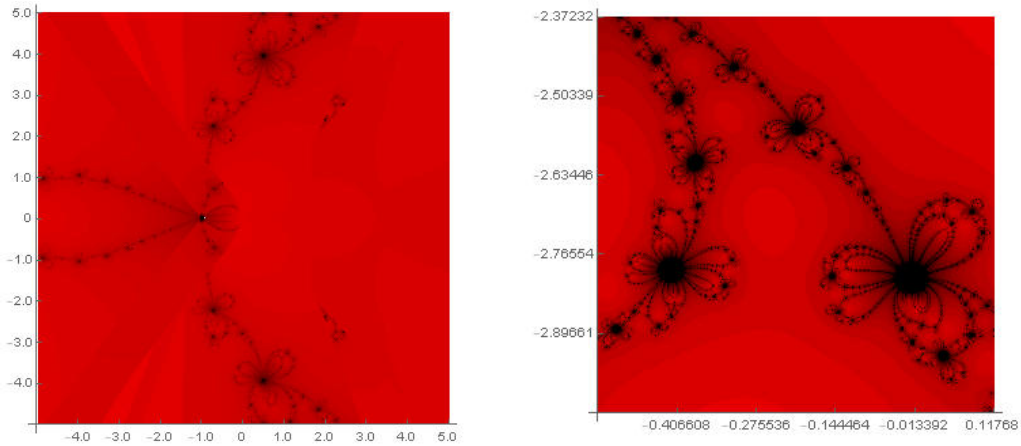


Figure 8: Parameter plane P_4 for the critical point $r_7(c)$ and detail

Now, we are going to obtain the adapted Traub's method for solving the equation $F(X) = X^4 - A$ and obtaining $A^{\frac{1}{4}}$.
The first step of Traub's scheme defines a sequence of iterates $\{Y_k\}_{k\geq 0}$ by

$$Y_k = X_k + E_k$$

where E_k is to be determined. For obtaining the E_k, we use the Fréchet derivative of the function F as the following form:

$$F(X_k + E_k) - F(X_k) = L_F(X_k, E_k) + o(||E_k||).$$

We assume that $F(X_k + E_k) = 0$, and ignore $o(||E_k||)$ and by using the fact that $L_f(X, E) = \sum_{i=1}^{n} X^{n-i}EX^{i-1}$ for the matrix function $f(X) = X^n$, we obtain that

$$-X_k^4 + A = X_k^3E_k + X_k^2E_kX_k + X_kE_kX_k^2 + E_kX_k^3,$$

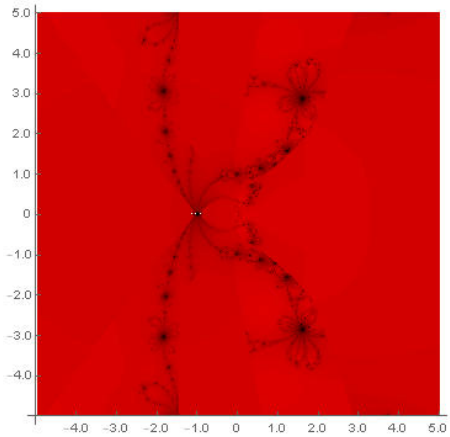
and by assuming $X_kE_k = E_kX_k$, we get the following relation:

$$E_k = \frac{1}{4}(X_k^{-3}A - X_k),$$

and therefore the first step of adapted Traub's method for finding $A^{\frac{1}{4}}$ is

$$Y_k = \frac{1}{4}(X_k^{-3}A + 3X_k)$$

where X_k^{-3} denotes the 3th power of the inverse of X_k.



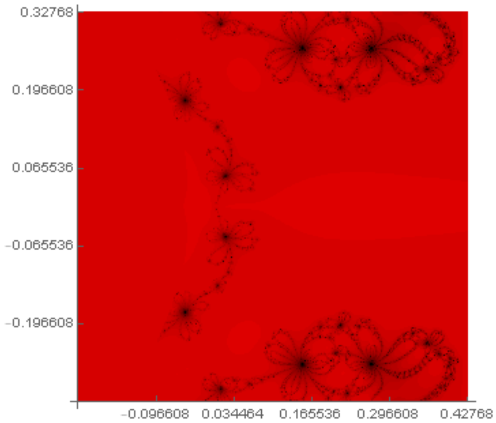


Figure 9: Parameter plane P_5 for the critical point $r_9(c)$ and detail

The first step of Traub's method that is the same Newton's method is quadratically converges to $A^{\frac{1}{4}}$ when the matrix A is non-singular, and $X_0A = AX_0$ where X_0 is initial guess. By the relation

$$X_{k+1} = Y_k + E_k,$$

we can obtain the sequence $\{X_k\}$ for the second step of the adapted Traub's method. By assuming $F(Y_k + E_k) = 0$, ignoring $o(||E_k||)$, letting $L_F(X_k, E_k)$ instead of $L_F(Y_k, E_k)$, and assuming $X_kE_k = E_kX_k$ in the Freéchet derivative of X_{k+1}, we again obtain

$$E_k = \frac{1}{4}X_k^{-3}(A - Y_k^4).$$

Therefore, we get the second step of the adapted Traub's method by

$$X_{k+1} = \frac{1}{4}(X_k^{-3}A + 3X_k) + \frac{1}{4}X_k^{-3}(A - Y_k^4) = \frac{1}{4}(3X_k + 2X_k^{-3}A - X_k^{-3}Y_k^4).$$

For enhancing the initial speed of convergence, we scale the iterates of the Traub method. We replace X_k by μ_kX_k in the Traub method that μ_k is given by

$$\mu_k = |\frac{Det(X_k)}{Det(A)^{\frac{1}{4}}}|^{-\frac{1}{n}}, \tag{4.1}$$

where n is the size of the matrix A. So, the scaled Traub's method for 4th root of a matrix A is given by:

$$\begin{cases} Y_k = \frac{1}{4}(\mu_k^{-3}X_k^{-3}A + 3\mu_kX_k) \\ X_{k+1} = \frac{1}{4}(3\mu_kX_k + 2\mu_k^{-3}X_k^{-3}A - \mu_k^{-3}X_k^{-3}Y_k^4). \end{cases} \tag{4.2}$$

Now, we examine Traub's method and scaled Traub's method for different sizes of matrices and different initial guess.

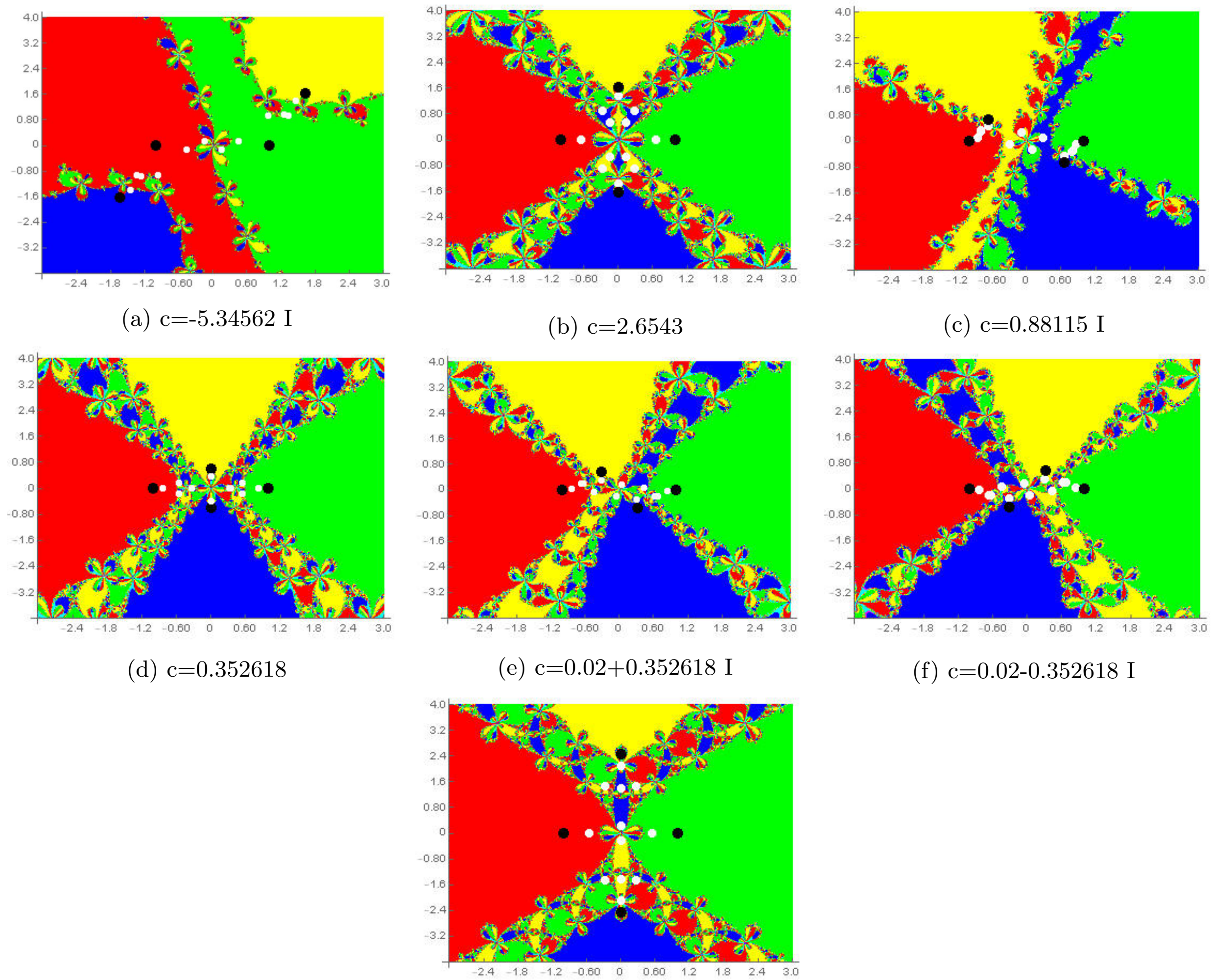
Example 1. We are going to apply Traub's method and scaled Traub's method on the non-singular and diagonalizable matrix A that is given by:

$$\begin{pmatrix} 4 & 1 & 1 & 1 \\ 2 & 4 & 2 & 1 \\ 1 & 0 & 4 & 1 \\ 0 & 2 & 1 & 4 \end{pmatrix}, \tag{4.3}$$

with 100 significant digits, and a stopping criterion using the 2-norm, $||X_k^4 - A|| < 10^{-40}$.

In 1, we show the results of applying of Traub's method and scaled Traub's method on the matrix A for different initial guess, which they are number of iterations, the error in the last iteration, and approximated computational order of convergence (ACOC) that is:

$$ACOC = \frac{\ln(\frac{||X_{k+1}-X_k||}{||X_k-X_{k-1}||})}{\ln(\frac{||X_k-X_{k-1}||}{||X_{k-1}-X_{k-2}||})}. \tag{4.4}$$



(a) c=-5.34562 I (b) c=2.6543 (c) c=0.88115 I

(d) c=0.352618 (e) c=0.02+0.352618 I (f) c=0.02-0.352618 I

(g) c=6

Figure 10: Dynamic planes for different values of c

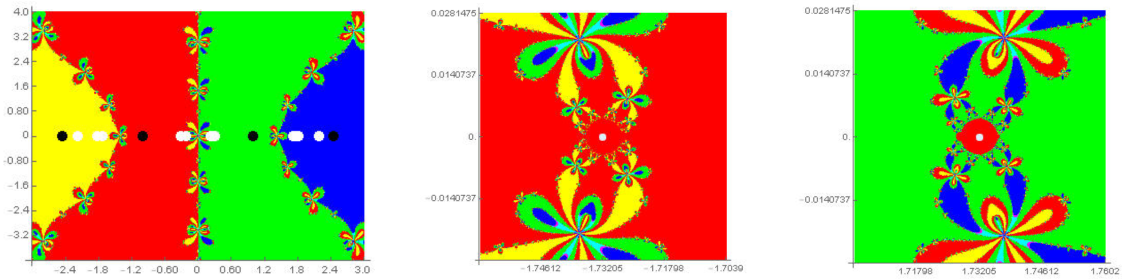
We remark that I_4 is the identity matrix of size 4×4 and $||X||$ is the 2-norm of matrix X.

1 shows that Traub's method converges to the 4th root of the matrix A for all of the initial guess. When Traub's scheme has a large number of iterations, or order of convergence of it has been less than three, scaled Traub's method can overcome these issues. Hence, we illustrate the stability of Traub's method.

Example 2. In the following example, we present the matrix B of different size $n \times n$. The matrix B is the total of a upper triangular matrix, Hilbert matrix, and a random factor of Identity matrix in different size b that its Mathematica code is given by:

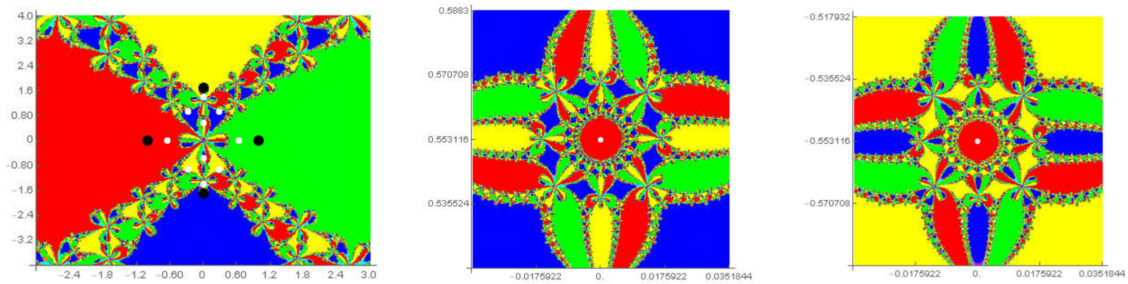
```
b =number;
g[i_, j_] := (j - i)/10 /; i < j
g[i_, j_] := 1 /; i == j
g[i_, j_] := 0 /; i > j
G[b_] := Table[g[i, j], {i, 1, b}, {j, 1, b}];
SeedRandom[1234];
K[b_] := HilbertMatrix[b] + RandomReal[{1, b}]IdentityMatrix[b];
B[b_] := G[b]+K[b];
```

In 2, we show the number of iterations, ACOC, and the error in the last iteration for different size of matrix B. The stopping criterion is $||X_k^4 - B|| < 10^{-12}$ with 100 significant digits. Moreover, we suppose that the initial guess



(a) c=-6 (b) Details (c) Details

Figure 11: Dynamic plane for c=-6 and details



(a) c=2.83593 (b) Details (c) Details

Figure 12: Dynamic plane for c=2.83593 and details

is the matrix B[b] for each size b. Also, we show the condition number for any size of matrix B, according to

$$\kappa(B) = ||B||||B^{-1}||.$$

This example also confirms the stability of Traub's method due to convergence of Traub's method of order three with independence of size and condition number of matrix B.

5 Semi-local convergence of the method (1.2)

In this section, we are going to present the semi-local convergence of Traub's method. According to [7], if the Lipschitz condition

$$||F'(x_0)^{-1}(F'(x) - F'(y))|| \leq M||x-y||, \quad \text{for every } x, y \in D \tag{5.1}$$

as well as $||F'(x_0)^{-1}F(x_0)|| \leq s_0$ holds for some $M > 0$ and $s_0 > 0$, then the sufficient semi-local convergence condition for the method (1.2) is given by the well-known Newton–Kantorovich hypothesis [28]:

$$h = Ms_0 \leq \frac{1}{2}. \tag{5.2}$$

In this paper, we use the following Lipschitz condition:

$$||F'(x_0)^{-1}([x, y; F] - [z, t; F])|| \leq L(||x - z|| + ||y - t||), \quad \text{for every } x, y, z, t \in D \tag{5.3}$$

for some $L > 0$. Also, $[., .; F]$ is divided difference of order one. Based on the relation $[x, x; F] = F'(x)$, it is concluded that $2L = M$. Hence,

$$h = Ls_0 \leq \frac{1}{4}, \tag{5.4}$$

and therfore, based on the [28, 7, 10] and references there in, the error estimates $||x_{n+1} - x_n||$ and $||x_n - x^*||$ are more precise by the relation (5.3). That is why we study the semi-local convergence of the method (1.2). Now, we want to present the majorizing sequences that they are obtained by the relations (5.28) and (5.29).

Table 1

Traub's method				scaled Traub's method			
X_0	Iter	ACOC	Error	X_0	Iter	ACOC	Error
A	8	3.0000	2.21e-69	A	5	3.0000	8.96e-45
$\frac{A+I_4}{10}$	39	3.0053	1.14e-67	$\frac{A+I_4}{10}$	5	3.0014	1.41e-76
I_4	9	3.0000	3.3e-101	I_4	5	2.9991	5.9e-101
$12I_4$	10	2.9613	2.23e-79	$12I_4$	5	2.9991	5.9e-101

Table 2

Matrix B		Traub's method			scaled Traub's method		
b	$\kappa(B)$	Iter	ACOC	Error	Iter	ACOC	Error
4	1.39202	7	2.9758	2.6808e-15	4	2.9936	1.8414e-15
8	1.39294	8	2.8898	1.8879e-15	4	2.9940	1.8828e-15
16	1.6127	10	3.0833	8.9267e-15	4	2.9987	8.9533e-15
32	2.30588	11	3.2679	2.5402e-14	4	2.9755	2.9755e-14
64	4.04788	12	3.0424	5.8682e-14	4	2.8597	9.9800e-14
128	7.59455	13	3.2088	1.3662e-13	4	2.9376	1.4862e-13

Lemma 5.1. (**majorizing sequences for the method** (1.2)) Let $L \geq 0$, $L_0 \geq 0$ and $s_0 \geq 0$ be parameters. We define the polynomial q by

$$q(t) = L(t+2)(t-1) + 2L_0t^3. \tag{5.5}$$

q(t) has a unique root α in the interval $(0,1)$. We suppose that

$$0 < \frac{L(t_1+s_0)}{1-2L_0t_1} < \alpha < 1-2L_0s_0. \tag{5.6}$$

Define the sequence $\{t_n\}$ for $n = 1, 2, \dots$ by

$$t_0 = 0, \qquad t_1 = s_0(1+Ls_0), \tag{5.7}$$

$$\begin{aligned} s_n &= t_n + \frac{L(t_n - t_{n-1} + s_{n-1} - t_{n-1})}{1-2L_0t_n}(t_n - s_{n-1}), \\ t_{n+1} &= s_n + \frac{L(s_n - t_n)}{1-2L_0t_n}(s_n - t_n). \end{aligned} \tag{5.8}$$

Then the sequence $\{t_n\}$ is an increasing and bounded above by $t^{**} = \frac{s_0}{1-\alpha}$.Therefore, $\{t_n\}$ converges to its least upper bound t^*. Moreover, the following relations are hold:

$$t_{n+1} - s_n \leq \alpha(s_n - t_n) \leq \alpha^{2n+1}(s_0 - t_0), \tag{5.9}$$

$$s_n - t_n \leq \alpha(t_n - s_{n-1}) \leq \alpha^{2n}(s_0 - t_0), \tag{5.10}$$

$$t_n \leq s_n \leq t_{n+1}, \tag{5.11}$$

for $n = 0, 1, \dots$.

Proof . First, we shall show that $q(t)$ has roots. We have that $q(0) = -2L < 0$ and $q(1) = 2L_0 > 0$. Hence, by intermediate value Theorem, q(t) has roots in the interval $(0,1)$. $q'(t) = L(2t+1) + 6L_0t^2 > 0$ for all of the points in the interval (0,1). So, q is an increasing function in the interval (0,1). Hence, the graph of q only intersects the x-axis once in the interval (0,1). Therefore, q has a unique root in the interval (0,1). Denote this root by α. Next, we want to prove that $\{t_n\}$ is an increasing and bounded sequence. For showing this, it is enough that we present the relations (5.9)-(5.11). The relations (5.9)-(5.11) are true if the following relations are true for $k = 0, 1, 2, \ldots$:

$$0 < \frac{L(s_k - t_k)}{1 - 2L_0 t_k} < \alpha, \tag{5.12}$$

$$0 < \frac{L(t_{k+1} - t_k + s_k - t_k)}{1 - 2L_0 t_{k+1}} < \alpha, \tag{5.13}$$

$$t_k \leq s_k \leq t_{k+1}. \tag{5.14}$$

We prove (5.12)-(5.14) by induction on k. By the relation (5.7) and the left hand sides of inequalities (5.6), we have

$$0 < Ls_0 \frac{2 + Ls_0}{1 - 2L_0 t_1} < \alpha < 1. \tag{5.15}$$

Because $\frac{2+Ls_0}{1-2L_0t_1} > 1$, $Ls_0 < \alpha$. Hence, the relation (5.12) is true for $k = 0$. By the relation (5.7) and the left hand sides of inequalities (5.6), the relations (5.13) and (5.14) are true for k=0, too. Using the hypotheses of induction, we suppose that the relations (5.12)-(5.14) are true for $k = 1, 2, \ldots, n$. Using these assumptions and by (5.9) and (5.10), we have the following estimates:

$$\begin{aligned} s_k &\leq t_k + \alpha^{2k}(s_0 - t_0) \leq s_{k-1} + \alpha^{2k-1}(s_0 - t_0) + \alpha^{2k}(s_0 - t_0) \\ &\leq (s_0 - t_0) + \cdots + \alpha^{2k}(s_0 - t_0) = \frac{1 - \alpha^{2k+1}}{1 - \alpha}(s_0 - t_0) \leq \frac{s_0}{1 - \alpha} = t^{**}, \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} t_{k+1} &\leq s_k + \alpha^{2k+1}(s_0 - t_0) \leq t_k + \alpha^{2k}(s_0 - t_0) + \alpha^{2k+1}(s_0 - t_0) \\ &\leq (s_0 - t_0) + \cdots + \alpha^{2k+1}(s_0 - t_0) = \frac{1 - \alpha^{2k+2}}{1 - \alpha}(s_0 - t_0) \leq \frac{s_0}{1 - \alpha} = t^{**}. \end{aligned} \tag{5.17}$$

By (5.6) and the hypotheses of induction, we get that

$$0 < \frac{1}{1 - 2L_0 t_1} < \frac{1}{1 - 2L_0 t_k}, \tag{5.18}$$

and

$$0 < \frac{L(s_k - t_k)}{1 - 2L_0 t_k} < \frac{L(t_{k+1} - t_k + s_k - t_k)}{1 - 2L_0 t_{k+1}}. \tag{5.19}$$

We get the left hand side of inequalities (5.12) and (5.13) by the relation (5.19). Now, we shall show the relations (5.12)-(5.14) for $k > n$. By (5.19) we only show that the relation (5.13) holds. We must show that the following relation is true for each $k = 1, 2, \ldots$.

$$\frac{L(\frac{1-\alpha^{2k+2}}{1-\alpha} - \frac{1-\alpha^{2k}}{1-\alpha} + \alpha^{2k})(s_0 - t_0)}{1 - 2L_0(1 + \cdots + \alpha^{2k+1})(s_0 - t_0)} < \alpha, \tag{5.20}$$

or

$$\frac{L\alpha^{2k-1}(\alpha + 2)(s_0 - t_0)}{1 - 2L_0(1 + \cdots + \alpha^{2k+1})(s_0 - t_0)} < 1. \tag{5.21}$$

We define the function $f_k(t)$ on the interval (0,1) by the following relation:

$$f_k(t) = Lt^{2k-1}(t + 2)(s_0 - t_0) + 2L_0(1 + \cdots + t^{2k+1})(s_0 - t_0) - 1. \tag{5.22}$$

The relation (5.21) is true, if $f_k(\alpha) < 0$ is true for each $k = 1, 2, \ldots$. For this aim, we decide to make the following relationship:

$$f_{k+1}(t) - f_k(t) = t^{2k-1}(s_0 - t_0)(t + 1)q(t). \tag{5.23}$$

Using (5.5), we get that for each $k = 1, 2, \ldots$

$$f_{k+1}(\alpha) = f_k(\alpha) = f_\infty(\alpha), \tag{5.24}$$

where $f_\infty(t) = lim_{k\to\infty} f_k(t)$. By the right hand of the inequalities (5.6), we get that

$$f_\infty(\alpha) = \frac{2L_0(s_0 - t_0)}{1-\alpha} - 1 < 0. \tag{5.25}$$

We deduce that $f_k(\alpha) < 0$ for each $k = 1, 2, \ldots$ so that the relations (5.9)-(5.11) are true. Hence, $\{t_n\}$ is a increasing and bounded above by t^{**}, so it converges to least upper bound t^*. Therefore, the proof of the Lemma is complete. □

Now, by using the Lemma 1, we shall present the semi-local convergence of the method(1.2). The open and closed ball $U(s, r)$ in the Banach space X is the set $\{x \in X \mid ||x - s|| < r\}$ and $\{x \in X \mid ||x - s|| \le r\}$, respectively, such that $||.||$ is a norm in the Banach space X. For example, in the Banach space $\mathbb{R}^n$, the norm $||x - s||$ is the Euclidean-norm $(\sum_{i=1}^n (x_i - s_i)^n)^{1/n}$ where x and s are n-dimensional vectors in $\mathbb{R}^n$.

Theorem 5.2. Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator and $[.,.; F]$ be a divided difference of order one for operator F on $D \times D$. X and Y are Banach spaces and D is a convex subset of X. Moreover, suppose that there exist $x_0 \in D$, $L_0 > 0$, $L > 0$, and $s_0 > 0$ with $L_0 \le L$ such that for every x, y, z, and $t \in D$

$$F'(x_0)^{-1} \in L(Y, X), \tag{5.26}$$

$$||F'(x_0)^{-1}F(x_0)|| \le s_0, \tag{5.27}$$

$$||F'(x_0)^{-1}([x, y; F] - F'(x_0))|| \le L_0(||x - x_0|| + ||y - x_0||), \tag{5.28}$$

$$||F'(x_0)^{-1}([x, y; F] - [z, t; F])|| \le L(||x - z|| + ||y - t||), \tag{5.29}$$

and all of the hypotheses of Lemma 1 are confirmed. Also, $F'(x) = [x, x; F]$. Then the sequence $\{x_n\}$, generated by the method(1.2), converges to $x^* \in \overline{U}(x_0, t^*) \subseteq D$ and remains in $\overline{U}(x_0, t^*)$. Moreover, x^* is the unique solution of $F(x) = 0$ in the $\overline{U}(x_0, t^*)$

Proof . By induction on n, we shall show that

$$||x_{n+1} - y_n|| \le t_{n+1} - s_n \tag{5.30}$$

and

$$||y_n - x_n|| \le s_n - t_n. \tag{5.31}$$

For $n = 0$, by (5.27), we have

$$||y_0 - x_0|| = ||F'(x_0)^{-1}F(x_0)|| \le s_0 = s_0 - t_0 \le t^*. \tag{5.32}$$

So, $y_0 \in \overline{U}(x_0, t^*)$, and (5.31) holds for $n = 0$. Using the first and second substeps of the method(1.2) and relations (5.7) and (5.28), we get that

$$\begin{aligned} ||x_1 - y_0|| = ||F'(x_0)^{-1}F(y_0)|| &\le ||F'(x_0)^{-1}([y_0, x_0; F] - F'(x_0))||||y_0 - x_0|| \\ &\le L_0(||y_0 - x_0||)||y_0 - x_0|| \\ &\le L(||y_0 - x_0||)||y_0 - x_0|| = t_1 - s_0. \end{aligned} \tag{5.33}$$

Hence,

$$||x_1 - x_0|| \le ||x_1 - y_0|| + ||y_0 - x_0|| \le t_1 - s_0 + s_0 - t_0 = t_1 \le t^*, \tag{5.34}$$

so, $x_1 \in \overline{U}(x_0, t^*)$, and (5.30) holds for $n = 0$. By (5.6) and (5.28), we get that

$$||F'(x_0)^{-1}(F'(x_1) - F'(x_0))|| \le L_0(||x_1 - x_0|| + ||x_1 - x_0||) = 2L_0t_1 < 1. \tag{5.35}$$

It follows by the Banach lemma on invertible operators that $F'(x_1)^{-1}$ exists and

$$||F'(x_1)^{-1}F'(x_0)|| \le \frac{1}{1 - 2L_0||x_1 - x_0||}. \tag{5.36}$$

Also, by the second substep of the method(1.2) and (5.28), we deduce that

$$\begin{aligned}||F'(x_0)^{-1}F(x_1)|| &= ||F'(x_0)([x_1,y_0;F]-F'(x_0))(x_1-y_0)|| \\ &\leq L_0(||x_1-x_0||+||y_0-x_0||)||x_1-y_0|| \\ &\leq L(||x_1-x_0||+||y_0-x_0||)||x_1-y_0||.\end{aligned} \tag{5.37}$$

Now, by (5.36), (5.37), and the first substep of the method(1.2), we get that

$$\begin{aligned}||y_1-x_1|| &\leq ||F'(x_1)^{-1}F'(x_0)F'(x_0)^{-1}F(x_1)|| \\ &\leq \frac{L(||x_1-x_0||+||y_0-x_0||)}{1-2L_0||x_1-x_0||}||x_1-y_0|| \\ &\leq \frac{L(t_1-t_0+s_0-t_0)}{1-2L_0t_1}(t_1-s_0) \\ &= s_1-t_1,\end{aligned} \tag{5.38}$$

and

$$||y_1-x_0|| \leq ||y_1-x_1||+||x_1-x_0|| \leq s_1-t_1+t_1-t_0 = s_1 \leq t^*. \tag{5.39}$$

So, $y_1 \in \overline{U}(x_0,t^*)$, and (5.31) holds for $n=1$. Also, using the first substep of the method(1.2), we get that

$$\begin{aligned}||F'(x_0)^{-1}F(y_1)|| &= ||F'(x_0)^{-1}([x_1,y_1;F]-F'(x_1))(y_1-x_1)|| \\ &\leq L(||y_1-x_1||)||y_1-x_1||,\end{aligned} \tag{5.40}$$

and using the second substep of the method(1.2), we have

$$\begin{aligned}||x_2-y_1|| &\leq ||F'(x_1)^{-1}F'(x_0)||||F'(x_0)^{-1}F(y_1)|| \\ &\leq \frac{L(||y_1-x_1||)||y_1-x_1||}{1-2L_0||x_1-x_0||} \\ &\leq \frac{L(s_1-t_1)(s_1-t_1)}{1-2L_0t_1} \\ &= t_2-s_1.\end{aligned} \tag{5.41}$$

Hence, (5.30) holds for$n=1$, and

$$||x_2-x_0|| \leq ||x_2-y_1||+||y_1-x_1||+||x_1-x_0|| \leq t_2-s_1+s_1-t_1+t_1-t_0 = t_2 \leq t^*, \tag{5.42}$$

so, $x_2 \in \overline{U}(x_0,t^*)$. If we replace the role of x_1, y_1, x_2 with x_k, y_k, x_{k+1}, we obtain that for each $k = 0,1,2,\ldots$ the relations (5.31) and (5.30) are true. Therefore, we get that

$$||x_{k+1}-x_k|| \leq ||x_{k+1}-y_k||+||y_k-x_k|| \leq t_{k+1}-s_k+s_k-t_k = t_{k+1}-t_k \tag{5.43}$$

so that $\{x_k\}$ is a Cauchy sequence. Also, we have for each $k = 1,2,\ldots$

$$\begin{aligned}||x_k-x_0|| &\leq ||x_k-y_{k-1}||+||y_{k-1}-x_{k-1}||+\cdots+||x_1-y_0||+||y_0-x_0|| \\ &\leq t_k-s_{k-1}+s_{k-1}-t_{k-1}+\cdots+t_1-s_0+s_0-t_0 = t_k \leq t^*,\end{aligned} \tag{5.44}$$

so, $x_k \in \overline{U}(x_0,t^*)$. Thus, we deduce that $\{x_k\}$ is a Cauchy sequence in the closed subset, $\overline{U}(x_0,t^*)$, of the Banach space X so that $\{x_k\}$ converges to x^* in the $\overline{U}(x_0,t^*)$.Moreover, by the second substep of the method(1.2), we obtain that

$$||F'(x_0)^{-1}F(x_{k+1})|| \leq L(||x_{k+1}-x_k||+||y_k-x_k||)||x_{k+1}-y_k||, \tag{5.45}$$

and by letting $k \to \infty$ and continuity of F, we get that

$$F(x^*) = \lim_{k\to\infty} F(x_{k+1}) = 0 \tag{5.46}$$

so that x^* is a solution of $F(x)=0$. If y^* be another solution of $F(x)=0$ in the $\overline{U}(x_0,t^*)$, we have by (5.25) that

$$\begin{aligned}||F'(x_0)^{-1}([x^*,y^*;F]-F'(x_0))|| &\leq L_0(||x^*-x_0||+||y^*-x_0||) \\ &\leq L_0(t^*+t^*) \leq L_0(\frac{2s_0}{1-\alpha}) \\ &\leq 1\end{aligned} \tag{5.47}$$

so that by the Banach lemma on invertible operators $[x^*, y^*; F]^{-1}$ exists. Therefore, it follows from the following relation that $x^* = y^*$:

$$[x^*, y^*; F](x^* - y^*) = F(x^*) - F(y^*) = 0. \tag{5.48}$$

Hence, the proof of Theorem is completed. □

6 Local convergence of the method(1.2)

For the local convergence, we also use the Lipschitz conditions (6.10) and (6.11) based on the divided difference. In the reference [7], the following relations are used:

$$||F'(x^*)^{-1}(F'(x) - F'(x^*))|| \leq l'_0||x - x^*||, \tag{6.1}$$

and

$$||F'(x^*)^{-1}(F'(x) - F'(y))|| \leq l'||x - y||. \tag{6.2}$$

By definition the divided differences, it is deduced that $2l_0 = l'_0$ and $2l = l'$. In [7], then they obtained the following convergence radius for the method (1.2)

$$R = \frac{2}{2l'_0 + 5l'}. \tag{6.3}$$

In this paper, based on the relations (6.10) and (6.11), we obtain the following convergence radius:

$$r = \frac{2}{4l_0 + (1+\sqrt{5})l} = \frac{2}{2l'_0 + \frac{1}{2}(1+\sqrt{5})l'} > \frac{2}{2l'_0 + 5l'} = R. \tag{6.4}$$

So, we enlarge the convergence domain of the method (1.2) in this paper. In order to find the convenient convergence radius r, we introduce some functions that they also have essential role to verify the local convergence. Let $l > 0$ and $l_0 > 0$ with $l_0 \leq l$ be parameters. Define functions g_1 and h_1 on the interval $(0, \frac{1}{2l_0})$ by

$$g_1(t) = \frac{lt}{1 - 2l_0t} \qquad and \qquad h_1(t) = g_1(t) - 1. \tag{6.5}$$

We have $h_1(0) = -1$ and $h_1(t) \to +\infty$ as $t \to \frac{1}{l_0}^-$. Hence, h_1 has root, $r_1 = \frac{1}{l+2l_0}$, in the interval $(0, \frac{1}{2l_0})$ by intermediate value Theorem. Then we define the functions g_2 and h_2 on the interval $(0, r_1)$ by

$$g_2(t) = \frac{lt(2 + g_1(t))g_1(t)}{1 - 2l_0t} \qquad and \qquad h_2(t) = g_2(t) - 1. \tag{6.6}$$

Also, $h_2(0) = -1$ and $h_2(r_1) = \frac{3lr_1}{1-2l_0r_1} - 1 = \frac{\frac{1}{l+2l_0}(3l+2l_0)-1}{1-2l_0r_1} > 0$ so that h_2 has roots in the interval $(0, r_1)$ by intermediate value theorem. Define the smallest of these roots by r. Also, $g'_1(t) > 0$ and $g'_2(t) > 0$ on the interval $(0, r)$. Hence, g_1 and g_2 are increasing functions on the interval $(0, r)$, so we have on the interval $(0, r)$

$$0 \leq g_1(t) < 1, \tag{6.7}$$

$$0 \leq g_2(t) < 1. \tag{6.8}$$

In the other hand, we have $g_2(r) = 1$ and $g_2(r) = g_1(r)^2(2 + g_1(r))$ so that $g_1(r) = \frac{2}{1+\sqrt{5}}$. Hence, $r = \frac{2}{4l_0+(1+\sqrt{5})l}$.

Theorem 6.1. Let $F : D \subseteq X \to Y$ be Fréchet-differentiable function. Suppose parameters l, l_0, r and functions g_1, g_2 are the same defined previously. Also, suppose there exists $x^* \in D$ such that we have for every $x, y, z, t \in D$

$$F(x^*) = 0 \quad , \quad F'(x^*)^{-1} \in L(Y, X), \tag{6.9}$$

$$||F'(x^*)^{-1}([x, y; F] - F'(x^*))|| \leq l_0(||x - x^*|| + ||y - x^*||), \tag{6.10}$$

$$||F'(x^*)^{-1}([x, y; F] - [z, t; F])|| \leq l(||x - z|| + ||y - t||), \tag{6.11}$$

and

$$U(x^*, r) \subseteq D, \tag{6.12}$$

where

$$r = \frac{2}{4l_0 + (1+\sqrt{5})l}. \tag{6.13}$$

Then the sequence $\{x_n\}$ generated by the method(1.2) remains in $U(x^*, r)$ and converges to x^* provided that $x_0 \in U(x^*, r)$. Moreover, the following estimates hold for every $n = 0, 1, 2, \ldots$

$$||y_n - x^*|| \leq g_1(||x_n - x^*||)||x_n - x^*|| \leq ||x_n - x^*||, \tag{6.14}$$

and

$$||x_{n+1} - x^*|| \leq g_2(||x_n - x^*||)||x_n - x^*|| \leq ||x_n - x^*|| < r. \tag{6.15}$$

Proof . By induction on n, we will show the estimates (6.14) and (6.15) are true. First, by hypotheses, we have $x_0 \in U(x^*, r)$, and by the (6.10), we get that

$$\begin{aligned} ||F'(x^*)^{-1}(F'(x_0) - F'(x^*))|| &\leq 2l_0||x_0 - x^*|| \\ &\leq \frac{4l_0}{4l_0 + (1+\sqrt{5})l} \\ &< 1 \end{aligned} \tag{6.16}$$

so that by the Banach lemma on invertible operators $F'(x_0)^{-1}$ exists and

$$||F'(x_0)^{-1}F'(x^*)|| \leq \frac{1}{1 - 2l_0||x_0 - x^*||}. \tag{6.17}$$

Also, by first substep of the method(1.2), (6.5), and (6.7) we have

$$\begin{aligned} ||y_0 - x^*|| &= ||x_0 - x^* - F'(x_0)^{-1}F(x_0)|| \\ &\leq ||F'(x_0)^{-1}F'(x^*)||||F'(x^*)^{-1}(F'(x_0) - [x_0, x^*; F])||||x_0 - x^*|| \\ &\leq \frac{l||x_0 - x^*||}{1 - 2l_0||x_0 - x^*||}||x_0 - x^*|| \\ &= g_1(||x_0 - x^*||)||x_0 - x^*|| \leq ||x_0 - x^*|| < r. \end{aligned} \tag{6.18}$$

Hence, the estimate (6.14) holds for n=0, and $y_0 \in U(x^*, r)$.

Next, using second substep of method(1.2), (6.6), and (6.8) we have

$$\begin{aligned} ||x_1 - x^*|| &= ||y_0 - x^* - F'(x_0)^{-1}F(y_0)|| \\ &\leq ||F'(x_0)^{-1}F'(x^*)||||F'(x^*)^{-1}(F'(x_0) - [y_0, x^*; F])||||y_0 - x^*|| \\ &\leq \frac{l(||x_0 - y_0|| + ||x_0 - x^*||)}{1 - 2l_0||x_0 - x^*||}||y_0 - x^*|| \\ &\leq \frac{l(||x_0 - x^*|| + ||y_0 - x^*|| + ||x_0 - x^*||)}{1 - 2l_0||x_0 - x^*||}||y_0 - x^*|| \\ &\leq \frac{l(g_1(||x_0 - x^*||)||x_0 - x^*|| + 2||x_0 - x^*||)}{1 - 2l_0||x_0 - x^*||}g_1(||x_0 - x^*||)||x_0 - x^*|| \\ &= g_2(||x_0 - x^*||)||x_0 - x^*|| < ||x_0 - x^*|| < r. \end{aligned} \tag{6.19}$$

Hence, (6.15) holds for n=0, and $x_1 \in U(x^*, r)$. Suppose (6.14) and (6.15) are true for $n = 1, 2, \ldots, k$, so verify them for $n = k + 1$. Therefore, we get that

$$\begin{aligned} ||y_{k+1} - x^*|| &= ||x_{k+1} - x^* - F'(x_{k+1})^{-1}F(x_{k+1})|| \\ &\leq ||F'(x_{k+1})^{-1}F'(x^*)||||F'(x^*)^{-1}(F'(x_{k+1}) - [x_{k+1}, x^*; F])||||x_{k+1} - x^*|| \\ &\leq \frac{l||x_{k+1} - x^*||}{1 - 2l_0||x_{k+1} - x^*||}||x_{k+1} - x^*|| \\ &= g_1(||x_{k+1} - x^*||)||x_{k+1} - x^*|| \leq ||x_{k+1} - x^*|| < r. \end{aligned} \tag{6.20}$$

Hence, the estimate (6.14) holds for n=k+1, and $y_{k+1} \in U(x^*, r)$. Also, we have

$$\begin{aligned}
||x_{k+2} - x^*|| &= ||y_{k+1} - x^* - F'(x_{k+1})^{-1}F(y_{k+1})|| \\
&\leq ||F'(x_{k+1})^{-1}F'(x^*)||||F'(x^*)^{-1}(F'(x_{k+1}) - [y_{k+1}, x^*; F])||||y_{k+1} - x^*|| \\
&\leq \frac{l(||x_{k+1} - y_{k+1}|| + ||x_{k+1} - x^*||)}{1 - 2l_0||x_{k+1} - x^*||}||y_{k+1} - x^*|| \\
&\leq \frac{l(||x_{k+1} - x^*|| + ||y_{k+1} - x^*|| + ||x_{k+1} - x^*||)}{1 - 2l_0||x_{k+1} - x^*||}||y_{k+1} - x^*|| \\
&\leq \frac{l(g_1(||x_{k+1} - x^*||)||x_{k+1} - x^*|| + 2||x_{k+1} - x^*||)}{1 - 2l_0||x_{k+1} - x^*||}g_1(||x_{k+1} - x^*||)||x_{k+1} - x^*|| \\
&= g_2(||x_{k+1} - x^*||)||x_{k+1} - x^*|| < ||x_{k+1} - x^*|| < r.
\end{aligned} \tag{6.21}$$

Hence, (6.15) holds for n=k+1, and $x_{k+2} \in U(x^*, r)$. Finally, we have g_2 is an increasing function, and using (6.15) we have that

$$||x_{n+1} - x^*|| \leq ||x_n - x^*||, \tag{6.22}$$

so

$$g_2(||x_{n+1} - x^*||) \leq g_2(||x_n - x^*||). \tag{6.23}$$

By (6.22), (6.23), and induction on n, the following estimate

$$||x_{n+1} - x^*|| \leq g_2(||x_0 - x^*||)^{n+1}||x_0 - x^*|| \tag{6.24}$$

is true for $n = 0, 1, \ldots$. Hence, by letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} x_{n+1} = x^*. \tag{6.25}$$

Therefore, the Theorem was proved. □

7 Examples

In this section, we confirm the theoretical results in Sections 5 and 6 by using numerical examples.

Example 3. Let $F : \mathbb{R}^3 \to: \mathbb{R}^3$ be a Fréchet-differentiable function where

$$F(x_1, x_2, x_3) = (e^{x_1} - 1, e^{x_2} - 1, e^{x_3} - 1) \quad , \quad x_i \in (-1, 1), \quad i = 1, 2, 3.$$

The Fréchet-derivative of function F is given by

$$\begin{pmatrix} e^{x_1} & 0 & 0 \\ 0 & e^{x_2} & 0 \\ 0 & 0 & e^{x_3} \end{pmatrix}, \tag{7.1}$$

and divided difference of F is given by

$$\begin{pmatrix} \frac{e^{x_1} - e^{y_1}}{x_1 - y_1} & 0 & 0 \\ 0 & \frac{e^{x_2} - e^{y_2}}{x_2 - y_2} & 0 \\ 0 & 0 & \frac{e^{x_3} - e^{y_3}}{x_3 - y_3} \end{pmatrix}. \tag{7.2}$$

Let $X_0 = (0.1, 0.1, 0.1)$, and $||.||$ be max-norm. For computing L in the relation 5.29, by using $e^x \approx 1 + x + \frac{x^2}{2}$, we have

$$\begin{aligned}
||F'(X_0)^{-1}([X, Y; F] - [Z, T; F])|| &\leq ||F'(X_0)^{-1}||Max_{1\leq i\leq 3}(|\frac{e^{x_i} - e^{y_i}}{x_i - y_i} - \frac{e^{z_i} - e^{t_i}}{z_i - t_i}|) \\
&\leq 0.904837Max_{1\leq i\leq 3}(1 + \frac{1}{2}(x_i + y_i) - 1 - \frac{1}{2}(z_i + t_i)) \\
&\leq \frac{0.904837}{2}(||X - Z|| + ||Y - T||),
\end{aligned} \tag{7.3}$$

also for computing L_0 in the relation 5.28, we get that

$$\begin{aligned}||F'(X_0)^{-1}([X,Y;F]-[X_0,X_0;F])|| &\leq ||F'(X_0)^{-1}||Max_{1\leq i\leq 3}(|\frac{e^{x_i}-e^{y_i}}{x_i-y_i}-F'(X_0)|)\\ &\leq 0.904837Max_{1\leq i\leq 3}(1+\frac{1}{2}(x_i+y_i)-1-\frac{1}{2}(0.2))\\ &\leq \frac{0.904837}{2}(||X-X_0||+||Y-X_0||).\end{aligned} \tag{7.4}$$

Therefore $L=L_0\approx 0.452419$ and by relation (5.27), we have

$$||F'(X_0)^{-1}F(X_0)||=s_0\approx 0.0951626.$$

Moreover, using (5.5) and (5.7), we deduce that $\alpha\approx 0.722714$, $t_1\approx 0.0992597$, respectively. Now, we confirm our unique condition for semi-local convergence:

$$0<\frac{L(t_1+s_0)}{1-2L_0t_1}\approx 0.0966398<\alpha\approx 0.722714<1-2L_0s_0\approx 0.913893. \tag{7.5}$$

Example 4. Let us consider the following equation that is a nonlinear integral equation of mixed Hammerstein type.

$$F(x)(s)=x(s)-\int_0^1 G(s,t)(\frac{4}{7}x(t)^3+\frac{9}{21}x(t)^2)dt,\quad x\in C[0,1],\quad s\in[0,1]. \tag{7.6}$$

where F is a Fréchet-differentiable operator that $F:C[0,1]\to C[0,1]$, and $C[0,1]$ is the space of continuous functions on $[0,1]$ with the norm-max. Also, the kernel G is Green's function defined by

$$G(s,t)=\begin{cases}(1-s)t & \text{if}\quad t\leq s\\ (1-t)s & \text{if}\quad s\leq t.\end{cases} \tag{7.7}$$

For $y\in C[0,1]$ and $s\in[0,1]$, the Fréchet derivative of F is defined by

$$F'(x)(y)(s)=y(s)-\int_0^1 G(s,t)(\frac{12}{7}x(t)^2+\frac{18}{21}x(t))y(t)dt. \tag{7.8}$$

It is clear that $x^*(s)=0$ is the root of equation $F(x)=0$ and $||F'(x^*)^{-1}||=1$. By using the relations $[x,y;F]=\int_0^1 F'(x\theta+y(1-\theta))d\theta$ and $||\int_0^1\int_0^1 G(s,t)dtds||\leq\frac{1}{8}$, for computing the l_0 in the relation (6.10), we have that

$$\begin{aligned}||[x,y;F]-[x^*,x^*;F]|| &= \int_0^1 F'(x\theta+y(1-\theta))d\theta-\int_0^1 F'(x^*\theta+x^*(1-\theta))d\theta\\ &= ||\int_0^1\int_0^1 G(s,t)(\frac{12}{7}[(x\theta+y(1-\theta))^2-(x^*\theta+x^*(1-\theta))^2]\\ &+\frac{18}{21}[(x\theta+y(1-\theta))-(x^*\theta+x^*(1-\theta))]])y(t)dtd\theta||\\ &\leq \int_0^1\int_0^1 G(s,t)[\frac{12}{7}(||x\theta+y(1-\theta)||+||x^*\theta+x^*(1-\theta)||)(||x\theta\\ &+y(1-\theta)||-||x^*\theta+x^*(1-\theta)||)+\frac{18}{21}(||x-x^*||\theta+||y-x^*||(1-\theta))]\\ &\leq \int_0^1\frac{1}{8}[\frac{12}{7}+(\frac{18}{21})](||x-x^*||\theta+||y-x^*||(1-\theta))d\theta\\ &\leq \frac{9}{56}(||x-x^*||+||y-x^*||),\end{aligned} \tag{7.9}$$

so, $l_0 = \frac{9}{56}$. Also, for computing l from relation (6.11), we get that

$$\begin{aligned}
||[x,y;F]-[u,v;F]|| &= \int_0^1 F'(x\theta+y(1-\theta))d\theta - \int_0^1 F'(u\theta+v(1-\theta))d\theta \\
&= || \int_0^1 \int_0^1 G(s,t)(\frac{12}{7}[(x\theta+y(1-\theta))^2-(u\theta+v(1-\theta))^2] \\
&+ \frac{18}{21}[(x\theta+y(1-\theta))-(u\theta+v(1-\theta))])y(t)dtd\theta|| \\
&\leq \int_0^1 \int_0^1 G(s,t)[\frac{12}{7}(||x\theta+y(1-\theta)||+||u\theta+v(1-\theta)||)(||x\theta \\
&+ y(1-\theta)||-||u\theta+v(1-\theta)||)+\frac{18}{21}(||x-u||\theta+||y-v||(1-\theta))] \\
&\leq \int_0^1 \frac{1}{8}[\frac{24}{7}+(\frac{18}{21})](||x-u||\theta+||y-v||(1-\theta))d\theta \\
&\leq \frac{15}{56}(||x-u||+||y-v||),
\end{aligned} \tag{7.10}$$

hence, $l = \frac{15}{56}$. Therefore, we deduce the following large radius of convergence:

$$r = \frac{2}{4l_0 + (1+\sqrt{5})l} = 1.3248. \tag{7.11}$$

Example 5. Let $X = Y = C[0,1]$ be the space of continuous functions defined on the interval $[0,1]$ with max-norm. Define F on X by

$$F(x)(s) = x(s) - 5\int_0^1 s\theta x(\theta)^3 d\theta, \tag{7.12}$$

and we have

$$F'(x(y))(s) = y(s) - 15\int_0^1 s\theta x(\theta)^2 y(\theta)d\theta \quad for\, all\, y \in X. \tag{7.13}$$

We obtain by hypotheses of Theorem 1 that $l = 15$ and $l_0 = 7.5$. Hence, the radius of convergence is $\frac{2}{4l_0+(1+\sqrt{5})l} \approx 0.0254644$ while the radius ball of convergence that Magrenan et. al. have obtained in [31], the earliest work in this matter, is $\frac{1}{45} \approx 0.222222$. So, we obtain a larger radius of ball convergence than earlier studies.

8 Conclusions

In this paper we have studied the behavior of Traub's method. We have shown the stability behavior of Traub's method on a big family of fourth-degree polynomials. In parameter planes, there exist small black regions. Also, the strange fixed points are attractive in very small regions of complex plane. There exists no basin attraction for infinity. All in all, the behavior of Traub's method is very stable.Meanwhile, the application of Traub's method on functions of matrices for computing 4th root of a matrix confirms the good stability of it as well. Moreover, we have presented semi-local and local convergence of Traub's method based on the divided differences. In semi-local convergence, we have obtained more precision in the error estimates, and in local convergence we have obtained a larger convergence ball than other studies while the numerical examples confirm our results.

References

[1] S. Amat, C. Bermúdez, M.A. Hernández-Verón, and E. Martínez, *On an efficient k-step iterative method for nonlinear equations*, J. Comput. Appl. Math. **302** (2016), 258–271.

[2] S. Amat, S. Busquier, and S. Plaza, *Review of some iterative root-finding methods from a dynamical point of view*, Scientia **10** (2004), no. 3, 35.

[3] ———, *Chaotic dynamics of a third-order Newton-type method*, J. Math. Anal. Appl. **366** (2010), no. 1, 24–32.

[4] S. Amat, J.A. Ezquerro, and M.A. Hernández-Verón, *On a new family of high-order iterative methods for the matrix pth root*, Numer. Linear Alg. Appl. **22** (2015), no. 4, 585–595.

[5] S. Amat, M.A. Hernández, and N. Romero, *Semilocal convergence of a sixth order iterative method for quadratic equations*, Appl. Numer. Math. **62** (2012), no. 7, 833–841.

[6] S. Amat, Á.A. Magreñán, and N. Romero, *On a two-step relaxed Newton-type method*, Appl. Math. Comput. **219** (2013), no. 24, 11341–11347.

[7] I.K. Argyros, R. Behl, and S.S. Motsa, *Unifying semilocal and local convergence of Newton's method on Banach space with a convergence structure*, Appl. Numer. Math. **115** (2017), 225–234.

[8] I.K. Argyros, Y.J. Cho, and S. Hilout, *On the midpoint method for solving equations*, Appl. Math. Comput. **216** (2010), no. 8, 2321–2332.

[9] I.K. Argyros, A. Cordero, Á.A. Magreñán, and J.R. Torregrosa, *Third-degree anomalies of Traub's method*, J. Comput. Appl. Math. **309** (2017), 511–521.

[10] I.K. Argyros and S. Hilout, *Extending the Newton–Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math. **234** (2010), no. 10, 2993–3006.

[11] ______, *Weaker convergence conditions for the Secant method*, Appl. Math. **59** (2014), no. 3, 265–284.

[12] P. Bakhtiari, A. Cordero, T. Lotfi, K. Mahdiani, and J.R. Torregrosa, *Widening basins of attraction of optimal iterative methods*, Nonlinear Dyn. **87** (2017), no. 2, 913–938.

[13] R. Behl, S. Amat, Á.A. Magreñán, and S.S. Motsa, *An efficient optimal family of sixteenth order methods for nonlinear models*, J. Comput. Appl. Math. **354** (2019), 271–285.

[14] R. Behl, A. Cordero, S.S. Motsa, and J.R. Torregrosa, *Stable high-order iterative methods for solving nonlinear models*, Appl. Math. Comput. **303** (2017), 70–88.

[15] N.J. Bini, D.A.and Higham and B. Meini, *Algorithms for the matrix pth root*, Numer. Algorithms **39** (2005), no. 4, 349–378.

[16] P. Blanchard, *Complex analytic dynamics on the Riemann sphere*, Bull. Amer. Math. Soc. **11** (1984), no. 1, 85–141.

[17] J. Chen, I.K. Argyros, and R.P. Agarwal, *Majorizing functions and two-point Newton-type methods*, J. Comput. Appl. Math. **234** (2010), no. 5, 1473–1484.

[18] A. Cordero, F. Soleymani, J.R. Torregrosa, and F.K. Haghani, *A family of Kurchatov-type methods and its stability*, Appl. Math. Comput. **294** (2017), 264–279.

[19] A. Cordero, F. Soleymani, J.R. Torregrosa, and M.Z. Ullah, *Numerically stable improved Chebyshev–Halley type schemes for matrix sign function*, J. Comput. Appl. Math. **318** (2017), 189–198.

[20] A. Cordero and J.R. Torregrosa, *A sixth-order iterative method for approximating the polar decomposition of an arbitrary matrix*, J. Comput. Appl. Math. **318** (2017), 591–598.

[21] J.A. Ezquerro, M.A. Hernández, and N. Romero, *Newton-type methods of high order and domains of semilocal and global convergence*, Appl. Math. Comput. **214** (2009), no. 1, 142–154.

[22] J.A. Ezquerro, M.A. Hernández, and M.A. Salanova, *Recurrence relations for the midpoint method*, Tamkang J. Math. **31** (2000), no. 1, 33–42.

[23] P. Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **47** (1919), no. 48, 1920.

[24] C.-H. Guo, *On Newton's method and Halley's method for the principal pth root of a matrix*, Linear Alg. Appl. **432** (2010), no. 8, 1905–1922.

[25] N.J. Higham, *Functions of matrices: theory and computation*, vol. 104, Siam, 2008.

[26] B. Iannazzo, *On the Newton method for the matrix pth root*, SIAM J. Matrix Anal. Appl. **28** (2006), no. 2, 503–523.

[27] G. Julia, *Memoire sur l'iteration des fonctions rationnelles*, J. Math. Pures Appl. **8** (1918), 47–245.

[28] L.V. Kantorovich, *Functional analysis and applied mathematics*, Uspekhi Mate. Nauk **3** (1948), no. 6, 89–185.

[29] S.K. Khattri, *How to increase convergence order of the Newton method to $2\times m$?*, Appl. Math. **59** (2014), no. 1,

15–24.

[30] T. Liu, X. Qin, and P. Wang, *Local convergence of a family of iterative methods with sixth and seventh order convergence under weak conditions*, Int. J. Comput. Meth. **16** (2019), no. 08, 1850120.

[31] A.A. Magrenan Ruiz and I.K. Argyros, *Two-step Newton methods*, J. Complexity **30** (2014), no. 4, 533–553.

[32] M. Moccari and T. Lotfi, *On a two-step optimal Steffensen-type method: Relaxed local and semi-local convergence analysis and dynamical stability*, J. Math. Anal. Appl. **468** (2018), no. 1, 240–269.

[33] ______, *Using majorizing sequences for the semi-local convergence of a high-order and multipoint iterative method along with stability analysis*, J. Math. Exten. **15** (2020).

[34] P.J. Psarrakos, *On the mth roots of a complex matrix*, Electron. J. Linear Alg. **9** (2002), 32–41.

[35] J.R. Sharma, D. Kumar, I.K. Argyros, and Á.A. Magreñán, *On a bi-parametric family of fourth order composite newton–jarratt methods for nonlinear systems*, Mathematics **7** (2019), no. 6, 492.

[36] W.T. Shaw, *Complex analysis with Mathematica*, Cambridge University Press, 2006.

[37] J.F. Traub, *Iterative methods for the solution of equations*, vol. 312, American Mathematical Soc., 1982.