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Some novel tripled fixed point results in ternary algebras

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Abstract

The purpose of this paper is to provide the concept tripled fixed point theorem in a tripled lattice. For this, we prove that every tripled lattice with the fixed point property is complete.

Keywords: Triple fixed point, Triple coincidence point, tripled partially ordered set, tripled lattice, tripled complete lattice
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1 Introduction

One of the most famous theorems in analysis, Banach fixed point theorem and its applications are that well known. Many authors have extended this theorem by introducing more general contractive conditions, that imply the existence of a fixed point. Existence of fixed points in ordered metric spaces was investigated in 2004 by Ran and Reurings [18], and then by Nieto and Lopez [16]. For some other results in ordered metric spaces, see [5, 3, 4]. Bhashkar and Lakshmikantham in [10] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. Later, various results on coupled fixed point have been obtained, see e.g.[6, 2, 14].

Recently, Berinde and Borcut [9] obtained extensions to the concept of the tripled fixed points and the tripled coincidence fixed points and also obtained the tripled fixed points theorems and tripled coincidence fixed points theorems for contractive type mappings in partially ordered metric spaces. Research onthe tripled fixed point was continued by Abbas, Aydi and Karapinar [1], Aydi and Karapinar [7], Borcut [11], Rao and Kishore [19] and [15].

Partially ordered sets (or poset) are generalizations of ordered sets. A partially ordered set is a set together with a binary relation that indicates that, for some pairs of elements in the set, one of the elements precedes the other.

Definition 1.1. A set X is a partially ordered set if it has a binary relation $x \preceq y$ defined on it that satisfies

1. reflexivity: $x \preceq x$ for all $x \in X$;
2. Antisymmetry: If $x \preceq y$, and $y \preceq x$, then $x = y$ for all $x, y \in X$;
3. Transitivity: If $x \preceq y$ and $y \preceq z$, then $x \preceq z$ for all $x, y, z \in X$.

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Examples of partially ordered sets include the integers and real numbers with their ordinary ordering, subsets of a given set ordered by inclusion and natural numbers ordered by divisibility. For more details about partially ordered sets and their properties we refer the reader to [20] and [21]. Now we recall the following definitions from [12]

Definition 1.2. Let X be a nonempty set. A tripled partial order relation is a triple relation $\preceq_3$ on X (i.e.$\preceq_3 \subseteq X \times X \times X$) which satisfies the following conditions:

1. (reflexivity) $(x, x, x) \in \preceq_3$
2. (antisymmetry) if

$$(x, y, z) \in \preceq_3, (y, z, x) \in \preceq_3, (z, x, y) \in \preceq_3, (x, z, y) \in \preceq_3 \; (y, x, z) \in \preceq_3$$

and $(z, y, x) \in \preceq_3$, then $x = y = z$,

3. (transitivity) if $(x, y, z) \in \preceq_3, (y, z, t) \in \preceq_3$ and $(z, t, w) \in \preceq_3$, then $(x, z, w) \in \preceq_3$, for all $x, y, z, t, w \in X$. A set with a triple partial order $\preceq_3$ is called a *tripled partially ordered set*. We denote this a tripled partially ordered set by $(X, \preceq_3)$.

Definition 1.3. Let $(X, \preceq_3)$ be a tripled partially ordered set and $x, y, z \in X$. Elements x, y and z are said to be *comparable elements* of X if one of the following cases hold.

(1) $(x, y, z) \in \preceq_3$,

(2) $(y, z, x) \in \preceq_3$,

(3) $(z, x, y) \in \preceq_3$,

(4) $(x, z, y) \in \preceq_3$,

(5) $(y, x, z) \in \preceq_3$,

(6) $(z, y, x) \in \preceq_3$

When x, y and z are elements of X such that non of the above relations hold, they are called *incomparable.*

Definition 1.4. Let X be a tripled partially ordered set.

(i) A subset A of X is bounded from above if there exists a $u \in X$, called an *upper bound* of A, such that $(x, y, u) \in \preceq_3$, for all $x, y \in X$.

(ii) A subset A of X is bounded from below if there exists a $v \in X$, called a *lower bound* of A, such that $(v, x, y) \in \preceq_3$, for all $x, y \in X$.

(iii) A subset A of X is bounded from middling if there exists a $h \in X$, called a *middle bound* of A, such that $(x, h, y) \in \preceq_3$, for all $x, y \in X$.

(iv) A subset A of X is called *bounded* if it is bounded from above, bounded from below and bounded from middle.

Definition 1.5. Let $(X, \preceq_3)$ be a tripled partially ordered set and $A \subseteq X$.

(i) If $u \in X$ is an upper bound of A such that $(x, y, u) \in \preceq_3$, for all upper bounds x, y of A, then u is called the *supremum* of A, denoted by $u = \sup A$.

(ii)If $v \in X$ is an lower bound of A such that $(v, x, y) \in \preceq_3$, for all lower bounds x, y of A, then v is called the *infimum* of A, denoted by $v = \inf A$.

Definition 1.6. Let $(X, \preceq_3)$ be a tripled partially ordered set and let $f : X \longrightarrow X$ be a mapping. Then

(1) f is monotone of type 1, $(f(a), f(b), f(c)) \in \preceq_3$ if $(a, b, c) \in \preceq_3$.

(2) f is monotone of type 2, $(f(a), f(c), f(b)) \in \preceq_3$ if $(a, b, c) \in \preceq_3$.

(3) f is monotone of type 3, $(f(c), f(b), f(a)) \in \preceq_3$ if $(a, b, c) \in \preceq_3$.

(4) f is monotone of type 4, $(f(c), f(a), f(b)) \in \preceq_3$ if $(a, b, c) \in \preceq_3$.

(5) f is monotone of type 5, $(f(b), f(a), f(c)) \in \preceq_3$ if $(a, b, c) \in \preceq_3$.

(6) f is monotone of type 6, $(f(b), f(c), f(a)) \in \preceq_3$ if $(a, b, c) \in \preceq_3$.

Example 1.7. Let $S = \{(pn + 1, pn + 2, pn + 3) : n \in \mathbb{N}\}$ where p is a prime number, with order $\leq$. Define $f : S \to \mathbb{N}_0 \times \mathbb{N}_0$ $(\mathbb{N}_0 = \mathbb{N} \cup \{0\})$ by $f(x) =$ *remainder of division of* x *to* p. Then f is a monotone of type $1 - 6$.

2 Tripled fixed point theorem in the tripled lattices

We start with the following definitions.

Definition 2.1. A *tripled lattice* is a tripled partially ordered set $(X, \preceq_3)$ in which every pair of elements $x, y, z \in X$ has a least upper bound, denoted by $w = \sup\{x, y, z\}$, and a greatest lower bound, denoted by $v = \inf\{x, y, z\}$.

Example 2.2. Given any set X its power set $\mathcal{P}(X)$ (the set of all its subsets) is a tripled partially ordered set under inclusion $\subseteq$. Moreover, it is a tripled lattice with least upper bound, $\cup$, and greatest lower bound, $\cap$.

Definition 2.3. A *complete the tripled lattice* is a tripled partially ordered set $(X, \preceq_3)$ with the property that every subset $A \subseteq X$ (finite or infinite) has a least upper bound denoted by $\bigvee A$, and a greatest lower bound denoted by $\bigwedge A$. $\bigvee X$ is called the top element and $\bigwedge X$ is called the bottom element .

Example 2.4. If X is any set, then $\mathcal{P}(X)$ is a complete the tripled lattice under $\subseteq$. Indeed if $A := \{A_i,\ i \in I\}$ is a subset of $\mathcal{P}(X)$, then its least upper bound is

$$\bigvee_i A = \bigcup_{i\in I} A_i$$

and its greatest lower bound is

$$\bigwedge_i A = \bigcap_{i\in I} A_i$$

Of course a complete the tripled lattice is a lattice, since every pair of elements has a least upper and greatest lower bound. However the converse is not true in general.

Definition 2.5. Given a partially ordered set X, and $a, b \in X$, with $(a, b) \in (X, \preceq_3)$, we call an *interval* in X, the set $[a, b] \subseteq X$ defined by

$$[a, b] := \{x \in X : a \preceq_3 x \preceq_3 b\}\,.$$

Intervals are important because of the following proposition:

Proposition 2.6. If X is a complete the tripled lattice, then for any $a, b \in X$, with $(a, b) \in (X, \preceq_3)$, $[a, b]$ is a complete the tripled lattice.

Proof . Consider any subset $S \subseteq [a, b]$. Since X is complete, S has a least upper bound $s_1 \in X$. We will show that $s_1 \in [a, b]$. Indeed, since b is an upper bound for $[a, b]$, b is also an upper bound for S. Hence $(a, s_1, b) \in\preceq_3$ *or* $(s_1, a, b) \in\preceq_3$. In addition, since a is a lower bound for $[a, b]$, a is a lower bound for S, for any $x \in S$, we get $(a, x, s_1) \in\preceq_3$. Therefore $(a, s_1, b) \in\preceq_3$ and so $s_1 \in [a, b]$. Similarly, the greatest lower bound s_0 of S is in $[a, b]$. $\square$

Definition 2.7. let $F : X \times X \times X \longrightarrow X$. An element $(x, y, z) \in X^3$ is called a *triple fixed point* of F if

$$F(x, y, z) = x, \quad F(y, z, x) = y, \quad F(z, x, y) = z.$$

Theorem 2.8. If $(X, \preceq_3)$ is a complete the tripled lattice and $f : X \longrightarrow X$ is an order preserving function, then f has a fixed point. In fact, the set of fixed points of f is again a complete the tripled lattice.

Proof . Let X be a completethe tripled lattice and $f : X \longrightarrow X$ be an order preserving map and F the set of fixed points of f. Consider the set

$$\begin{aligned} B = \{&(x, y, z) \in X : (f(x), f(y), f(z)) \succeq_3 (x, y, z), (f(x), f(z), f(y)) \succeq_3 (x, z, y), (f(z), f(y), f(x)) \succeq_3 (z, y, x),\\ &(f(z), f(x), f(y)) \succeq_3 (z, x, y), (f(y), f(x), f(z)) \succeq_3 (y, x, z), (f(y), f(z), f(x)) \succeq_3 (y, z, x)\ \}\,. \end{aligned}$$

Evidently, $B \neq \emptyset$, since $\bigwedge X \in B$ (X as a complete the tripled lattice has a minimum). Now, consider the element $b = (b_1, b_2, b_3) = \bigvee B$. Since $(x, y, b) \in\preceq_3$ for all $x, y \in B$, and f is order preserving,

$$\begin{aligned}(f(b_1), f(b_2), f(b_3)) &\succeq_3 \bigvee_{x,y,z\in B} (f(x), f(y), f(z))\\ &\succeq_3 \bigvee_{x,y,z\in B} (x, y, z) = (b_1, b_2, b_3)\end{aligned}$$

Therefore, $(b_1, b_2, b_3) \in B$. But this implies that

$$\begin{gathered}(b_1, b_2, b_3) \preceq_3 (f(b_1), f(b_2), f(b_3))\\ (b_1, b_3, b_2) \preceq_3 (f(b_1), f(b_3), f(b_2))\\ (b_3, b_2, b_1) \preceq_3 (f(b_3), f(b_2), f(b_1))\\ (b_3, b_1, b_2) \preceq_3 (f(b_3), f(b_1), f(b_2))\\ (b_2, b_1, b_3) \preceq_3 (f(b_2), f(b_1), f(b_3))\\ (b_2, b_3, b_1) \preceq_3 (f(b_2), f(b_3), f(b_1))\end{gathered}$$

and since f is monotone

$$\begin{gathered}f(b_1, b_2, b_3) \preceq_3 f(f(b_1), f(b_2), f(b_3))\\ f(b_1, b_3, b_2) \preceq_3 f(f(b_1), f(b_3), f(b_2))\\ f(b_3, b_2, b_1) \preceq_3 f(f(b_3), f(b_2), f(b_1))\\ f(b_3, b_1, b_2) \preceq_3 f(f(b_3), f(b_1), f(b_2))\\ f(b_2, b_1, b_3) \preceq_3 f(f(b_2), f(b_1), f(b_3))\\ f(b_2, b_3, b_1) \preceq_3 f(f(b_2), f(b_3), f(b_1)).\end{gathered}$$

Hence

$$\begin{gathered}(f(b_1), f(b_2), f(b_3)) \in B, (f(b_1), f(b_3), f(b_2)) \in B,\\ (f(b_1), f(b_3), f(b_2)) \in B, (f(b_1), f(b_3), f(b_2)) \in B,\\ (f(b_2), f(b_1), f(b_3)) \in B, (f(b_2), f(b_3), f(b_1)) \in B\end{gathered}$$

so

$$\begin{gathered}(f(b_1), f(b_2), f(b_3)) \preceq_3 (b_1, b_2, b_3)\\ (f(b_1), f(b_3), f(b_2)) \preceq_3 (b_1, b_3, b_2)\\ (f(b_3), f(b_2), f(b_1)) \preceq_3 (b_3, b_2, b_1)\\ (f(b_3), f(b_1), f(b_2)) \preceq_3 (b_3, b_1, b_2)\\ (f(b_2), f(b_1), f(b_3)) \preceq_3 (b_2, b_1, b_3)\\ (f(b_2), f(b_3), f(b_1)) \preceq_3 (b_2, b_3, b_1)\end{gathered}$$

We conclude that

$$\begin{gathered}(f(b_1), f(b_2), f(b_3)) = (b_1, b_2, b_3)\\ (f(b_1), f(b_3), f(b_2)) = (b_1, b_3, b_2)\\ (f(b_3), f(b_2), f(b_1)) = (b_3, b_2, b_1)\\ (f(b_3), f(b_1), f(b_2)) = (b_3, b_1, b_2)\\ (f(b_2), f(b_1), f(b_3)) = (b_2, b_1, b_3)\\ (f(b_2), f(b_3), f(b_1)) = (b_2, b_3, b_1)\end{gathered}$$

i.e. $b = (b_1, b_2, b_3)$ is a triple fixed point. It is clear that $F \subseteq B$, and since b is an upper bound of B, it must be an upper bound of F. In other words, b is the largest triple fixed point.

A similar argument shows that the greatest lower bound of

$$A = \left\{ \begin{array}{c} (x,y,z) \in X : \\ (x,y,z) \succeq_3 ((f(x),f(y),f(z)), (x,z,y) \succeq_3 ((f(x),f(z),f(y)), \\ (z,y,x) \succeq_3 ((f(z),f(y),f(x)), (z,x,y) \succeq_3 ((f(z),f(x),f(y)), \\ (y,x,z) \succeq_3 ((f(y),f(x),f(z)), (y,z,x) \succeq_3 ((f(y),f(z),f(x)) \end{array} \right\} \tag{2.1}$$

is also a triple fixed point. Call this triple fixed point $a = (a_1, a_2, a_3)$ (i.e. f has at least two fixed points). Again, $F \subseteq A$, and since a is a lower bound of A, it must be a lower bound of F, i.e. a is the smallest triple fixed point.

It remains to prove that F is, in fact, a complete the tripled lattice. In other words, we have to show if $S \neq \emptyset$ is any subset of F, then S has a least upper bound and a greatest lower bound, both in F. By assumption X is a complete the tripled lattice, hence S has a least upper bound $s_1 \in X$. But $S \subseteq X$, so $(a, s_1, b) \in \preceq_3$. Now, consider the interval $[s_1, b]$. By Proposition 2.6 $[s_1, b]$ is a complete the tripled lattice. We will prove that f maps $[s_1, b]$ into itself. This will imply (by the same argument used at the beginning of the proof) that f has a smallest triple fixed point in $[s_1, b]$, which we denote by $\overline{s}$. We claim that $\overline{s}$ is the least upper bound of S in F. Indeed, $\overline{s}$ is an upper bound since it is in $[s_1, b]$, and it is the least upper bound of S in F, because it is $\preceq_3$ to any other triple fixed point in $[s_1, b]$, and therefore $\preceq_3$ to any triple fixed point that is an upper bound of S. If s_1 is a triple fixed point then $\overline{s} = s_1$.

Now, to see that f maps $[s_1, b]$ into itself, take any $x \in [s_1, b]$. Since b is a triple fixed point of f, and by assumption f is order preserving, the fact that $(x, y, z) \preceq_3 (b_1, b_2, b_3) = b$, implies $(f(x), f(y), f(z)) \preceq_3 (f(b_1), f(b_2), f(b_3)) = (b_1, b_2, b_3) = b$. Similarly, $x \in [s_1, b]$ implies $(x, y, z) \succeq_3 (s_1, x, b)$. Hence $(x, y, z)_3 \succeq_3 (s, x, b)$ for any $s \in S$. Thus, since f is an order preserving map and every $s \in S$ is a triple fixed point,

$$(f(x), f(y), f(z)) \preceq_3 f(s_1, x, b) = (s_1, x, b) = (s, x, b)$$

for any $s \in S$. Therefore $(f(x), f(y), f(z))$ is an upper bound of S, and since s_1 is the least upper bound, $(f(x), f(y), f(z)) \succeq_3 (s_1, x, b)$. Hence, $(s_1, x, b) \preceq_3 (f(x), f(y), f(z)) \preceq_3 (b_1, b_2, b_3) = b$, i.e. $(f(x), f(y), f(z)) \in [s_1, b]$.

The proof that S has a greatest lower bound in F uses a similar argument. Therefore, S has a least upper bound and a greatest lower bound in F. Thus, F is a complete the tripled lattice. ☐

Corollary 2.9. Let x, y, z be in $\mathbb{R}$ with $(x, y, z) \in \preceq_3$. Since the closed interval $[x, y]$ is a complete the tripled lattice relative to $\preceq_3$, every monotone increasing map $f : [x, y] \longrightarrow [x, y]$ must have a greatest triple fixed point and a least triple fixed point. Note here f need not be continuous.

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