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# Some new results on zero near rings

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**Abstract**

In this paper, we considered investigating some properties of zero near rings and novel results in this regard are obtained. In addition, we defined a zero divisor near ring and studied its relation to the zero near ring. In particular, we investigated some conditions on characteristics of elements of a near ring under which it becomes a zero near ring.

Keywords: Near Rings, Zero Near Rings, Near Fields, Zero Divisors, Zero Divisor Near Rings, Idempotent Near Rings, Nilpotent Near Rings, Nil Near Rings
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## 1 Introduction

The theory of near rings and near fields can be traced back to the beginning of the twentieth century. The idea of a near field started in 1905 where the American mathematician L. Dickson considered it for the first time and presented a first example of a near field which is not a field. He also stated and proved the essential properties of near fields. Dickson started his work on division rings by modifying their multiplication and leaving the addition unchanged and he obtained near fields which are not division rings, these near fields are known as Dickson near fields [5].

The study of near rings has witnessed a great deal of interest in different directions. In 1959, Berman and Silverman considered and investigated some properties of near rings in [4]. Beidleman [2] studied in his PhD thesis the algebraic theory for the near ring modules and obtained important results in this area. The distributively generated near rings with descending chain condition is considered by the same author in [3]. The relation between certain finite groups and near rings is investigated in [6] and the authors examined some near rings with identities on specific finite groups. Also, Fain [9] studied some structures of near rings and obtained and proved some relevant theorems in this regard.

In 1960s and 1970s Ligh considered the study of near rings in a series of papers [17, 20, 18, 19, 21, 22]. In [17] the author examined the distributively generated near rings and proved some of their important properties. The division near rings (near fields) are investigated in [20]. The near rings with descending chain condition are studied in [18]. Also, Ligh considered studying the properties and structure of Boolean near rings in [19]. The commutative near rings and their relevant characteristics are investigated in [21, 22].

Holcombe in 1970, studied the primitive near rings and their characteristics in his PhD thesis in [13]. The homomorphism of near rings is studied in [30]. Heatherly in 1978 [12] examined and considered the structure of

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the additive groups in finite near field and proved that the additive group of a finite near field is abelian. The IFP near rings and their structures are studied in [24]. In 1982, Roberts studied the generalized distributivity in near rings in [26]. Some important results on near rings and near fields are presented in [5]. In [7], Dheena studied some properties of some near fields. Samman investigated in his PhD thesis in 1998 the endomorphisms of seminear rings in [27].

A significant number of studies and research papers about near rings and near fields are presented in the conferences held in 1985 and 1997 in [5] and [10]. The strongly reduces near rings is studied in [8]. Recently, Shyryaeu in 2017 studied the zero symmetric idempotent near rings with abelian additive groups [29]. Ayansa in 2018 studied the structure and commutativity of near rings in [1]. In [31] the ideals in near rings is studied. Rasovic and Dasic [25] investigated generalization of division near rings. Also, Hashemi and Shokuhifar in 2019 studied some type elements of zero symmetric near ring of polynomials [11]. In [14] the authors considered studying some properties of near fields. The zero near rings and near fields are examined in [15] and some novel results are obtained.

In this paper, we considered studying some important properties of zero near rings. In addition, we defined a zero divisor near rings and we considered examining its relation to the zero near rings. Also, we examined the relationship between idempotency, nilpotency of the elements of zero near rings. This paper is organised as follows. In section 2 we give the necessary and relevant definitions and preliminaries of the topic. The novel theorems and corollaries are proved in section 3. The conclusions are given in section 4.

## 2 Basic and Relevant Definitions on Near Rings

In this section we give the basic, essential and relevant definitions of near ring theory which are necessary for the work in this paper.

**Definition 2.1 (Left Near Ring).** [23] An algebraic system $(\mathcal{N}, +, .)$ where $\mathcal{N}$ is a nonempty set with two binary operations " $+$ " and " $\cdot$ " is called a left near ring if it satisfies the following conditions:

1. $(\mathcal{N}, +)$ is a group.
2. $(\mathcal{N}, .)$ is a semigroup.
3. $n_3\,(n_1 + n_2) = n_3n_1 + n_3n_2,\ \forall\, n_1, n_2, n_3 \in \mathcal{N}$ (left distributive law).
4. For simplicity, we use $ab$ instead of $a.b$.

Note that we consider in this paper the left near rings and left near fields and the results for the right near rings and right near fields are analogous.

**Remark 2.2.**
- Notice that the definition in above is for the left near ring and the right near ring can be defined in similar way the only difference is that the distributive law is right but not left i.e., $(n_1 + n_2)\,n_3 = n_1n_3 + n_2n_3,\ \forall\, n_1, n_2, n_3 \in \mathcal{N}$.
- The additive group $(\mathcal{N}, +)$ in the near ring $(\mathcal{N}, +, .)$ is not necessary to be an abelian group.
- For simplicity of notation, we will refer to a near ring $(\mathcal{N}, +, .)$ simply as a near ring $\mathcal{N}$ unless, we want to avoid any confusion between the set $\mathcal{N}$ and the near ring $\mathcal{N}$ with two binary operations $(+)$ and $(.)$.
- The near ring $\mathcal{N}$ is referred to it as a **trivial near ring** if $\mathcal{N} = \{0\}$, otherwise if $\mathcal{N} \neq \{0\}$, then it is a **non-trivial near ring**.

**Definition 2.3 (Zero-Symmetric Near Ring).** [23] A left near ring $\mathcal{N}$ is called **zero-symmetric** if $0x = 0,\ \forall\, x \in \mathcal{N}$, this means that the left distributive law yields $x0 = 0$. The set of all zero-symmetric elements of $\mathcal{N}$ is denoted by $\mathcal{N}_0 = \{x \in \mathcal{N} : 0x = 0\}$ is called the zero-symmetric part of $\mathcal{N}$. If $\mathcal{N} \in \eta$ and $\mathcal{N} = \mathcal{N}_0$, then $\mathcal{N}$ is said to be a zero-symmetric near ring. The zero-symmetric near ring is sometimes referred to as **C-ring**.

**Definition 2.4 (Abelian and Commutative Near Ring).** [28] Let $\mathcal{N}$ be a near ring, then $\mathcal{N}$ is said to be an abelian near ring if $a + b = b + a,\ \forall\, a, b \in \mathcal{N}$ and $\mathcal{N}$ is said to be a commutative near ring if $ab = ba,\ \forall\, a, b \in \mathcal{N}$.

**Remark 2.5.**
- A near ring $\mathcal{N}$ with identity is abelian if $n(-1) = -n,\ \forall\, n \in \mathcal{N}$.
- The statement in (1) does not hold if the near ring $\mathcal{N}$ has no identity element.

**Definition 2.6 (Nilpotent Elements).** [16] Let $\mathcal{N}$ be a near ring, an element $n \in \mathcal{N}$ is called a nilpotent element if there exists a least positive integer $k$ such that $n^k = 0$.

**Definition 2.7 (Nilpotent and Nil Set).** [16] Let $\mathcal{N}$ be a near ring, then:

1. A nonempty subset $S$ of $\mathcal{N}$ is called **nilpotent** if there exists a positive integer $k$ such that $S^k = \{0\}$.
2. A nonempty subset $S$ of $\mathcal{N}$ is called nil if every element of $S$ is a nilpotent element.

Note that if $\mathcal{N}$ itself is a nilpotent set, then the near ring $(\mathcal{N}, +, .)$ is called a **nilpotent near ring** and also, if $\mathcal{N}$ is a nil set, then the near ring $(\mathcal{N}, +, .)$ is called a **nil near ring**. Observe that every nilpotent near ring is a nil near ring but the converse is not true.

**Definition 2.8 (Idempotent Elements and Idempotent Near Ring).** [28] Let $\mathcal{N}$ be a near ring, an element $n \in \mathcal{N}$ is called an idempotent if $n^2 = n, \forall n \in \mathcal{N}$. The near ring $\mathcal{N}$ such that all its elements are idempotent is referred to as an idempotent near ring.

**Definition 2.9 (Boolean Near Ring).** [19] Let $(\mathcal{N}, +, .)$ be a near ring, then $\mathcal{N}$ is called a Boolean near ring if $n^2 = n$, $\forall n \in \mathcal{N}$, i.e., $\mathcal{N}$ is an idempotent near ring.

**Definition 2.10 (Zero Divisors).** [28] A non-zero element $n$ in a near ring $\mathcal{N}$ is called:

1. **Left zero divisor** if there exists a non-zero element $a \in \mathcal{N}$ such that $na = 0$.
2. **Right zero divisor** if there exists a non-zero element $b \in \mathcal{N}$ such that $bn = 0$.

Note that if $n$ is both left and right zero divisor, then it is simply called a **zero divisor**. It is worth mentioning that the zero element also is considered as a zero divisor.

**Definition 2.11 (Integral Near Ring).** [16] If the near ring $\mathcal{N}$ has no zero divisors, then $\mathcal{N}$ is called an integral near ring, if for all $a, b$ in $\mathcal{N}$ if $ab = 0$, then either $a = 0$ or $b = 0$.

**Definition 2.12 (Annihilator).** [1] Let $A$ be a subset of a commutative near ring $\mathcal{N}$, then the annihilator of $A$ is the set of all elements $n$ of $\mathcal{N}$ such that $an = 0, \forall\, a \in A$ and is denoted by $Ann(A)$. So, $Ann(A) = \{n \in \mathcal{N} : an = 0, \forall\, a \in A\}$.

**Definition 2.13 (Zero Divisor Near Ring).** If all elements of the near ring $\mathcal{N}$ are left zero divisors then it is said to be a **left zero divisor near ring** and similarly, if all elements of the near ring $\mathcal{N}$ are right zero divisors then it is said to be a **right zero divisor near ring**. A near ring $\mathcal{N}$ is called a **zero divisor near ring** if all non-zero elements of $\mathcal{N}$ are both left and right zero divisors.

**Definition 2.14 (Ideal of a Near Ring).** [28] Let $I$ be a nonempty subset of $\mathcal{N}$, where $(\mathcal{N}, +, .)$ is a near ring, then:

- $I$ is said to be a left ideal of $\mathcal{N}$ if $(I, +)$ is a normal subgroup of $(\mathcal{N}, +)$ i.e., $I$ is a left ideal of $\mathcal{N}$ if and only if $(I, +) \trianglelefteq (\mathcal{N}, +)$ and $\mathcal{N}I \subseteq I$.
- $I$ is said to be a right ideal of $\mathcal{N}$ if $(I, +) \trianglelefteq (\mathcal{N}, +)$ and $(a + i)b - ab \in I, \forall\, a, b \in \mathcal{N}$ and $\forall\, i \in I$.
- $I$ is said to be an ideal of $\mathcal{N}$ if it is both a left and a right ideal of $\mathcal{N}$.

**Definition 2.15 (Prime Ideal).** [16] Let $P$ be an ideal of the near ring $\mathcal{N}$. $P$ is called a prime ideal if for all ideals $I$ and $J$ of $\mathcal{N}$, $IJ \subseteq P$ implies that $I \subset P$ or $J \subset P$.

**Definition 2.16 (Prime Near Ring).** [28] $\mathcal{N}$ is called a prime near ring if $\{0\}$ is a prime ideal.

**Definition 2.17 (Simple Near Ring).** [28] A near ring $\mathcal{N}$ is called simple if $\mathcal{N}$ has no non-trivial ideals i.e., $\mathcal{N}$ has $\{0\}$ and $\mathcal{N}$ are the only ideals of $\mathcal{N}$.

**Definition 2.18 (Zero Near Ring).** [28] A near ring $\mathcal{N}$ is said to be a zero near ring if $\mathcal{N}\mathcal{N} = \{0\}$, or in other words if $n_1 n_2 = 0$, $\forall\, n_1, n_2 \in \mathcal{N}$.

**Definition 2.19 (Near Field).** [28] A near field (**division near ring**) is a nonempty set $F$ together with two binary operations, "$+$" (called addition) and "." (called multiplication) and two identity elements, 0 for addition and 1 for multiplication such that it satisfies the following conditions:

- $(F, +)$ is an abelian group.
- $(F^* = F \setminus \{0\}, .)$ is a group.
- $c(a + b) = ca + cb$, $\forall\, a, b, c \in F$ (The left distributive law).

Note that the near field is an abelian near ring with identity such that each non-zero element has a multiplicative inverse.

For example, the only near field $\mathcal{N}$ for which 1 is not a multiplicative identity is defined as follows. Let $\mathcal{N} = \{0, 1\}$ with addition and multiplication as defined below:

Table 1: Addition Table

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

Table 2: Multiplication Table

| . | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0 | 1 |

We know that any field is a near field and the only field of two elements is the field of integers modulo 2.

**Remark 2.20.** Note that every ring is a near ring but the converse is not true and a near ring which is not a ring sometimes called a **non-ring**. Also, every field is a near field but the converse is not true and a near field which is not a field sometimes called a **non-field**.

## 3 Some Relevant Results and Theorems on Zero Near Rings

In this section, we present some relevant results in near ring theory to the zero near rings and prove some new results in this regard.

**Lemma 3.1.** [28] Let $\mathcal{N}$ be a left near ring, then the following conditions hold:

***(i)*** $x0 = 0$, $\forall x \in \mathcal{N}$.

***(ii)*** $x(-y) = -xy$, $\forall x, y \in \mathcal{N}$.

**Proof .**

***(i)*** Let $x, y \in \mathcal{N}$, then $x0 = x(0 + 0) = x0 + x0$, this implies that $x0 = x0 + x0$, so $x0 - x0 = x0 + x0 - x0$, therefore $0 = x0$, $\forall x \in \mathcal{N}$.

***(ii)*** Let $x, y \in \mathcal{N}$, then $xy + x(-y) = x(y + (-y)) = x0 = 0$, hence $x(-y) = -xy$.

□

**Theorem 3.2.** [23] If a near ring $\mathcal{N}$ is simple, then either $\mathcal{N}$ is prime or $\mathcal{N}$ is a zero near ring.

**Proof .** Assume that $\mathcal{N}$ a simple near ring and it is not a zero near ring. Then $\mathcal{N}\mathcal{N} \neq \{0\}$. We have to prove that (0) is a prime ideal of $\mathcal{N}$. Suppose $I$ and $J$ are ideals of $\mathcal{N}$ such that $IJ \subseteq (0)$. Since $I$ and $J$ are ideals of $\mathcal{N}$, and $\mathcal{N}$ is simple, it follows that $I, J \in \{(0), \mathcal{N}\}$. If $I = \mathcal{N}$ and $J = \mathcal{N}$, and $\mathcal{N}\mathcal{N} \subseteq IJ \subseteq (0)$, which is a contradiction. Hence, $I = (0)$ or $J = (0)$. So, (0) is a prime ideal of $\mathcal{N}$. Therefore, $\mathcal{N}$ is a prime near ring. □

**Theorem 3.3.** Let $\mathcal{N}$ be a near ring, then $\mathcal{N}$ is a zero near ring if and only if $\mathcal{N}$ is a zero divisor and nilpotent (nil) near ring with $k = 2$.

**Proof .** Assume that $\mathcal{N}$ be a zero near ring, and let $a, b \in \mathcal{N}$ such that $a \neq b$, then we have $ab = 0$, so $\mathcal{N}$ is a zero divisor near ring. If $a = b$, then we obtain $bb = b^2 = 0$. Hence, $\mathcal{N}$ is a nilpotent (nil) near ring with $k = 2$. Conversely, let $\mathcal{N}$ be a zero divisor and nilpotent (nil) near ring with $k = 2$. This implies that $ab = 0$, $\forall a, b \in \mathcal{N}$ (if $a \neq b$). If $a = b$, this results in $bb = 0$. So, $\mathcal{N}$ is a zero near ring. □

**Theorem 3.4.** If $\mathcal{N}$ is a non-trivial zero near ring then $\mathcal{N}$ has no idempotent element, or in other words, a non-trivial zero near ring is not an idempotent near ring.

**Proof .** Let $\mathcal{N}$ be a non-trivial zero near ring, we will prove the theorem by contradiction. Let $a \in \mathcal{N}$ be an idempotent element, then $a^2 = a$ implies that $a^2 - a = a(a-1) = 0$. So, either $a = 0$ or $a - 1 = 0$, if $a = 0$ this implies that $\mathcal{N} = \{0\}$ a contradiction since $\mathcal{N} \neq \{0\}$. Also, $1 \notin \mathcal{N}$ since if $1 \in \mathcal{N}$ this results in $1a = a, \forall\, a \in \mathcal{N}$, so we have $1a = a \neq 0$ a contradiction, hence $\mathcal{N}$ has no idempotent element. □

**Theorem 3.5.** The near ring $\mathcal{N}$ is a zero near ring if and only if $Ann(\mathcal{N}) = \mathcal{N}$.

**Proof .** Let $\mathcal{N}$ is a zero near ring, so, if $n \in \mathcal{N}$, then $nm = 0$, $\forall\, n, m \in \mathcal{N}$. Hence, $n \in Ann(\mathcal{N})$, which implies that $\mathcal{N} \subset Ann(\mathcal{N})$. Now, let $n \in Ann(\mathcal{N})$, then $nm = 0$, $\forall m \in \mathcal{N}$ results in $Ann(\mathcal{N}) \subset \mathcal{N}$ since $\mathcal{N}$ is a zero near ring. Therefore, we have $Ann(\mathcal{N}) = \mathcal{N}$. Conversely, assume that $\mathcal{N}$ is a near ring such that $Ann(\mathcal{N}) = \mathcal{N}$ which leads to $\mathcal{N}$ is a zero near ring. □

**Corollary 3.6.** A non-trivial zero near ring $\mathcal{N}$ can not be a Boolean near ring.

**Proof .** Let $\mathcal{N}$ be a non-trivial zero near ring, then by Theorem 3.4 the zero near ring is not an idempotent near ring and since the Boolean near ring is an idempotent near ring then $\mathcal{N}$ is not a Boolean near ring. □

**Corollary 3.7.** If $\mathcal{N}$ is a zero near ring then $\mathcal{N}$ has no multiplicative identity.

**Proof .** Let $\mathcal{N}$ be a zero near ring with a multiplicative identity, so $a1 = a, \forall\, a \in \mathcal{N}$, hence $a1 = a \neq 0$ a contradiction, since $\mathcal{N}$ a zero near ring. So, $\mathcal{N}$ has no multiplicative identity. □

**Corollary 3.8.** If $\mathcal{N}$ is a non-trivial zero near ring then $\mathcal{N}$ can not be an integral near ring.

**Proof .** By the definition of the zero near ring, the non-zero elements of $\mathcal{N}$ are divisors of zero, so $\mathcal{N}$ is not an integral near ring. □

**Corollary 3.9.** If $\mathcal{N}$ is a zero near ring then $\mathcal{N}$ can not be a near field (a division near ring).

**Proof .** From Corollary 3.7, we have $\mathcal{N}$ has no multiplicative identity, so $\mathcal{N}$ is not a near field, since it has no multiplicative identity. □

**Remark 3.10.**

- A zero near ring is a nilpotent near ring with $k = 2$.
- The non-zero elements of a zero near ring are divisors of zero. So, a zero near ring is a zero divisor near ring but the converse is not true.

## 4 Conclusions

In this paper, we investigated the structure of zero near rings and some properties and features of them are presented. Also, we defined zero divisor near rings and studied the connection between them and zero near rings. In particular, some new results about zero near rings are obtained and proved under certain conditions.

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