Int. J. Nonlinear Anal. Appl. 14 (2023) 4, 313–322
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.24178.2685

Equivalence relations on approximation theory

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(Communicated by Michael Th. Rassias)

Abstract

In this paper, we define relations between the best approximation and the worst approximation. We show that these relations are equivalence relations if the sets are Chebyshev or uniquely remotal. We obtain cosets sets of best approximation and cosets sets of worst approximation. We obtain some results on these sets, for example, compactness and weakly compactness. Finally, we consider the semi-inner products (Lumer-Giles) and semi-inner(usual).

Keywords: Chebyshev sets, Uniquely remotal sets, Cosets best approximation sets, Cosets worst approximation sets, Equivalence relations, Semi-inner product
2020 MSC: 41A65, 41A52, 46N10

1 Introduction

Approximation theory, which mainly consists of theory of nearest points (best approximation) and theory of farthest points (worst approximation), is an old and rich branch of analysis. The theory is as old as Mathematics itself. Starting in 1853, a Russian mathematician P.L. Chebyshev made significant contributions in the theory of best approximation. The Weierstrass approximation theorem of 1885 by K. Weierstrass is well known.The study was followed in the first half of the 20th Century by L.N.H. Bunt (1934) , T.S . Motzkin (1935) and B. Jessen (1940). B. Jessen was the first to make significant contributions in the theory of farthest points. This theory is less developed as compared to the theory of best approximation .

Let $(X, \|.\|)$ be a normed linear space, W a non-empty subset of X. A point $y_0 \in W$ is said to be a best approximation point (nearest point) for $x \in X$, if

$$\|x - y_0\| \leq \|x - y\|,$$

for each $y \in W$.
For each $x \in X$, put

$$P_W(x) = \{y_0 \in W : \ \|x - y_0\| = dist(x, W) = \inf_{y \in W} \|x - y\|\}.$$

For each $x \in X$, if $P_W(x)$ is non-empty (a singleton), we say that W is proximinal (Chebyshev). For each $x \in X \backslash W$, if $P_W(x) = \emptyset$, we say that W is anti-proximinal. Suppose $g \in W$, we set

$$P_g = \{x \in X : \ g \in P_W(x)\},$$

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Received: August 2021 *Accepted:* December 2021

(see [2, 5, 9, 16]).

Let X be a normed linear space and W a bounded non-empty subset of X. A point $q(x) \in W$ is said to be a farthest point for $x \in X$, if

$$\|x - q(x)\| \geq \|x - y\|,$$

for each $y \in W$. For each $x \in X$, put

$$F_W(x) = \{y_0 \in W : \ \|x - y_0\| = \delta(W, x) = \sup_{y \in W} \|x - y\|\}.$$

For each $x \in X$, if $F_W(x)$ is non-empty (a singleton), we say that W is remotal (uniquely remotal). For each $x \in X$, if $F_W(x) = \emptyset$, we say that W is anti-remotal. Suppose $g \in W$, we set

$$F_g = \{x \in X : \ g \in F_W(x)\},$$

(see [4, 5, 7, 9, 10, 13, 14, 15]).

2 Equivalence relations on approximate sets

In this section we define two equivalence relations on approximate sets. We obtains some results on these relations.

Definition 2.1. Let $(X, \|.\|)$ be a normed space, W a proximinal subset in X and $x, y \in X$. We define two relations on X, with
(i)

$$x \rhd_1 y \Leftrightarrow P_W(x) = P_W(y), \ dist(x, W) = dist(y, W)$$

(ii)

$$x \rhd_2 y \Leftrightarrow \text{for some } g \in W : \ g \in P_W(x) \cap P_W(y), \ dist(x, W) = dist(y, W)$$

It is clear that if W is a Chebyshev subset of X, then $\rhd_1 = \rhd_2$.
We denoted the equivalence class of $x \in X$ under relation $\rhd_1(\rhd_2)$ by $[x]_1([x]_2)$

Theorem 2.2. Let $(X, \|.\|)$ be a normed space, W a proximinal subset in X. The relations $\rhd_1$ is equivalence relation.

Proof . These relation is reflexive and symmetric. We show that transitivity relation $\rhd_1$. For all elements $a, b, c \in X$, if $a \ \rhd_1 b$ and $b \ \rhd_1 c$, then $P_W(a) = P_W(b) = P_W(c)$ and $dist(a, W) = dist(b, W) = dist(c, W)$. It follows that $a \rhd_1 c$. □

Theorem 2.3. Let $(X, \|.\|)$ be a normed space, W a proximinal subset in X. Then for every $x \in X$, there exists a $g \in W$ such that $[x]_1 = P_g$.

Proof . Suppose $x \in X$, we have

$$\begin{aligned} [x]_1 &= \{y \in X : \ x \rhd_1 y\} \\ &= \{y \in X : \text{for some } g \in W \ g \in P_W(x) \cap P_W(y), dist(x, W) = dist(y, W)\} \\ &= \{y \in X : g \in P_W(y), dist(x, W) = dist(y, W)\} \\ &= \{y \in X : y \in P_g, dist(x, W) = dist(y, W)\}. \end{aligned}$$

□

Theorem 2.4. Let $(X, \|.\|)$ be a normed space, W a proximinal subset in X. Then for every $x \in X$, there exists a $g \in W$ such that $[x]_2 = P_g$.

Proof . Suppose $x \in X$, we have

$$\begin{aligned} [x]_2 &= \{y \in X : \ x \rhd_1 y\} \\ &= \{y \in X : \text{for some } g \in W \ g \in P_W(x) = P_W(y), \ dist(x, W) = dist(y, W)\} \\ &= \{y \in X : \ g \in P_W(y), \ dist(x, W) = dist(y, W)\} \\ &= \{y \in X : y \in P_g, \ dist(x, W) = dist(y, W)\}. \end{aligned}$$

□

Example 2.5. Suppose $X = \mathbb{R}^2$ with the norm $\|(x,y)\| = \sqrt{x^2+y^2}$ and $W = \{(x,0): \ x \in \mathbb{R}\}$. Since W is closed and convex, W is Chebyshev and for every $(x,y) \in X$, $dist((x,y),W) = |y|$. Therefore

$$\begin{aligned}[(0,1)]_1 &= [(0,1)]_2 \\ &= \{(x,y): P_W(0,1) = P_W(x,y),\ dist((0,1),W) = dist((x,y),W)\} \\ &= \{(x,y): P_W(x,y) = \{(0,0), dist((x,y),W) = 1\}\} \\ &= \{(0,1),(0,-1)\}.\end{aligned}$$

Theorem 2.6. Let $(X, ||.||)$ be a normed linear space and W a Chebyshev subspace of X, $x, y \in X$ and $g_0 \in W$. If $g_0 = P_W(x)$ and $y \in [g_0, x)$, then $g_0 = P_W(y)$ and $x \lhd_i y$ for i=1,2. (where $[g_0,x) = \{\lambda g_0 + (1-\lambda)x: \ \lambda \geq 0\}$.)

Proof . Since $g_0 = P_W(x)$ and $y \in [g_0, x)$, for some $\lambda > 0$, $y = \lambda g_0 + (1-\lambda)x$ and $\|x - g_0\| = d(x,W)$. Therefore

$$\begin{aligned}\|y-g_0\| &= = \|\lambda g_0 + (1-\lambda)x - \lambda g_0 - (1-\lambda)g_0\| \\ &= \|(1-\lambda)(x-g_0)\| \\ &= (1-\lambda)\|x-g_0\| \\ &= d((1-\lambda)x, W) \\ &= d(\lambda g_0 + (1-\lambda)x, W) \\ &= d(y,W).\end{aligned}$$

Therefore $g_0 = P_W(y)$, $dist(x,W) = dist(y,W)$ and $x \lhd_i y$. □

Definition 2.7. [13, 16] For any two elements x and y in normed linear space X, x is said to be orthogonal to y in the sense of Birkhorff-James, written as $x \perp y$, if $\|x + \lambda y\| \geq \|x\|$ for every real scaler λ.

Definition 2.8. [4, 8] Let $(X, \|.\|)$ be a normed linear space and G a nonempty subset of X. Then the set $\{x \in X: x \perp G(B)\}$ is called the Birkhoff orthogonal complement of G, denoted by $G^{\perp}(B)$.

Lemma 2.9. ([4],[8]) Let $(X, \|.\|)$ be a normed linear space and W a subspace of X, $x \in X$ and $g_0 \in W$. Then the following statemnts are equivalence:

i) $y_0 \in P_W(x)$,

ii) $x - g_o - \lambda(y - y_0) \in W^{\perp}(B)$ for every $y \in W$ and every $\lambda \in [0,1]$,

iii) $x - g_0 \in W^{\perp}(B)$.

Remark 2.10. Let $(X, \|.\|)$ be a normed linear space and W a subspace of X, α is a scaler and $x \in W^{\perp}(B)$. Then $\alpha x \in W^{\perp}(B)$.

Theorem 2.11. Let $(X, \|.\|)$ be a normed linear space and W a proximinal subspace of X, $W^{\perp}(B)$ a convex set and $x, y \in X$. If $x \rhd_i y$ for every $i = 1,2$, then $x - y \in W^{\perp}(B)$.

Proof . Suppose $x \rhd_i y$ for every $i = 1,2$, then there exists a $g_0 \in P_W(x) \cap P_W(y)$ and $dist(x,W) = dist(y,W)$. From Lemma 2.1 and Remark 2.1, $x - y = 2\frac{x-g_0+y-g_0}{2} \in W^{\perp}(B)$. □

3 Equivalence relations on worst approximate sets

In this section we define two equivalence relations on worst best approximate sets. We obtains some results on these relations.

Definition 3.1. Let $(X, \|.\|)$ be a normed space, W a bounded subset in X and $x, y \in X$. We define two relations on X, with
(i)

$$x \lhd_1 y \Leftrightarrow F_W(x) = F_W(y),\ \delta(x,W) = \delta(y,W)$$

(ii)

$$x \lhd_2 y \Leftrightarrow g \in W : g \in F_W(x) \cap F_W(y),\ \delta(x,W) = \delta(y,W)$$

It is clear that if W is a uniquely remotal subset of X, then $\lhd_1 = \lhd_2$. We denoted the equivalence class of $x \in X$ under ralation $\lhd_1(\lhd_2)$ by $[x]'_1([x]'_2)$.

Theorem 3.2. Let $(X, \|.\|)$ be a normed space, W a remotal subset in X. The relations $\lhd_1$ is equivalance relation.

Proof . These relation is reflexive and symmetric. We show that trnsitivity relation $\lhd_1$. For all elements $a, b, c \in X$, if $a \lhd_1 b,\ \delta(a, W) = \delta(b, W)$ and $b \lhd_1 c,\ \delta(b, W) = \delta(c, W)$, then $F_W(a) = F_W(b) = F_W(c),\ \delta(a, W) = \delta(b, W) = \delta(c, W)$. It follows that $a \lhd_1 c$. □

Theorem 3.3. Let $(X, \|.\|)$ be a normed space, W a remotal subset in X. Then for every $x \in X$, there exists a $g \in W$ such that $[x]'_2 = F_g$.

Proof . Suppose $x \in X$, we have

$$\begin{aligned} [x]'_2 &= \{y \in X:\ x \lhd_2 y\} \\ &= \{y \in X: \text{for some } g \in W\ g \in F_W(x) \cap F_W(y),\ \delta(x, W) = \delta(y, W)\} \\ &= \{y \in X: g \in F_W(y),\ \delta(x, W) = \delta(y, W)\} \\ &= \{y \in X: y \in F_g,\ \delta(x, W) = \delta(y, W)\}. \end{aligned}$$

□

Theorem 3.4. Let $(X, \|.\|)$ be a normed space, W a remotal subset in X. Then for every $x \in X$, there exists a $g \in W$ such that $[x]'_1 = F_g$.

Proof . Suppose $x \in X$, we have

$$\begin{aligned} [x]'_1 &= \{y \in X:\ x \lhd_1 y\} \\ &= \{y \in X: \text{for some } g \in W\ g \in F_W(x) = F_W(y),\ \delta(x, W) = \delta(y, W)\} \\ &= \{y \in X:\ g \in F_W(y),\ \delta(x, W) = \delta(y, W)\} \\ &= \{y \in X: y \in F_g,\ \delta(x, W) = \delta(y, W)\}. \end{aligned}$$

□

Theorem 3.5. Let $(X, ||.||)$ be a normed linear space and W a uniquely remotal subset of X, $x, y \in X$ and $g_0 \in W$. If $g_0 = F_W(x)$, $\alpha g_0 + W = W$ for every scalar α and $y \in [g_0, x)$, then $g_0 = F_W(y)$ and $x \lhd_i y$ for i=1,2. (where $[g_0, x) = \{\lambda g_0 + (1-\lambda)x:\ \lambda \geq 0\}$.)

Proof . Since $g_0 = F_W(x)$ and $y \in [g_0, x)$, for some $\lambda > 0$, $y = \lambda g_0 + (1-\lambda)x$ and $\|x - g_0\| = \delta(x, W)$. Therefore

$$\begin{aligned} \|y - g_0\| &= = \|\lambda g_0 + (1-\lambda)x - \lambda g_0 - (1-\lambda)g_0\| \\ &= \|(1-\lambda)(x - g_0)\| \\ &= (1-\lambda)\|x - g_0\| \\ &= \delta((1-\lambda)x, W) \\ &= \delta(\lambda g_0 + (1-\lambda)x, W) \\ &= \delta(y, W). \end{aligned}$$

Therefore $g_0 = P_W(y)$, $\delta(x, W) = \delta(y, W)$ and $x \lhd_i y$. □

Definition 3.6. Let $\{x_n\}_{n\in L}$ and $\{y_n\}_{n\in L}$ are bounded sequences in the Banach space X, $\{x_n\}_{n\in L}$ is said to be farthest orthogonal to $\{y_n\}_{n\in L}$ and denote by $\{x_n\}_{n\in L} \perp_F \{y_n\}_{n\in L}$ if and only if there exist $z_k \in \{x_n\}_{n\in L} \cup \{y_n\}_{n\in L}$ such that

$$\|z_k\| \geq \|\sum_{n\in L} (-1)^n (x_n - y_n)\|.$$

Also for $W \subset X$ and $\{x_n\}_{n\in L}$, we write $\{x_n\}_{n\in L} \perp W$ if $\{x_n\}_{n\in L} \perp \{y_n\}_{n\in L}$ for all $\{y_n\}_{n\in L} \subset W$.

It should be noted that if $0 \in W$, then $x_0 \perp_F W$ if and only if $0 \in F_W(x_0)$, therefore $g_0 \in F_W(x)$ if and only if $x - g_0 \perp_F W$.

Theorem 3.7. Let $(X, \|.\|)$ be a normed linear space and W a proximinal subspace of X, $W^{\perp}(B)$ a convex set and $x, y \in X$. If $x \rhd_i y$ for every $i = 1, 2$, then $x - y \in W^{\perp}(B)$.

Proof . Suppose $x \lhd_i y$ for every $i = 1, 2$, then there exists a $g_0 \in P_W(x) \cap P_W(y)$. From Lemma 2.1 and Remark 2.1, $x - y = 2\frac{x-g_0+y-g_0}{2} \in W^{\perp}(B)$. □

4 Properties of the sets P_g and F_g

In this section we are bring some propeties of P_g and F_g.

Theorem 4.1. Let $(X, \|.\|)$ be a normed linear space.

i) If W is a subset of X. we have

$$\cup_{g \in W} P_g = \{x \in X : for\ some\ g \in W : \ \|x - g\| = d(x, W)\},$$

and if W is proximinal

$$X = \cup_{g \in W} P_g.$$

If W is a bounded subset of X. We have

$$\cup_{g \in W} F_g = \{x \in X : \ for\ some\ g \in W : \ \|x - g\| = \delta(x, W)\},$$

also if W is remotal,

$$X = \cup_{g \in W} F_g.$$

Proof . We have

$$\begin{aligned} x \in \cup_{g \in W} P_g &\iff \text{for some } g \in W \ x \in P_g \\ &\iff g \in W \ \|x - g\| = d(x, W). \end{aligned}$$

If W is proximinal and $x \in X$, for some $g \in W$, $x \in P_g$. Therefore

$$X = \cup_{g \in W} P_g.$$

Also

$$\begin{aligned} x \in \cup_{g \in W} F_g &\iff \text{for some } g \in W \ x \in F_g \\ &\iff g \in W \ \|x - g\| = \delta(x, W). \end{aligned}$$

If W is remotal and $x \in X$, for some $g \in W$, $x \in F_g$. Therefore

$$X = \cup_{g \in W} F_g.$$

□

Theorem 4.2. Let $(X, \|.\|)$ be a normed linear space. If W is a bounded subset of X. We have

$$\cap_{g \in W} P_g = \cap_{g \in W} F_g = \{x \in X : \delta(x, W) = d(x, W)\},$$

Proof .

$$\begin{aligned} x \in \cap_{g \in W} P_g &\iff \forall g \in W \ x \in P_g \\ &\iff \forall g \in W \forall w \in W \ \|x - g\| \leq \|x - w\| \\ &\iff \forall g \in W \forall w \in W \ \|x - g\| \leq \|x - w\| \\ &\iff \delta(x, W) \leq d(x, W) \\ &\iff \forall g \in W \ x \in F_g \\ &\iff x \in \cap_{g \in W} F_g. \end{aligned}$$

□

Theorem 4.3. Let $(X, \|.\|)$ be a normed linear space. If W is a subset of X. Then

i) if $W = -W$, then $-P_g = P_{-g}$ and $-F_g = F_{-g}$;

ii) for every $g \in W$, F_g and P_g are closed sets;

iii) for every $g \in W$, $P_g \cap W = \{g\}$;

iv) if for every $g \in W$, $P_g \cap W = \{g\}$, then $W = \{g\}$;

v) the priximinal set W is Chebyshev if and only if for every $g_1, g_2 \in W$ and $g_1 \neq g_2$, we have $P_{g_1} \cap P_{g_2} = \emptyset$;

vi) the remotal set W is uniquely remotal if and only if for every $g_1, g_2 \in W$ and $g_1 \neq g_2$, we have $F_{g_1} \cap F_{g_2} = \emptyset$;
vii) if W is convex proximinal and P_0 is convex, then W is Chebyshev

Proof . The parts i), ii), iii) and iv) are trivial. We proof v), suppose W is Chebyshev, $g_1, g_2 \in W$, $g_1 \neq g_2$ and $x \in P_{g_1} \cap P_{g_2}$. Then $g_1, g_2 \in P_W(x)$ and that is a contraction. On converse, if $g_1, g_2 \in W$ and $g_1 \neq g_2$ and $P_{g_1} \cap P_{g_2} = \emptyset$. Suppose for $x \in X$, there exist $g_1, g_2 \in P_W(x)$. Then $x \in P_{g_1} \cap P_{g_2}$ and $g_1 = g_2$. It follows that W is Chebyshev.

vi) Suppose W is uniquely remotal, $g_1, g_2 \in W$, $g_1 \neq g_2$ and $x \in F_{g_1} \cap F_{g_2}$. Then $g_1, g_2 \in P_W(x)$ and that is a contraction. On converse, if $g_1, g_2 \in W$ and $g_1 \neq g_2$ and $F_{g_1} \cap F_{g_2} = \emptyset$. Suppose for $x \in X$, there exist $g_1, g_2 \in P_W(x)$. Then $x \in F_{g_1} \cap F_{g_2}$ and $g_1 = g_2$. It follows that W is Chebyshev.

vii) By (iii) for $P_0 \cap W = \{0\}$. Suppose $x \in X$ and $g_1, g_2 \in P_W(x)$. Then

$$g_1 - g_2 = 2\frac{x - g_1 - (x - g_2)}{2} \in W \cap P_0.$$

It follows that $g_1 = g_2$. □

Definition 4.4. [9, 12] Let X be a Banach space and W a closed subspace of X. W is called quasi-Chebyshev(weakly-Chebyshev) if for every $x \in X$, the set $P_W(x)$ is a non-empty compact(weakly compact) subset of X.

Lemma 4.5. [9, 12] Let X be a Banach space and W a proximinal hyperplane subspace of X. Then the following statements are equivalent:

i) W is quasi-Chebyshev(weakly-Chebyshev).

ii) for every $g \in W$ and for every sequence $\{x_n\}_{n\geq 1}$ with $\|x_n\| = 1$ and $0 \in P_W(x_n)$ has a convergent subsequence(weakly convergent subsequence).

Theorem 4.6. Let X be a Banach space and W a proximinal hyperplane subspace of X. Then the following statements are equivalent:

i) W is quasi-Chebyshev(weakly-Chebyshev).

ii) for every $g \in W$ and for every sequence $\{x_n\}_{n\geq 1}$ with $\|x_n\| = 1$ and $x_n \in P_n$ has a convergent subsequence(weakly convergent subsequence).

Proof . *i) ⇒ ii)*. Suppose $g \in W$ and $\{x_n\}_{n\geq 1}$ is a sequence with $\|x_n\| = 1$ and $x_n \in P_n$. Then for every $n \geq 1$, $\|x_n - g\| = d(x_n, W)$, therefore

$$\|\frac{x_n - g}{d(x_n, W)}\| = 1 = \frac{d(x_n, W)}{d(x_n, W)} = d(\frac{x_n - g}{d(x_n, W)}, W).$$

Therefore $0 \in P_W(\frac{x_n - g}{d(x_n,W)}$ and $\|\frac{x_n - g}{d(x_n,W)}\| = 1$. From Lemma 4.1, the sequence $\{\frac{x_n - g}{d(x_n,W)}\}_{n\geq 1}$ has a convergent subsequence(weakly convergent subsequence). There exists a $x_0 \in X$ such that

$$\frac{x_{n_k} - g}{d(x_{n_k}, W)} \to x_0 (\frac{x_{n_k} - g}{d(x_{n_k}, W)} \rightharpoonup x_0).$$

Also

$$d(x_{n_k}, W) \leq \|x_{n_k}\| = 1.$$

Then the sequence $\{d(x_{n_k}, W)\}$ has a convergent subsequece $\{d(x_{n_{k_l}}, W)\}$. Therefore exists $k_0 \in \mathbb{R}$ such that

$$d(x_{n_{k_l}}, W) \to k_0 \ as \ l \to \infty.$$

Since

$$x_{n_{k_l}} - g = d(x_{n_{k_l}}, W)\frac{x_{n_{k_l}} - g}{d(x_{n_{k_l}}, W)}.$$

We have

$$x_{n_{k_l}} \to g + k_0x_0(x_{n_{k_l}} \rightharpoonup \ g + k_0x_0).$$

It follows that the sequence $\{x_n\}_{n\geq1}$ has a convergent subsequence(weakly convergent subsequence). $ii) \Rightarrow i)$. We set $g = 0$ and for every sequence $\{x_n\}_{n\geq1}$ with $\|x_n\| = 1$ and $x_n \in P_0$ has a convergent subsequence(weakly convergent subsequence). From Lemma 4.1, W is quasi-Chebyshev(weakly Chebyshev). □

Theorem 4.7. Let X be a Banach space and W a proximinal subspace of X. Then the following statements are equivalent:

i) W is quasi-Chebyshev(weakly-Chebyshev).

ii) for every $g \in W$, for every subspace of X of form $W_x = W + span\{x\}$ and for every sequence $\{x_n\}_{n\geq1} \subset W_x$ with $\|x_n\| = 1$ and $x_n \in P_n^{W_x}$ has a convergent subsequence(weakly convergent subsequence).

Proof . $i) \to ii)$ If W is quasi-Chebyshev(weakly Chebyshev) in X. Then W quasi-Chebyshev (weakly-Chebyshev) in every W_x $(x \in X\backslash W)$, Since $codim(W) = 1$ in every W_x. From Theorem 4.4, for every sequence $\{x_n\}_{n\geq1} \subset W_x$ with $\|x_n\| = 1$ and $x_n \in P_n^{W_x}$ has a convergent subsequence(weakly convergent subsequence).
$ii) \to i)$ Assume that we have (ii), $codim(W) = 1$ in every W_x. Also W is proximinal in W_x and $X = \cup_{x\in X\backslash W} W_x$. It follows that W is quasi-Chebyshev in X. □

Theorem 4.8. Let X be a Banach space and W a proximinal subspace of X. If for every $g \in W$, P_g is compact(weakly compact). Then W is quasi-Chebyshev(weakly-Chebyshev).

Proof . Suppose $g \in W$ and P_g is compact(weakly compact) in X. We know that $P_g = g + P_0$. If $x \in X$ and $\{g_n\}_{n\geq1}$ is a sequence in $P_W(x)$, then $\{x - g_n\}_{n\geq1} \subset P_0$. Therefore there exists a convergence subsequence(weakly convergence sequence) $\{x - g_{n_k}\}_{k\geq1}$. It follows that $\{g_{n_k}\}_{k\geq1}$ is a convergence sequence(weakly convergence sequence). Therefore From Lemma 4.1, W is quasi-Chebyshev(weakly-Chebyshev). □

Theorem 4.9. Let X be a Banach space, W a remotal subset of X and for every $g \in W$, $W - g = W$. If for every $g \in W$, F_g is compact (weakly compact). Then $F_W(x)$ is compact (weakly compact).

Proof . Suppose $g \in W$ and F_g is compact(weakly compact). Since $W - g = W$ we have $F_g = g + F_0$, becasuse

$$\begin{aligned} x \in F_g \iff \|x - g\| &= \delta(x, W) \\ &= \delta(x - g, W - g) \\ &= \delta(x - g, W). \end{aligned}$$

If $\{g_n\}_{n\geq1}$ is a sequence in $F_W(x)$. Then $\{x - g_n\}_{n\geq1}$ is a sequence in F_0. Since F_0 is compact, there exists a convergence subsequence(weakly convergence subsequence) $\{x - g_{n_k}\}_{k\geq1}$ and $\{g_{n_k}\}_{k\geq1}$. Therefore $F_W(x)$ is compact(weakly compact). □

Example 4.10. Let $(X, \|.\|)$ be a normed space, $W = \{x \in X : \|x\| = 1\}$ and $x \in X$. We show that

$$F_g = \{-\lambda g : \lambda \geq 1\},$$

and

$$P_g = \{\lambda g : \lambda \geq 1\}.$$

If $g \in W$, put $x = -\lambda g$ for every $\lambda \geq 1$. Therefore $q(x) = g$ and $x \in F_g$. If $x \in F_g$, since $q(x) = -\frac{x}{\|x\|} = g$. Therefore $x = -\|x\|g$ and $\|x\| \geq 1$. It follows that

$$F_g = \{-\lambda g : \lambda \geq 1\}.$$

Put $x = \lambda g$, for every $\lambda \geq 0$. Therefore nearest point $(x) = g$ and $x \in P_g$. If $x \in P_g$, since nearest point $(x) = \frac{x}{\|x\|} = g$ and $\|x\| \geq 1$. Therefore $x = \|x\|g$ and $\|x\| \geq 1$. It follows that

$$P_g = \{\lambda g : \lambda \geq 1\}.$$

Suppose $g = F_W(x)$, then $x \in F_g$. Therefore for some $\lambda_0 \geq 1 : x = -\lambda_0 g$.

$$\begin{aligned} \lambda x + (1-\lambda)g &= -\lambda\lambda_0 g + g - \lambda g \\ &= -(-1+\lambda+\lambda\lambda_0)g, \end{aligned}$$

Note that $-1+\lambda+\lambda\lambda_0 \geq 1$. Therefore

$$F_W(\lambda x + (1-\lambda)g) = g$$

and W is a sunrise set. Suppose $g = P_W(x)$, then $x \in P_g$. Therefore for some $\lambda_0 \geq 1$ we have $x = \lambda_0 g$. For $\lambda \geq 0$, we have

$$\begin{aligned} \lambda x + (1-\lambda)g &= \lambda\lambda_0 g + g - \lambda g \\ &= (\lambda\lambda_0 - \lambda + 1)g. \end{aligned}$$

Note that $\lambda\lambda_0 - \lambda + 1 \geq 0$. Therefore $P_W(\lambda x + (1-\lambda)g) = g$ and W is a sun set in X.

5 Applications

In what follows, we assume that X is a linear space over the real or complex number field K. The following concept was introduced in 1961 by G. Lumer [9] but the main properties of it were discovered by J.R. Giles [2,10]. In this introductory section we give the definition of this concept and point out the main facts which are derived directly from the definition.

Definition 5.1. The mapping $[.,.] : X \times X \to K$ will be called the semi-inner product in the sense of Lumer-Giles, for short, if the following properties are satisfied:

(i) $[x+y, z] = [x, z] + [y, z]$ for all$x, y \in X$;

(ii) $[\lambda x, y] = \lambda[x, y]$ for all $x, y \in X$ and λ a scalar in K;

(iii) $[x, x] \geq 0$ for all $x \in X$ and $[x, x] = 0$ implies that $x = 0$;

(iv) $|[x, y]|^2 \leq [x, x][y, y]$ for all $x, y \in X$;

(v) $[x, \lambda y] = \bar{\lambda}[x, y]$ for all $x, y \in X$ and λ a scalar in K.

Now, we will state the first result.

Lemma 5.2. Let X be a linear space and $[.,.]$ a semi-norm on X. Then the following statements are true:

(i) The mapping $x \to [x, x]^{\frac{1}{2}}$ is a norm on X;

(ii) For. every $y \in X$ the functional $x \to [x, y] \in K$ is a continuous linear functional on X

In following khnown Lemmas we bring some theorms about best approximation in semi-inner proudact spaces.

Lemma 5.3. Let H be a Hilbert space, C a non-empty closed convex subset of H, $x \in H$ and $w \in C$. Then the following conditions are equivalent:

i) $w = P_C(x)$;

ii) $[x-w, y-w] \leq 0$ for every $y \in C$.

Lemma 5.4. Let X be a Banach space and $[.,.] : X \times X \to R$ a semi-inner product on X which generates the norm. Let C be a nonempty closed convex set, $x \in X$ and $w \in C$. Then the following statments are equivalent:

$w \in P_C(x)$;

$[z-w, x-w] \leq 0$ for every $z \in C$.

Theorem 5.5. Let H be a Hilbert space, W a non-empty closed convex subset of H, $x \in H$ and $w \in W$. Then the following conditions are equivalent: Then the following statments are equivalent:

i) $x \in P_w$;

ii) $dist^2(x, W) \leq [x - z, x - w]$.

Proof . If $w \in P_W(x)$, then from Lemma 5.2, $[z - w, x - w] \leq 0$ for every $z \in W$. Therefore

$$\begin{aligned} \|x - w\|^2 &= [x - w, x - w] \\ &= [x - w + z - z, x - w] \\ &= [x - z, x - w] + [z - w, x - w] \\ &\leq [x - z, x - w]. \end{aligned}$$

Also for every $z \in W$,

$$\begin{aligned} \|x - w\|^2 &\leq [x - z, x - w] \\ &\leq \|x - z\| \|x - w\|. \end{aligned}$$

Therefore $\|x - w\| \leq \|x - z\|$ and $w \in P_W(x)$. □

Theorem 5.6. Let X be a Banach space and $[.,.] : X \times X \to R$ a semi-inner product on X which generates the norm. Let W be a nonempty bounded subset of X, $x \in X$, $w \in W$ and for every $z \in W$, $[w - z, x - w] = 0$. Then the following statments are equivalent:

i) $x \in F_w$;

ii) $\delta^2(x, W) = [x - z, x - w]$.

Proof . Suppose for $z \in W$ $[w - z, x - w] = 0$, then

$$\begin{aligned} x \in F_w &\Leftrightarrow \delta^2(x, W) = \|x - w\|^2 \\ &\Leftrightarrow \delta^2(x, W) = [x - w, x - w] \\ &\Leftrightarrow \delta^2(x, W) = [x - z - w + z, x - w] \\ &\Leftrightarrow \delta^2(x, W) = [x - z, x - w] - [w - z, x - w] \\ &\Leftrightarrow \delta^2(x, W) = [x - z, x - w]. \end{aligned}$$

□

The point of mentioning that a question is "**open**" is to:

Theorem 5.7. Let X be a Banach space and W a proximinal subspace of X. Then the following statements are equivalent:

i) W is quasi-Chebyshev(weakly-Chebyshev).

ii) for every $g \in W$ and for every sequence $\{x_n\}_{n \geq 1}$ with $\|x_n\| = 1$ and $x_n \in P_n$ has a convergent subsequence(weakly convergent subsequence).

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