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An example for the nonstability of multicubic mappings

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Abstract

In this paper, we present a counterexample for the nonstability of multicubic mappings. In other words, we show that Corollary 3.5 of [A. Bodaghi and B. Shojaee, On an equation characterizing multi-cubic mappings and its stability and hyperstability, Fixed Point Theory. 22 (2021), No. 1, 83–92] does not hold when $\alpha = 3n$.

Keywords: Banach space, Hyers-Ulam stability, multicubic mapping
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1 Introduction

A functional equation $\mathcal{F}$ is said to be *stable* if any function f satisfying the equation $\mathcal{F}$ approximately must be near to an exact solution of $\mathcal{F}$. In two last decades, the stability problem for functional equations which has been initiated by the celebrated question of Ulam [11] for group homomorphisms (answered by Hyers [7], Aoki [1] and Th. M. Rassias [10] for Banach algebras), was studied for multivariable mappings. One of them is the multicubic mapping. Let V and W be vector spaces over the rational numbers $\mathbb{Q}$, $n \in \mathbb{N}$. A mapping $f : V^n \longrightarrow W$ is called *n-cubic* or *multicubic* if f satisfies

$$C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x) \tag{1.1}$$

in each variable [3]. Indeed, f is multicubic if

$$\begin{aligned} &f(v_1, \dots, v_{i-1}, 2v_i + v'_i, v_{i+1}, \dots, v_n) + f(v_1, \dots, v_{i-1}, 2v_i - v'_i, v_{i+1}, \dots, v_n) \\ &= 2f(v_1, \dots, v_{i-1}, v_i + v'_i, v_{i+1}, \dots, v_n) + 2f(v_1, \dots, v_{i-1}, v_i - v'_i, v_{i+1}, \dots, v_n) + 12f(v_1, \dots, v_n) \end{aligned}$$

for all $i \in \{1, \dots, n\}$. In [3], the authors unified the system of functional equations defining a multicubic mapping to a single equation, namely, multi-cubic functional equation (Proposition 2.1). Moreover, they studied the Hyers-Ulam stability of such mappings. A lot of information about miscellaneous versions of multicubic mappings and their stabilities in various spaces are available in [2], [4], [6] and [9].

In this paper, we show that the stability result in Corollary 3.5 of [3] for multicubic mappings is not valid for $\alpha = 3n$.

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2 Main results

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, \infty), n \in \mathbb{N}$. For any $l \in \mathbb{N}_0, m \in \mathbb{N}$, $t = (t_1, \ldots, t_m) \in \{-1, 1\}^m$ and $x = (x_1, \ldots, x_m) \in V^m$ we write $lx := (lx_1, \ldots, lx_m)$ and $tx := (t_1x_1, \ldots, t_mx_m)$, where lx stands, as usual, for the lth power of an element x of the commutative group V.

From now on, let V and W be vector spaces over $\mathbb{Q}$, $n \in \mathbb{N}$ and $x_i^n = (x_{i1}, x_{i2}, \cdots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. We will write x_i^n simply x_i when no confusion can arise. Given $x_1, x_2 \in V^n$. Put

$$\mathcal{M}^n = \{\mathfrak{N}_n = (N_1, \ldots, N_n) | \, N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\},$$

where $j \in \{1, \ldots, n\}$. For $k \in \mathbb{N}_0$ with $0 \leq k \leq n$, consider

$$\mathcal{M}_k^n := \{\mathfrak{N}_n \in \mathcal{M}^n | \, \mathrm{Card}\{N_j : \, N_j = x_{1j}\} = k\}.$$

The upcoming result was proved in [3, Proposition 2.2], which shows that every multicubic mapping can be described a single equation.

Proposition 2.1. If a mapping $f : V^n \longrightarrow W$ is multi-cubic, then f satisfies the equation

$$\sum_{q\in\{-1,1\}^n} f(2x_1 + qx_2) = \sum_{k=0}^{n} 2^{n-k}12^k f(\mathcal{M}_k^n), \tag{2.1}$$

where $f(\mathcal{M}_k^n) := \sum_{\mathfrak{N}_n \in \mathcal{M}_k^n} f(\mathfrak{N}_n)$.

Recall from [3] that a mapping $f : V^n \longrightarrow W$ has *the r-power condition* in the jth variable if

$$f(z_1, \ldots, z_{j-1}, 2z_j, z_{j+1}, \ldots, z_n) = 2^r f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n),$$

for all $(z_1, \cdots, z_n) \in V^n$. Note that 3-power condition is also called the *cubic condition.*

The following proposition is a direct consequence of main result in [3], which shows that the functional equation (2.1) is stable. In fact, we improve Corollary 3.5 from [3].

Proposition 2.2. Given $\delta > 0$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 3n$. Let V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality

$$\left\| \sum_{q\in\{-1,1\}^n} f(2x_1 + qx_2) - \sum_{k=0}^{n} 2^{n-k}12^k f(\mathcal{M}_k^n) \right\| \leq \sum_{i=1}^{2}\sum_{j=1}^{n} \|x_{ij}\|^\alpha \delta,$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{C} : V^n \longrightarrow W$ of *(2.1)* such that

$$\|f(x) - \mathcal{C}(x)\| \leq \begin{cases} \frac{\delta}{2^{4n}-2^{\alpha+n}} \sum_{j=1}^n \|x_{1j}\|^\alpha & \alpha < 3n, \\ \\ \frac{2^\alpha}{2^{\alpha+n}-2^{4n}} \delta \sum_{j=1}^n \|x_{1j}\|^\alpha & \alpha > 3n, \end{cases}$$

for all $x = x_1 \in V^n$. Moreover, if $\mathcal{C}$ has the cubic condition in each variable, then it is a multicubic mapping.

We bring an elementary lemma without the proof as follows.

Lemma 2.3. If a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies *(1.1)*, then it has the form $g(x) = cx^3$, for all $x \in \mathbb{R}$, where $c = f(1)$.

In the next result, we extend Lemma 2.3 for several variables functions. For doing this, we use an idea taken from the proof of [8, Theorem 13.4.3].

Proposition 2.4. Let $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous n-cubic function. Then, there exists a constant $c\in\mathbb{R}$ such that

$$f(x_1,\dots,x_n)=c\prod_{j=1}^{n}x_j^3 \tag{2.2}$$

for all $x_1,\dots,x_n\in\mathbb{R}$.

Proof . We argue the proof by induction on n. For $n=1$, (2.2) is valid in view of Lemma 2.3. Let (2.2) hold for a $n\in\mathbb{N}$. Assume that $f:\mathbb{R}^{n+1}\longrightarrow\mathbb{R}$ is a continuous $(n+1)$-cubic function. Fix the n variables $x_1,\dots,x_n$. Then, the function $y\mapsto f(x_1,\dots,x_n,y)$ as a function of y is cubic and continuous, and so by Lemma 2.3, there exists a constant $c\in\mathbb{R}$ such that

$$f(x_1,\dots,x_n,y)=cy^3,\qquad (y\in\mathbb{R}). \tag{2.3}$$

Note that c depends on $x_1,\dots,x_n$, and indeed

$$c=c(x_1,\dots,x_n). \tag{2.4}$$

Letting $y=1$ in (2.3) and applying (2.4), we get

$$c=c(x_1,\dots,x_n)=f(x_1,\dots,x_n,1).$$

Since f is $(n+1)$-cubic, it follows that c is an n-cubic function and hence by the induction hypothesis there exists a real number c' such that

$$c=c(x_1,\dots,x_n)=c'\prod_{j=1}^{n}x_j^3. \tag{2.5}$$

Now, the result follows from (2.3) and (2.5). □

Remark 2.5. *Note that in the proof of Proposition 2.4 only the continuity of g with respect to each variable separately was used. Therefore, the result is again true if and only if f is supposed separately continuous with respect to each variable. On the other hand, in virtue of the proof of Proposition 2.4, if the continuity condition of g is removed, then the result remains valid for a function $g:\mathbb{Q}^p\longrightarrow\mathbb{Q}$. We use this fact to make a non-stable example.*

Here, we present the main result of this paper that is a nonstable example for the multicubic mappings on $\mathbb{Q}^n$. Indeed, we show the hypothesis $\alpha\neq 3n$ cannot be removed in Proposition 2.2. Remember that the method of the proof is taken from [5].

Example 2.6. *Let $\delta>0$ and $n\in\mathbb{N}$ and consider $S\geq 6^n\sum_{k=0}^{n}6^k$. Put $\mu=\frac{2^{3n}-1}{2^{6n}S}\delta$. Define the function $\psi:\mathbb{Q}^n\longrightarrow\mathbb{Q}$ through*

$$\psi(r_1,\dots,r_n)=\begin{cases}\mu\prod_{j=1}^{n}r_j^3 & \text{for all } r_j \text{ with } |r_j|<1,\\ \mu & \text{otherwise.}\end{cases}$$

Moreover, define the function $f:\mathbb{Q}^n\longrightarrow\mathbb{Q}$ by

$$f(r_1,\dots,r_n)=\sum_{l=0}^{\infty}\frac{\psi(2^lr_1,\dots,2^lr_n)}{2^{3nl}},\qquad (r_j\in\mathbb{Q}).$$

It is obvious that ψ is bounded by μ. Indeed, for each $(r_1,\dots,r_n)\in\mathbb{Q}^n$, we have

$$|f(r_1,\dots,r_n)|\leq\frac{2^{3n}}{2^{3n}-1}\mu.$$

It follows from the last inequality that

$$|\mathbf{D}f\,(x_1,x_2)|\leq\mu S, \tag{2.6}$$

where

$$\mathbf{D}f\,(x_1,x_2):=f(2x_1+qx_2)-\sum_{k=0}^{n}2^{n-k}12^kf(\mathcal{M}_k^n)$$

in which $x_j = (x_{j1}, \dots, x_{jn}) \in Q^n$ *with* $j \in \{1,2\}$*. We wish to show that*

$$|\mathbf{D}f(x_1, x_2)| \le \delta \sum_{i=1}^{2} \sum_{j=1}^{n} |x_{ij}|^{3n}, \tag{2.7}$$

for all $x_1, x_2 \in \mathbb{Q}^n$*. We have three cases as follows:*
(i) If $x_1 = x_2 = 0$*, then it is clear that (2.7) holds.*
(ii) Let $x_1, x_2 \in \mathbb{Q}^n$ *with*

$$\sum_{i=1}^{2} \sum_{j=1}^{n} |x_{ij}|^{3n} < \frac{1}{2^{3n}}.$$

Thus, there exists a positive integer N *such that*

$$\frac{1}{2^{3n(N+1)}} < \sum_{i=1}^{2} \sum_{j=1}^{n} |x_{ij}|^{3n} < \frac{1}{2^{3nN}}, \tag{2.8}$$

and hence

$$|x_{ij}|^{3n} < \sum_{i=1}^{2} \sum_{j=1}^{n} |x_{ij}|^{3n} < \frac{1}{2^{3nN}}. \tag{2.9}$$

Relation (2.9) implies that $2^N |x_{ij}| < 1$ *for all* $i \in \{1,2\}$ *and* $j \in \{1, \dots, n\}$*. Therefore,* $2^{N-1}|x_{ij}| < 1$*. If* $y_1, y_2 \in \{x_{ij} | \, i \in \{1,2\}, \; j \in \{1, \dots, n\}\}$*, then*

$$2^{N-1} |\, y_1 \pm y_2| < 1, \qquad 2^{N-1} |\, 2y_1 \pm y_2| < 1.$$

Since ψ *is a multicubic function on* $(-1,1)^n$*, we have* $\mathbf{D}\psi\left(2^l x_1, 2^l x_2\right) = 0$ *for all* $l \in \{0, 1, 2, \dots, N-1\}$*. We conclude from the last equality and (2.8) that*

$$\begin{aligned} \frac{\left|\mathbf{D}f\left(2^l x_1, 2^l x_2\right)\right|}{\sum_{i=1}^{2} \sum_{j=1}^{n} |x_{ij}|^{3n}} &\le \sum_{l=N}^{\infty} \frac{\left|\mathbf{D}\psi\left(2^l x_1, 2^l x_2\right)\right|}{2^{3nl} \sum_{i=1}^{2} \sum_{j=1}^{n} |x_{ij}|^{3n}} \\ &\le \sum_{l=0}^{\infty} \frac{\mu S}{2^{3n(l+N)} \sum_{i=1}^{2} \sum_{j=1}^{n} |x_{ij}|^{3n}} \\ &\le \mu 2^{3n} S \sum_{l=0}^{\infty} \frac{1}{2^{3nl}} \\ &= \mu S \frac{2^{6n}}{2^{3n}-1} = \delta, \end{aligned}$$

for all $x_1, x_2 \in \mathbb{Q}^n$ *and thus (2.7) is true in this case.*
(iii) Assume that $\sum_{i=1}^{2} \sum_{j=1}^{n} |x_{ij}|^{3n} \ge \frac{1}{2^{3n}}$*. Using (2.6), we have*

$$\frac{\left|\mathbf{D}f\left(2^l x_1, 2^l x_2\right)\right|}{\sum_{i=1}^{2} \sum_{j=1}^{n} |x_{ij}|^{n}} \le 2^{3n} \frac{2^{3n}}{2^{3n}-1} \mu S = \delta.$$

Therefore, f *satisfies (2.7) for all* $x_1, x_2 \in \mathbb{Q}^n$*.*
Now, suppose the assertion is false, that is, there exist a number $b \in [0, \infty)$ *and a multicubic function* $\mathcal{C} : \mathbb{Q}^n \longrightarrow \mathbb{Q}$ *such that* $|f(r_1, \dots, r_n) - \mathcal{C}(r_1, \dots, r_n)| < b \prod_{j=1}^{n} r_j$ *for all* $(r_1, \dots, r_n) \in \mathbb{Q}^n$*. It follows now from Lemma 2.5 that there is a constant* $c \in \mathbb{R}$ *such that* $\mathcal{C}(r_1, \dots, r_n) = c \prod_{j=1}^{n} r_j^3$ *for all* $(r_1, \dots, r_n) \in \mathbb{Q}^n$ *and therefore*

$$|f(r_1, \dots, r_n)| \le (|c| + b) \prod_{j=1}^{n} |r_j|^3, \tag{2.10}$$

for all $(r_1, \dots, r_n) \in \mathbb{Q}^n$*. On the other hand, one can choose* $N \in \mathbb{N}$ *such that* $N\mu > |c| + b$*. If* $r = (r_1, \dots, r_n) \in \mathbb{Q}^n$ *such that* $r_j \in \left(0, \frac{1}{2^{N-1}}\right)$ *for all* $j \in \{1, \dots, n\}$*, then* $2^l r_j \in (0,1)$ *for all* $l = 0, 1, \dots, N-1$*. Hence*

$$
\begin{aligned}
|f(r_1, \dots, r_n)| &= \left| \sum_{l=0}^{\infty} \frac{\psi\left(2^l r_1, \dots, 2^l r_2\right)}{2^{3nl}} \right| \\
&= \left| \sum_{l=0}^{N-1} \frac{\mu 2^{3nl} \prod_{j=1}^{n} r_j^3}{2^{3nl}} \right| \\
&= N\mu \prod_{j=1}^{n} |r_j|^3 \\
&> (|c| + b) \prod_{j=1}^{n} |r_j|^3,
\end{aligned}
$$

that leads us to a contradiction with (2.10).

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