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Inequalities for the rational functions with no poles on the unit circle

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Abstract

Let $\mathcal{R}_n$ be the set of rational functions with prescribed poles. It is known that if $r \in \mathcal{R}_n$, such that $r(z) \neq 0$ in $|z| < 1$, then

$$\sup_{|z|=1} |r^{'}(z)| \leq \frac{|\mathcal{B}^{'}(z)|}{2} \sup_{|z|=1} |r(z)|$$

and in case $r(z) = 0$ in $|z| \leq 1$, then

$$\sup_{|z|=1} |r^{'}(z)| \geq \frac{|\mathcal{B}^{'}(z)|}{2} \sup_{|z|=1} |r(z)|,$$

where $\mathcal{B}(z)$ is the Blashke product. The main aim of this paper is to relax the condition that all poles of $r(z)$ lie outside the unit circle and instead assume their location anywhere off the unit circle in the complex plane $\mathbb{C}$. The results so obtained besides the above inequalities generalize some other well-known estimates for the derivative of rational functions $r \in \mathcal{R}_n$ with prescribed poles and restricted zeros.

Keywords: Inequalities, Polynomials, Rational functions, Off the unit circle, Poles, Zeros
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1 Introduction

Let $\mathcal{P}_n$ be the class of all polynomials $p(z) := \sum_{j=0}^{n} c_j z^j$ of degree at most n. Let $\mathcal{D}^-$ denote the region inside $U := \{z : |z| = 1\}$ and $\mathcal{D}^+$ the region outside U. For $a_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$, we write

$$w(z) := \prod_{j=1}^{n} (z - a_j) \quad ; \quad \mathcal{B}(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{a_j} z}{z - a_j} \right)$$

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and

$$\mathcal{R}_n = R_n(a_1, a_2, \ldots, a_n) := \left\{ \frac{p(z)}{w(z)} : \quad p \in \mathcal{P}_n \right\}.$$

Thus $\mathcal{R}_n$ is the set of all rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit at ∞. We observe that $\mathcal{B}(z) \in \mathcal{R}_n$.
A famous result due to Bernstien [4] states that if $p \in \mathcal{P}_n$, then

$$\max_{z\in U} |p'(z)| \leq n \max_{z\in U} |p(z)|.$$

In case $p(z) \neq 0$ for $z \in \mathcal{D}^-$, then it was conjectured by Erdös and latter proved by Lax [2] that

$$\max_{z\in U} |p'(z)| \leq \frac{n}{2} \max_{z\in U} |p(z)|,$$

whereas if $p(z) \neq 0$ for $z \in \mathcal{D}^+$, then Turán [6] proved that

$$\max_{z\in U} |p'(z)| \geq \frac{n}{2} \max_{z\in U} |p(z)|.$$

In the literature [1, 3, 5, 7], there exist several improvements and generalisations of the above results. Li, Mohapatra and Rodriguez [3] extended these inequalities to rational functions $r \in \mathcal{R}_n$ with prescribed poles $a_1, a_2, \ldots, a_n$ replacing z^n by Blashke product $\mathcal{B}(z)$. Among other things they proved the following results for rational functions with restricted poles.

Theorem A. Suppose $r \in \mathcal{R}_n$ and all the zeros of r lie in $U \cup \mathcal{D}^+$. Then for $z \in U$,

$$|r'(z)| \leq \frac{1}{2} |\mathcal{B}'(z)| \sup_{z\in U} |r(z)|.$$

Theorem B. Suppose $r \in \mathcal{R}_n$, where r has exactly n poles at $a_1, a_2, \ldots, a_n$ and all the zeros of r lie in $U \cup \mathcal{D}^-$, then for $z \in U$,

$$|r'(z)| \geq \frac{1}{2}\{|\mathcal{B}'(z)| - (n - m)\}|r(z)|,$$

where m is the number of zeros of r.

In the proofs of the above theorems and related results, it is assumed that either all the poles lie in $\mathcal{D}^-$ or in $\mathcal{D}^+$. However, in this paper we relax this condition and assume that the poles of $r \in \mathcal{R}_n$ lie anywhere off the unit circle in the complex plane.

Assume that $a = \{a_j\}_{j=1}^n,\ n \geq 1, |a_j| \neq 1, j = 1, 2, \ldots, n$ is an arbitrary finite sequence. $w(z) = w_1(z)w_2(z)$, where $w_1(z) = \prod_{a_j \in \mathcal{D}^-}(z - a_j)$ and $w_2(z) = \prod_{a_j \in \mathcal{D}^+}(z - a_j)$. Here we note that

$$w_1(z) \equiv 1, \qquad \text{if} \qquad a \subset \mathcal{D}^+$$

and

$$w_2(z) \equiv 1 \qquad \text{if} \qquad a \subset \mathcal{D}^-.$$

Also

$$\mathcal{B}_1(z) := \prod_{a_j \in \mathcal{D}^-} \left(\frac{1 - \overline{a_j} z}{z - a_j} \right) \text{ and } \mathcal{B}_2(z) := \prod_{a_j \in \mathcal{D}^+} \left(\frac{1 - \overline{a_j} z}{z - a_j} \right)$$

are the Blaskhe products whose poles with multiplicity counted are the enteries of the sequence inside or outside the unit circle.

2 Main results

Theorem 2.1. If $r \in \mathcal{R}_n$ has exactly n poles in $\mathbb{C}/U$ and all zeros of r lie in $U \cup \mathcal{D}^-$, then for $z \in U$

$$|r^{'}(z)| \geq \frac{1}{2}\Big\{\big||\mathcal{B}_2'(z)| - |\mathcal{B}_1'(z)|\big| - (n - s)\Big\}|r(z)|, \tag{2.1}$$

where s is the number of zeros of $r(z)$. The result is sharp and equality holds for

$$r(z) = \mathcal{B}_1(z)\mathcal{B}_2(z) + \lambda, \qquad \lambda \in U.$$

In particular, if $r(z)$ has exactly n zeros in $U \cup \mathcal{D}^-$, then we have the following result.

Corollary 2.2. Suppose $r \in \mathcal{R}_n$, be such that $r(z) \neq 0$ for $z \in \mathcal{D}^+$, having all poles off the unit circle, then for $z \in U$

$$|r^{'}(z)| \geq \frac{1}{2}\Big\{\big||\mathcal{B}_2^{'}(z)| - |\mathcal{B}_1^{'}(z)|\big|\Big\}|r(z)|. \tag{2.2}$$

Remark 2.3. If $r \in \mathcal{R}_n$ has all its poles in $\mathcal{D}^+$, then $\mathcal{B}_1(z) \equiv 1$ and $\mathcal{B}_2(z) = \mathcal{B}(z)$. Therefore, in this case Theorem 2.1 reduces to a result due to Li, Mohapatra and Rodriguez [3, Theorem 4].

As an improvement of Theorem 2.1, we next prove the following result.

Theorem 2.4. If $r \in \mathcal{R}_n$ has exactly n poles in $\mathbb{C}/U$ and all zeros of r lie in $U \cup \mathcal{D}^-$, then for $z \in U$

$$|r^{'}(z)| \geq \frac{1}{2}\left\{\big||\mathcal{B}_2^{'}(z)| - |\mathcal{B}_1^{'}(z)|\big| + s - n + \frac{|c_s| - |c_0|}{|c_s| + |c_0|}\right\}|r(z)|, \tag{2.3}$$

where s is the number of zeros of $r(z)$. The result is sharp and equality holds for

$$r(z) = \mathcal{B}_1(z)\mathcal{B}_2(z) + \lambda, \qquad \lambda \in U.$$

In particular, if $r(z)$ has exactly n zeros in $U \cup \mathcal{D}^-$, then we have the following result.

Corollary 2.5. Suppose $r \in \mathcal{R}_n$, is such that $r(z) \neq 0$ for $z \in \mathcal{D}^+$, then for $z \in U$

$$|r^{'}(z)| \geq \frac{1}{2}\left\{\big||\mathcal{B}_2^{'}(z)| - |\mathcal{B}_1^{'}(z)|\big| + \frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right\}|r(z)|. \tag{2.4}$$

Remark 2.6. If $r \in \mathcal{R}_n$, has all its poles in $\mathcal{D}^+$, then inequality (2.4) reduces to a result due to Wali and Shah [7, Corollary 2].

Theorem 2.7. Suppose that $r \in \mathcal{R}_n$ has exactly n poles in $\mathbb{C}/U$ and all zeros of r lie in $\mathcal{D}^+$, then for $z \in U$

$$\mathrm{Re}\left(\frac{zr^{'}(z)}{r(z)}\right) \leq \frac{1}{2}\left\{\big||\mathcal{B}_2^{'}(z)| - |\mathcal{B}_1^{'}(z)|\big| - (n-s) - \left(\frac{|c_0| - |c_s|}{|c_0| + |c_s|}\right)\right\},$$

where s is the number of zeros of $r(z)$. The result is sharp and equality holds for

$$r(z) = \mathcal{B}_1(z)\mathcal{B}_2(z) + \lambda, \qquad \lambda \in U.$$

In particular, if $r(z)$ has exactly n zeros in $\mathcal{D}^+$, then we have the following sharp result.

Corollary 2.8. Suppose that $r \in \mathcal{R}_n$ and all zeros of r lie in $\mathcal{D}^+$, then for $z \in U$

$$\mathrm{Re}\left(\frac{zr^{'}(z)}{r(z)}\right) \leq \frac{1}{2}\left\{\big||\mathcal{B}_2^{'}(z)| - |\mathcal{B}_1^{'}(z)|\big| - \left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)\right\}.$$

The result due to Wali and Shah [7, Lemma 2] is a special case of Theorem 2.7, if we assume that all poles lie in $\mathcal{D}^+$, that is, $\mathcal{B}_1(z) \equiv 1$ and $\mathcal{B}_2(z) = \mathcal{B}(z)$.

3 Lemmas

Lemma 3.1. Suppose that $r \in \mathcal{R}_n$, where r has exactly n poles all belong to $\mathbb{C}/U$ and all zeros of r lie in $U \cup \mathcal{D}^-$, then for all points on U such that $r(z) \neq 0$,

$$\mathrm{Re}\left(\frac{zr^{'}(z)}{r(z)}\right) \geq \frac{\big||\mathcal{B}_2^{'}(z)| - |\mathcal{B}_1^{'}(z)|\big|}{2} - \frac{n-s}{2}, \tag{3.1}$$

where s is the number of zeros of $r(z)$.

Proof . Since $r \in \mathcal{R}_n$, we can write

$$r(z) = \frac{p(z)}{w(z)}. \tag{3.2}$$

Let $z_1, z_2, \ldots, z_s$ be the zeros of $r(z)$, therefore $z_j,\ j = 1, 2, \ldots, s$ are also zeros of $p(z)$ and as such, we have

$$p(z) = \sum_{j=0}^{s} c_j z^j = c_s \prod_{j=1}^{s} (z - z_j), \quad z_j \in U \cup \mathcal{D}^-, j = 1, 2, \ldots, s.$$

By assumption all poles of $r(z)$ lie off the unit circle, therefore we can write (3.2) as

$$r(z) = \frac{p(z)}{w_1(z) w_2(z)}, \tag{3.3}$$

where $w_1(z) = \prod_{a_j \in \mathcal{D}^-} (z - a_j)$ and $w_2(z) = \prod_{a_j \in \mathcal{D}^+} (z - a_j)$.
Assume that n_1 poles with multiplicities counted lie inside U and remaining $n - n_1 = n_2(say)$ poles lie outside U. Also, we can write $p(z) = p_1(z)p_2(z)$ such that degree of $p_1(z) \leq$ degree of $w_1(z)$ and degree of $p_2(z) \leq$ degree of $w_2(z)$, so that

$$\begin{aligned} r(z) &= \frac{p_1(z)}{w_1(z)} . \frac{p_2(z)}{w_2(z)} \\ &= r_1(z) r_2(z), \end{aligned}$$

where $r_1 \in \mathcal{R}_{n_1}$ and $r_2 \in \mathcal{R}_{n_2}$.
Now

$$\begin{aligned} r_1(z) &= \frac{p_1(z)}{w_1(z)} \\ &= \frac{p_1(z)\mathcal{B}_1(z)}{\prod_{j=1}^{n_1} (1 - \overline{a_j} z)}, \quad a_j \in \mathcal{D}^-. \end{aligned}$$

Therefore, for $z \in U$

$$\mathrm{Re}\left(\frac{z r_1'(z)}{r_1(z)} \right) = \mathrm{Re}\left(\frac{z p_1'(z)}{p_1(z)} \right) + \mathrm{Re}\left(\frac{z \mathcal{B}_1'(z)}{\mathcal{B}_1(z)} \right) + \sum_{j=1}^{n_1} \mathrm{Re}\left(\frac{a_j}{z - a_j} \right). \tag{3.4}$$

Also we have

$$\mathcal{B}_1(z) = \prod_{j=1}^{n_1} \left(\frac{1 - \overline{a_j} z}{z - a_j} \right).$$

Therefore, for $z \in U$

$$\frac{\mathcal{B}_1'(z)}{\mathcal{B}_1(z)} = \sum_{j=1}^{n_1} \left\{ \frac{-\overline{a}_j z + |a_j|^2 - 1 + \overline{a}_j z}{(\overline{z - a_j})(z - a_j)} \right\}.$$

This further gives, for $z \in U$

$$\frac{z \mathcal{B}_1'(z)}{\mathcal{B}_1(z)} = \sum_{j=1}^{n_1} \frac{|a_j|^2 - 1}{|z - a_j|^2}, \quad a_j \in \mathcal{D}^-.$$

Since the right-hand side of above equation is a negative real number, therefore we can write for $z \in U$

$$\frac{z \mathcal{B}_1'(z)}{\mathcal{B}_1(z)} = -\left| \frac{z \mathcal{B}_1'(z)}{\mathcal{B}_1(z)} \right| = -|\mathcal{B}_1'(z)|. \tag{3.5}$$

This after using in equation (3.4), gives

$$\begin{aligned}\operatorname{Re}\left(\frac{zr_1^{'}(z)}{r_1(z)}\right) &= \operatorname{Re}\left(\frac{zp_1^{'}(z)}{p_1(z)}\right) - |\mathcal{B}_1^{'}(z)| + \sum_{j=1}^{n_1}\operatorname{Re}\left(\frac{a_j}{z-a_j}\right)\\ &= \operatorname{Re}\left(\frac{zp_1^{'}(z)}{p_1(z)}\right) - |\mathcal{B}_1^{'}(z)| - \frac{n_1}{2}\\ &+ \sum_{j=1}^{n_1}\operatorname{Re}\left(\frac{a_j}{z-a_j}+\frac{1}{2}\right)\\ &= \operatorname{Re}\left(\frac{zp_1^{'}(z)}{p_1(z)}\right) - \frac{|\mathcal{B}_1^{'}(z)|}{2} - \frac{n_1}{2}.\end{aligned} \tag{3.6}$$

Again

$$\begin{aligned}r_2(z) &= \frac{p_2(z)}{w_2(z)}\\ &= \frac{p_2(z)\mathcal{B}_2(z)}{\prod_{j=n_1+1}^{n}(1-\overline{a_j}z)}, \quad a_j \in \mathcal{D}^+.\end{aligned}$$

This gives

$$\operatorname{Re}\left(\frac{zr_2^{'}(z)}{r_2(z)}\right) = \operatorname{Re}\left(\frac{zp_2^{'}(z)}{p_2(z)}\right) + \operatorname{Re}\left(\frac{z\mathcal{B}_2^{'}(z)}{\mathcal{B}_2(z)}\right) + \sum_{j=n_1+1}^{n}\operatorname{Re}\left(\frac{a_j}{z-a_j}\right). \tag{3.7}$$

Also we have

$$\mathcal{B}_2(z) = \prod_{j=n_1+1}^{n}\left(\frac{z-a_j}{1-\overline{a_j}z}\right).$$

This gives, for $z \in U$

$$\frac{z\mathcal{B}_2^{'}(z)}{\mathcal{B}_2(z)} = \sum_{j=n_1+1}^{n}\frac{|a_j|^2-1}{|z-a_j|^2}, \quad a_j \in \mathcal{D}^+.$$

Therefore,

$$\frac{z\mathcal{B}_2^{'}(z)}{\mathcal{B}_2(z)} = \left|\frac{z\mathcal{B}_2^{'}(z)}{\mathcal{B}_2(z)}\right| = |\mathcal{B}_2^{'}(z)| \quad \text{for} \quad z \in U. \tag{3.8}$$

This after using in equation (3.7), gives

$$\begin{aligned}\operatorname{Re}\left(\frac{zr_2^{'}(z)}{r_2(z)}\right) &= \operatorname{Re}\left(\frac{zp_2^{'}(z)}{p_2(z)}\right) + |\mathcal{B}_2^{'}(z)| + \sum_{j=n_1+1}^{n}\operatorname{Re}\left(\frac{a_j}{z-a_j}\right).\\ &= \operatorname{Re}\left(\frac{zp_2^{'}(z)}{p_2(z)}\right) + |\mathcal{B}_2^{'}(z)| - \frac{(n-n_1)}{2}\\ &+ \sum_{j=n_1+1}^{n}\operatorname{Re}\left(\frac{a_j}{z-a_j}+\frac{1}{2}\right)\\ &= \operatorname{Re}\left(\frac{zp_2^{'}(z)}{p_2(z)}\right) + \frac{|\mathcal{B}_2^{'}(z)|}{2} - \frac{(n-n_1)}{2}\\ &= \operatorname{Re}\left(\frac{zp_2^{'}(z)}{p_2(z)}\right) + \frac{|\mathcal{B}_2^{'}(z)|}{2} - \frac{n_2}{2}.\end{aligned} \tag{3.9}$$

From equation (3.6) and (3.9), we get

$$\operatorname{Re}\left(\frac{zr_1'(z)}{r_1(z)}\right)+\operatorname{Re}\left(\frac{zr_2'(z)}{r_2(z)}\right)=\operatorname{Re}\left(\frac{zp_1^{'}(z)}{p_1(z)}\right)+\operatorname{Re}\left(\frac{zp_2^{'}(z)}{p_2(z)}\right)+\frac{|\mathcal{B}_2^{'}(z)|}{2}-\frac{|\mathcal{B}_1^{'}(z)|}{2}-\frac{n}{2}. \tag{3.10}$$

Equivalently,

$$\operatorname{Re}\left(\frac{zr^{'}(z)}{r(z)}\right)=\operatorname{Re}\left(\frac{zp^{'}(z)}{p(z)}\right)+\frac{|\mathcal{B}_2^{'}(z)|-|\mathcal{B}_1^{'}(z)|}{2}-\frac{n}{2}. \tag{3.11}$$

Now for $z_j \in U\cup\mathcal{D}^-, j=1,2,\dots,s$, we have for all $z\in U$, such that $z\neq z_j$ $j=1,2,\dots,s$.

$$\left|\frac{z}{z-z_j}\right|\geq\left|\frac{z}{z-z_j}-1\right|.$$

Therefore,

$$\operatorname{Re}\left(\frac{z}{z-z_j}\right)\geq\frac{1}{2},\quad j=1,2\dots,s.$$

This in particular gives for those points $z\in U$, such that $p(z)\neq 0$ and $z\neq z_j, j=1,2,\dots,s$

$$\begin{aligned}\operatorname{Re}\left(\frac{zp^{'}(z)}{p(z)}\right)&=\sum_{j=1}^{s}\operatorname{Re}\left(\frac{z}{z-z_j}\right),\\ &\geq\sum_{j=1}^{s}\frac{1}{2}\\ &=\frac{s}{2}.\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we get for $r(z)\neq 0$ and $z\in U$

$$\operatorname{Re}\left(\frac{zr^{'}(z)}{r(z)}\right)\geq\frac{\big||\mathcal{B}_2^{'}(z)|-|\mathcal{B}_1^{'}(z)|\big|}{2}-\frac{n-s}{2}.$$

□

4 Proofs of Theorems

Proof of Theorem 2.1. Suppose $r(z)\neq 0$ for $z\in U$, therefore it follows from Lemma 3.1

$$\operatorname{Re}\left(\frac{zr^{'}(z)}{r(z)}\right)\geq\frac{\big||\mathcal{B}_2^{'}(z)|-|\mathcal{B}_1^{'}(z)|\big|}{2}-\frac{n-s}{2}.$$

Using the fact that

$$\left|\frac{zr^{'}(z)}{r(z)}\right|\geq\operatorname{Re}\left(\frac{zr^{'}(z)}{r(z)}\right),$$

we get for $z\in U$,

$$|r'(z)|\geq\frac{1}{2}\left\{\big||\mathcal{B}_2^{'}(z)|-|\mathcal{B}_1^{'}(z)|\big|-(n-s)\right\}|r(z)|. \tag{4.1}$$

In case $r(z) = 0$, for $z \in U$, inequality (4.1) is trivially satisfied. Hence the result holds for all $z \in U$. To show equality in (4.1), we consider the rational function $r(z) = \mathcal{B}_1(z)\mathcal{B}_2(z) + \lambda$, $\lambda \in U$. So that

$$\begin{aligned} |r'(z)| &= |\mathcal{B}_1(z)\mathcal{B}_2'(z) + \mathcal{B}_2(z)\mathcal{B}_1'(z)| \\ &= \left|\frac{z\mathcal{B}_2'(z)}{\mathcal{B}_2(z)} + \frac{z\mathcal{B}_1'(z)}{\mathcal{B}_1(z)}\right| \\ &= \Big||\mathcal{B}_2'(z)| - |\mathcal{B}_1'(z)|\Big|. \end{aligned}$$

Therefore, it can be easily seen that equality in (2.1) holds for such type of rational functions. This completes the proof of Theorem 2.1.

□

Proof of Theorem 2.4. Suppose $r(z) \neq 0$ for $z \in U$. Let $z_1, z_2, \ldots, z_s$ be the zeros of $r(z)$, so that $z_j, j = 1, 2, \ldots, s$ are also zeros of $p(z)$ and we can write

$$p(z) = \sum_{j=0}^{s} c_j z^j = c_s \prod_{j=1}^{s} (z - z_j), \ z_j \in \mathcal{D}^-, j = 1, 2, \ldots, s.$$

Therefore from (3.11), we have for $z \in U$

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) = \sum_{j=1}^{s} \operatorname{Re}\left(\frac{z}{z - z_j}\right) + \frac{\big||\mathcal{B}_2'(z)| - |\mathcal{B}_1'(z)|\big|}{2} - \frac{n}{2}. \tag{4.2}$$

Now for $z_j \in \mathcal{D}^-$, we have for $z \in U$

$$\begin{aligned} \operatorname{Re}\left(\frac{z}{z - z_j}\right) &= \operatorname{Re}\left(\frac{e^{i\theta}}{e^{i\theta} - |z_j|e^{i\Phi}}\right) \\ &= \operatorname{Re}\left(\frac{\cos\theta + i\sin\theta}{(\cos\theta + i\sin\theta) - |z_j|(\cos\Phi + i\sin\Phi)}\right) \\ &= \operatorname{Re}\left(\frac{\cos\theta + i\sin\theta}{(\cos\theta - |z_j|\cos\Phi) + i(\sin\theta - |z_j|\sin\Phi)}\right) \\ &= \frac{1 - |z_j|\cos(\theta - \Phi)}{1 + |z_j|^2 - 2|z_j|\cos(\theta - \Phi)} \\ &\geq \frac{1}{1 + |z_j|}, \end{aligned}$$

if

$$1 + |z_j|(1 - |z_j|\cos(\theta - \Phi)) \geq 1 + |z_j|^2 - 2|z_j|\cos(\theta - \Phi).$$

That is, if

$$(|z_j| - |z_j|^2)(1 + \cos(\theta - \Phi)) \geq 0.$$

Equivalently

$$|z_j| \leq 1,$$

which is true.
Therefore, for $z_j \in U \cup \mathcal{D}^-$

$$\operatorname{Re}\left(\frac{z}{z - z_j}\right) \geq \frac{1}{1 + |z_j|}. \tag{4.3}$$

This gives from (4.2)

$$\begin{aligned}\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) &\geq \sum_{j=1}^{s}\frac{1}{1+|z_j|}+\frac{\big||\mathcal{B}_2'(z)|-|\mathcal{B}_1'(z)|\big|}{2}-\frac{n}{2}\\ &=\sum_{j=1}^{s}\frac{1-|z_j|}{2(1+|z_j|)}+\frac{\big||\mathcal{B}_2'(z)|-|\mathcal{B}_1'(z)|\big|}{2}-\frac{(n-s)}{2}.\end{aligned}$$

Using the fact that, if $\langle x_i\rangle_1^\infty$ is a sequence of real numbers, such that $0\leq x_i\leq 1,\ i=1,2,\ldots,n$, then

$$\sum_{i=1}^{n}\frac{1-x_i}{1+x_i}\geq\frac{1-\prod_{i=1}^{n}x_i}{1+\prod_{i=1}^{n}x_i},$$

we get by using Vitali's rule

$$\begin{aligned}\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) &\geq \frac{1}{2}\left\{\frac{1-\prod_{j=1}^{s}|z_j|}{1+\prod_{j=1}^{s}|z_j|}+\big||\mathcal{B}_2'(z)|-|\mathcal{B}_1'(z)|\big|-(n-s)\right\}\\ &=\frac{1}{2}\left\{\big||\mathcal{B}_2'(z)|-|\mathcal{B}_1'(z)|\big|-(n-s)+\frac{|c_s|-|c_0|}{|c_s|+|c_0|}\right\}.\end{aligned}\tag{4.4}$$

Finally, using the fact that

$$\left|\frac{zr'(z)}{r(z)}\right|\geq\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right),$$

we get for $z\in U$ and $r(z)\neq 0$

$$|r'(z)|\geq\frac{1}{2}\left\{\big||\mathcal{B}_2'(z)|-|\mathcal{B}_1'(z)|\big|-(n-s)+\frac{|c_s|-|c_0|}{|c_s|+|c_0|}\right\}|r(z)|.\tag{4.5}$$

In case $r(z)=0$, for $z\in U$, inequality (4.5) is trivially satisfied. Hence the result holds for all $z\in U$. This completes the proof of Theorem 2.4.

□

Proof of Theorem 2.7. Since $r\in\mathcal{R}_n$, therefore

$$r(z)=\frac{p(z)}{w(z)}.$$

Let $z_1,z_2,\ldots,z_s$ be the zeros of $r(z)$, therefore $z_j, j=1,2,\ldots,s$ are also zeros of $p(z)$ and we can write

$$p(z)=\sum_{j=0}^{s}c_jz^j=c_s\prod_{j=1}^{s}(z-z_j), z_j\in\mathcal{D}^+, j=1,2,\ldots,s.$$

Proceeding as in Lemma 3.1 and noting that $z_j\in\mathcal{D}^+$, we have

$$\begin{aligned}\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) &= \mathrm{Re}\left(\frac{zp'(z)}{p(z)}\right)+\frac{\big||\mathcal{B}_2'(z)|-|\mathcal{B}_1'(z)|\big|}{2}-\frac{n}{2}\\ &\leq\sum_{j=1}^{s}\frac{1}{1+|z_j|}+\frac{\big||\mathcal{B}_2'(z)|-|\mathcal{B}_1'(z)|\big|}{2}-\frac{n}{2}\\ &=\frac{1}{2}\left\{\sum_{j=1}^{s}\frac{1-|z_j|}{1+|z_j|}+\big||\mathcal{B}_2'(z)|-|\mathcal{B}_1'(z)|\big|+s-n\right\}.\end{aligned}$$

Using the fact that, if $\langle x_i \rangle_1^\infty$ is a sequence of real numbers, such that $x_i \geq 1$, then

$$\sum_{i=1}^{n} \frac{1-x_i}{1+x_i} \leq \frac{1-\prod_{i=1}^{n} x_i}{1+\prod_{i=1}^{n} x_i},$$

we get

$$\begin{aligned}\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) &\leq \frac{1}{2}\left\{\frac{1-\prod_{j=1}^{s}|z_j|}{1+\prod_{j=1}^{s}|z_j|} + \big||\mathcal{B}_2'(z)| - |\mathcal{B}_1'(z)|\big| + s - n\right\} \\ &= \frac{1}{2}\left\{\frac{|c_s|-|c_0|}{|c_s|+|c_0|} + \big||\mathcal{B}_2'(z)| - |\mathcal{B}_1'(z)|\big| + s - n\right\}.\end{aligned}$$

This completes the proof of Theorem 2.7.

□

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