

On some properties of k -circulant matrices with the generalized 3-primes numbers

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(Communicated by Choonkil Park)

Abstract

In this paper, we obtain explicit forms of the sum of entries, the maximum column sum matrix norm, the maximum row sum matrix norm, Euclidean norm, eigenvalues and the determinant of a k -circulant matrix with the generalized 3-primes numbers. We also study the spectral norm of this k -circulant matrix.

Keywords: 3-primes numbers, circulant matrix, k -circulant matrix, Tribonacci numbers, norm, spectral norm, determinant

2020 MSC: 11B39, 11B83, 15A18, 15A60, 15B36, 11C20

1 Introduction

In this section, we recall definitions and some properties of the generalized 3-primes sequence. A generalized 3-primes sequence $\{V_n\}_{n \geq 0} = \{V_n(V_0, V_1, V_2)\}_{n \geq 0}$ is defined by the third-order recurrence relations

$$V_n = 2V_{n-1} + 3V_{n-2} + 5V_{n-3} \quad (1.1)$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{3}{5}V_{-(n-1)} - \frac{2}{5}V_{-(n-2)} + \frac{1}{5}V_{-(n-3)}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (1.1) holds for all integers n .

Binet formula of generalized 3-primes numbers (Grahaml numbers) can be given as

$$V_n = \frac{b_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)}$$

where

$$b_1 = V_2 - (\beta + \gamma)V_1 + \beta\gamma V_0, \quad b_2 = V_2 - (\alpha + \gamma)V_1 + \alpha\gamma V_0, \quad b_3 = V_2 - (\alpha + \beta)V_1 + \alpha\beta V_0. \quad (1.2)$$

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Here, α, β and γ are the roots of the cubic equation $x^3 - 2x^2 - 3x - 5 = 0$. Moreover

$$\begin{aligned} \alpha &= \frac{2}{3} + \left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3}, \\ \beta &= \frac{2}{3} + \omega \left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \omega^2 \left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3}, \\ \gamma &= \frac{2}{3} + \omega^2 \left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \omega \left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3}, \end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 2, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -3, \\ \alpha\beta\gamma &= 5. \end{aligned}$$

Now we define two special cases of the sequence $\{V_n\}$. 3-primes sequence $\{G_n\}_{n \geq 0}$ and 3-primes-Lucas sequence $\{H_n\}_{n \geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} G_{n+3} &= 2G_{n+2} + 3G_{n+1} + 5G_n, & G_0 = 0, G_1 = 1, G_2 = 2, \\ H_{n+3} &= 2H_{n+2} + 3H_{n+1} + 5H_n, & H_0 = 3, H_1 = 2, H_2 = 10. \end{aligned}$$

The sequences $\{G_n\}_{n \geq 0}$ and $\{H_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{3}{5}G_{-(n-1)} - \frac{2}{5}G_{-(n-2)} + \frac{1}{5}G_{-(n-3)}, \\ H_{-n} &= -\frac{3}{5}H_{-(n-1)} - \frac{2}{5}H_{-(n-2)} + \frac{1}{5}H_{-(n-3)} \end{aligned}$$

for $n = 1, 2, 3, \dots$ respectively.

G_n (the sequence A335718) and H_n (the sequence A335719) are given in [2]. For more details for the generalized 3-primes numbers, see Soykan [2].

The following theorem presents sum formula of generalized 3-primes numbers.

Theorem 1.1. Let x be a nonzero real or complex number. For $n \geq 0$, we have the following formula: If $5x^3 + 3x^2 + 2x - 1 \neq 0$, then

$$\sum_{k=0}^n x^k V_k = \frac{\Theta_1(x)}{\Theta(x)}$$

where

$$\begin{aligned} \Theta_1(x) &= x^{n+3}V_{n+3} - (2x - 1)x^{n+2}V_{n+2} - (3x^2 + 2x - 1)x^{n+1}V_{n+1} - x^2V_2 + x(2x - 1)V_1 + (3x^2 + 2x - 1)V_0, \\ \Theta(x) &= 5x^3 + 3x^2 + 2x - 1. \end{aligned}$$

Proof. The desired result can be obtained by taking $r = 2, s = 3, t = 5$ in [2, Theorem 2.1. (a)]. \square

The following theorem presents sum formulas of generalized 3-primes numbers.

Theorem 1.2. For $n \geq 0$, we have the following formulas:

(a) $\sum_{i=0}^n V_i = \frac{1}{9}(V_{n+3} - 4V_{n+1} - V_{n+2} - V_2 + 4V_0 + V_1).$

- (b) $\sum_{i=0}^n iV_i = \frac{1}{81}((9n + 4)V_{n+3} - (9n + 13)V_{n+2} - 4(9n + 4)V_{n+1} + 5V_2 + 4V_1 - 20V_0)$.
- (c) $\sum_{i=0}^n V_i^2 = \frac{1}{495}(37V_{n+3}^2 + 253V_{n+2}^2 + 430V_{n+1}^2 - 182V_{n+2}V_{n+3} - 170V_{n+1}V_{n+3} + 260V_{n+1}V_{n+2} + 182V_1V_2 + 170V_0V_2 - 260V_0V_1 - 37V_2^2 - 253V_1^2 - 430V_0^2)$.
- (d) $\sum_{i=0}^n iV_i^2 = \frac{1}{245025}((18315n - 26674)V_{n+3}^2 + 11(11385n - 20951)V_{n+2}^2 + 50(4257n - 9080)V_{n+1}^2 - 2(45045n - 73042)V_{n+3}V_{n+2} - 10(8415n - 11279)V_{n+3}V_{n+1} + 10(12870n - 10757)V_{n+1}V_{n+2} - 236174V_1V_2 - 196940V_0V_2 + 236270V_0V_1 + 44989V_2^2 + 355696V_1^2 + 666850V_0^2)$.

Proof.

- (a) Take $x = 1, r = 2, s = 3, t = 5$ in [2, Theorem 2.1 (a)] or take $r = 2, s = 3, t = 5$ in [2, Theorem 2.1 (a)].
- (b) Take $x = 1, r = 2, s = 3, t = 5$ in [2, Theorem 2.1 (a)] or take $r = 2, s = 3, t = 5$ in [2, Theorem 2.1 (a)].
- (c) Take $x = 1, r = 2, s = 3, t = 5$ in [2, Theorem 3.1 (a)]. See also [2, Theorem 2.1].
- (d) Take $x = 1, r = 2, s = 3, t = 5$ in [2, Theorem 2.1 (a)] or take $r = 2, s = 3, t = 5$ in [2, Theorem 2.1 (a)]. \square

Note that, using the recurrence relation $V_{n+3} = 2V_{n+2} + 3V_{n+1} + 5V_n$, we can write the above theorem as follows.

Theorem 1.3. For $n \geq 0$, we have the following formulas:

- (a) $\sum_{i=0}^n V_i = \frac{1}{9}(V_{n+2} - V_{n+1} + 5V_n - V_2 + 4V_0 + V_1) = \frac{\Theta_1}{\Theta}$.
- (b) $\sum_{i=0}^n iV_i = \frac{1}{81}((9n - 5)V_{n+2} - (9n + 4)V_{n+1} + 5(9n + 4)V_n + 5V_2 + 4V_1 - 20V_0) = \frac{\Psi_1}{\Psi}$.
- (c) $\sum_{i=0}^n V_i^2 = \frac{1}{495}(37V_{n+2}^2 + 253V_{n+1}^2 + 925V_n^2 - 182V_{n+1}V_{n+2} - 170V_nV_{n+2} + 260V_nV_{n+1} + 182V_1V_2 + 170V_0V_2 - 260V_0V_1 - 37V_2^2 - 253V_1^2 - 430V_0^2) = \frac{\Delta_1}{\Delta}$.
- (d) $\sum_{i=0}^n iV_i^2 = \frac{1}{245025}((18315n - 44989)V_{n+2}^2 + 11(11385n - 32336)V_{n+1}^2 + 25(18315n - 26674)V_n^2 - 2(45045n - 118087)V_{n+2}V_{n+1} - 10(8415n - 19694)V_{n+2}V_n + 10(12870n - 23627)V_{n+1}V_n - 236174V_1V_2 - 196940V_0V_2 + 236270V_0V_1 + 44989V_2^2 + 355696V_1^2 + 666850V_0^2) = \frac{\Omega_1}{\Omega}$.

From the last theorem, we have the following corollary which gives sum formulas of 3-primes numbers (take $V_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 1.4. For $n \geq 0$, 3-primes numbers have the following properties:

- (a) $\sum_{i=0}^n G_i = \frac{1}{9}(G_{n+2} - G_{n+1} + 5G_n - 1)$.
- (b) $\sum_{i=0}^n iG_i = \frac{1}{81}((9n - 5)G_{n+2} - (9n + 4)G_{n+1} + 5(9n + 4)G_n + 14)$.
- (c) $\sum_{i=0}^n G_i^2 = \frac{1}{495}(37G_{n+2}^2 + 253G_{n+1}^2 + 925G_n^2 - 182G_{n+1}G_{n+2} - 170G_nG_{n+2} + 260G_nG_{n+1} - 37)$.
- (d) $\sum_{i=0}^n iG_i^2 = \frac{1}{245025}((18315n - 44989)G_{n+2}^2 + 11(11385n - 32336)G_{n+1}^2 + 25(18315n - 26674)G_n^2 - 2(45045n - 118087)G_{n+2}G_{n+1} - 10(8415n - 19694)G_{n+2}G_n + 10(12870n - 23627)G_{n+1}G_n + 63304)$.

Taking $V_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last theorem, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

Corollary 1.5. For $n \geq 0$, Lucas 3-primes numbers have the following properties:

- (a) $\sum_{i=0}^n H_i = \frac{1}{9}(H_{n+2} - H_{n+1} + 5H_n + 4)$.
- (b) $\sum_{i=0}^n iH_i = \frac{1}{81}((9n - 5)H_{n+2} - (9n + 4)H_{n+1} + 5(9n + 4)H_n - 2)$.
- (c) $\sum_{i=0}^n H_i^2 = \frac{1}{495}(37H_{n+2}^2 + 253H_{n+1}^2 + 925H_n^2 - 182H_{n+1}H_{n+2} - 170H_nH_{n+2} + 260H_nH_{n+1} - 1402)$.
- (d) $\sum_{i=0}^n iH_i^2 = \frac{1}{245025}((18315n - 44989)H_{n+2}^2 + 11(11385n - 32336)H_{n+1}^2 + 25(18315n - 26674)H_n^2 - 2(45045n - 118087)H_{n+2}H_{n+1} - 10(8415n - 19694)H_{n+2}H_n + 10(12870n - 23627)H_{n+1}H_n + 2709274)$.

2 Main results

Next, we recall some information on k -circulant matrix and Frobenius norm, spectral norm, maximum column length norm and maximum row length norm. Let $n \geq 2$ be an integer and k be any real or complex number. An $n \times n$ matrix $C_k = (c_{ij}) \in M_{n \times n}(\mathbb{C})$ is called a k -circulant matrix if it is of the form

$$C_k = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kc_2 & kc_3 & kc_4 & \cdots & c_0 & c_1 \\ kc_1 & kc_2 & kc_3 & \cdots & kc_{n-1} & c_0 \end{pmatrix}_{n \times n}.$$

k -circulant matrix C_k is denoted by $C_k = \text{Circ}_k(c_0, c_1, \dots, c_{n-1})$.

If $k = 1$ then 1-circulant matrix is called as circulant matrix and denoted by $C = \text{Circ}(c_0, c_1, \dots, c_{n-1})$. Circulant matrix was first proposed by Davis in [2].

The Frobenius (or Euclidean) norm and spectral norm of an $m \times n$ matrix $A = (a_{ij})_{m \times n} \in M_{m \times n}(\mathbb{C})$ are defined respectively as follows:

$$\|A\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad \|A\|_2 = \left(\max_{1 \leq i \leq n} |\lambda_i(A^*A)| \right)^{1/2}$$

where $\lambda_i(A^*A)$'s are the eigenvalues of the matrix A^*A and A^* is the conjugate of transpose of the matrix A . The following inequality holds for any matrix $A = (a_{ij})_{m \times n} \in M_{m \times n}(\mathbb{C})$ (see [2, Theorem 1 and Table 1]):

$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F. \tag{2.1}$$

It follows that

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n} \|A\|_2.$$

In literature there are other types of norms of matrices. The maximum column sum matrix norm of $n \times n$ matrix $A = (a_{ij})$ is $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$ and the maximum row sum matrix norm is $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$. The maximum column length norm $c_1(A)$ and the maximum row length norm $r_1(A)$ of $m \times n$ matrix $A = (a_{ij})$ are defined as follows:

$$c_1(A) = \max_{1 \leq j \leq n} \left(\sum_{i=1}^m |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad r_1(A) = \max_{1 \leq i \leq m} \left(\sum_{j=1}^n |a_{ij}|^2 \right)^{1/2}.$$

There is a relation between $\|\cdot\|_2$, $c_1(\cdot)$ and $r_1(\cdot)$ norms:

Lemma 2.1. [2] For any matrices $A = (a_{ij})_{m \times n} \in M_{m \times n}(\mathbb{C})$ and $B = (b_{ij})_{m \times n} \in M_{m \times n}(\mathbb{C})$, we have

$$\|A \circ B\|_2 \leq r_1(A)c_1(B)$$

and

$$\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2.$$

Furthermore, we have

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$$

where $A \circ B$ is the Hadamard product which is defined by

$$A \circ B = (a_{ij}b_{ij}),$$

and $A \otimes B$ is the Kronecker product which is defined by

$$A \otimes B = (a_{ij}B).$$

For more details on norm of matrices, see for example [2]. In the following Table 1, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (k -circulant, geometric circulant, semicirculant) matrices with the generalized m -step Fibonacci sequences which require sum formulas of second powers of numbers in m -step Fibonacci sequences ($m = 2, 3, 4$).

Table 1. Papers on the norms.

Name of sequence	Papers
second order↓	second order↓
Fibonacci, Lucas	[2,2,2]
Pell, Pell-Lucas	[2,2]
Jacobsthal, Jacobsthal-Lucas	[2,2,2,2]
third order↓	third order↓
Tribonacci, Tribonacci-Lucas	[2,2]
Padovan, Perrin	[2,2,2]
fourth order↓	fourth order↓
Tetranacci, Tetranacci-Lucas	[2]

We need the following two lemmas for our calculations.

Lemma 2.2. [2, Lemma 4] Let $C_k = Circ_k(c_0, c_1, \dots, c_{n-1})$ be an $n \times n$ k -circulant matrix. Then we have

$$\lambda_j(C_k) = \sum_{p=0}^{n-1} k^{\frac{p}{n}} \omega^{-jp} c_p = \sum_{p=0}^{n-1} \left(k^{\frac{1}{n}} \omega^{-j}\right)^p c_p$$

where $\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}, j = 0, 1, 2, \dots, n - 1$. Moreover, in this case

$$c_p = \frac{1}{n} \sum_{j=0}^{n-1} \left(k^{\frac{1}{n}} \omega^{-j}\right)^{-p} \lambda_j(C_k), \quad p = 0, 1, 2, \dots, n - 1.$$

Lemma 2.3. [2] Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of AA^* are $|\lambda_1|^2, |\lambda_2|^2, |\lambda_3|^2, \dots, |\lambda_n|^2$ where A^* is the conjugate of transpose of the matrix A .

Next, we define k -circulant matrix with generalized 3-primes numbers entries. Throughout this paper, the k -circulant matrix, whose entries are the generalized 3-primes numbers, will be denoted by $C_n(V)_k = Circ_k(V_0, V_1, \dots, V_{n-1})$:

Definition 2.4. An $n \times n$ k -circulant matrix with generalized 3-primes numbers entries is defined by

$$C_n(V)_k = Circ_k(V_0, V_1, \dots, V_{n-1}) = \begin{pmatrix} V_0 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ kV_{n-1} & V_0 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ kV_{n-2} & kV_{n-1} & V_0 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ kV_1 & kV_2 & kV_3 & \cdots & kV_{n-1} & V_0 \end{pmatrix}_{n \times n}. \tag{2.2}$$

We call this matrix as generalized 3-primes k -circulant matrix. We consider two special cases of generalized 3-primes k -circulant matrix, namely 3-primes k -circulant matrix: $C_n(G)_k = Circ_k(G_0, G_1, \dots, G_{n-1})$ and Lucas 3-primes k -circulant matrix:

$$C_n(H)_k = Circ_k(H_0, H_1, \dots, H_{n-1}).$$

We denote the sum of entries of $C_n(V)_k$ as $S(C_n(V)_k)$.

Lemma 2.5. The sum of entries of $C_n(V)_k$ is

$$S(C_n(V)_k) = \frac{1}{81}((9kn - 5k + 5)V_{n+2} - (9kn + 4k - 4)V_{n+1} - 4(9kn - 5k + 5)V_n + (5k - 9n - 5)V_2 + (9n + 4k - 4)V_1 + 4(9n - 5k + 5)V_0).$$

Proof . From the definition of $C_n(V)_k$, using Theorem 1.3, we obtain

$$\begin{aligned}
 S(C_n(V)_k) &= nV_0 + ((n - 1) + k)V_1 + ((n - 2) + 2k)V_2 + \dots + (1 + (n - 1)k)V_{n-1} \\
 &= \sum_{i=0}^{n-1} (n - i)V_i + k \sum_{i=1}^{n-1} iV_i \\
 &= n \sum_{i=0}^{n-1} V_i + (k - 1) \sum_{i=1}^{n-1} iV_i \\
 &= \frac{1}{81}((9kn - 5k + 5)V_{n+2} - (9kn + 4k - 4)V_{n+1} - 4(9kn - 5k + 5)V_n \\
 &\quad + (5k - 9n - 5)V_2 + (9n + 4k - 4)V_1 + 4(9n - 5k + 5)V_0).
 \end{aligned}$$

□

Taking $V_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$ and $V_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$, respectively in the last lemma, we obtain the following corollary.

Corollary 2.6. We have the following results:

(a) The sum of entries of $C_n(G)_k$ is

$$S(C_n(G)_k) = \frac{1}{81}((9kn - 5k + 5)G_{n+2} - (9kn + 4k - 4)G_{n+1} - 4(9kn - 5k + 5)G_n + (14k - 9n - 14)).$$

(b) The sum of entries of $C_n(H)_k$ is

$$S(C_n(H)_k) = \frac{1}{81}((9kn - 5k + 5)H_{n+2} - (9kn + 4k - 4)H_{n+1} - 4(9kn - 5k + 5)H_n + (36n - 2k + 2)).$$

Next, we present the maximum column sum matrix norm $\|C_n(V)_k\|_1$ and the maximum row sum matrix norm $\|C_n(V)_k\|_\infty$ of the matrix $C_n(V)_k = (c_{ij})$ under certain condition on the generalized 3-primes sequence V_n and k .

Theorem 2.7. Suppose that $V_p \geq 0$ for all the nonnegative integers p . Then we have the following formulas: If $k \geq 1$ then

$$\|C_n(V)_k\|_1 = \|C_n(V)_k\|_\infty = \frac{1}{9}(kV_{n+2} - kV_{n+1} - 4kV_n - kV_2 + kV_1 + (9 - 5k)V_0),$$

and if $k < 1$ then

$$\|C_n(V)_k\|_1 = \|C_n(V)_k\|_\infty = \frac{1}{9}(V_{n+2} - V_{n+1} - 4V_n - V_2 + 4V_0 + V_1).$$

Proof . Suppose that $k \geq 1$. Then from the definition of the matrix $C_n(V)_k = (c_{ij})$, using Theorem 1.3, we can write

$$\begin{aligned}
 \|C_n(V)_k\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^n |c_{ij}| = \max_{1 \leq j \leq n} \{|c_{1j}| + |c_{2j}| + |c_{3j}| + \dots + |c_{nj}|\} \\
 &= |c_{11}| + |c_{21}| + |c_{31}| + \dots + |c_{n1}| \\
 &= V_0 + kV_{n-1} + kV_{n-2} + \dots + kV_3 + kV_2 + kV_1 \\
 &= (V_0 - kV_0 - kV_n) + k \sum_{i=0}^n V_i \\
 &= (V_0 - kV_0 - kV_n) + \frac{k}{9}(V_{n+2} - V_{n+1} + 5V_n - V_2 + 4V_0 + V_1) \\
 &= (V_0 - kV_0 - kV_n) + \frac{k}{9}(V_{n+2} - V_{n+1} + 5V_n - V_2 + 4V_0 + V_1) \\
 &= \frac{1}{9}(kV_{n+2} - kV_{n+1} - 4kV_n - kV_2 + kV_1 + (9 - 5k)V_0)
 \end{aligned}$$

Similarly, we have

$$\|C_n(V)_k\|_\infty = \frac{1}{9}(kV_{n+2} - kV_{n+1} - 4kV_n - kV_2 + kV_1 + (9 - 5k)V_0).$$

Suppose now that $k < 1$. Then from the definition of the matrix $C_n(V)_k = (c_{ij})$, using Theorem 1.3, we can write

$$\begin{aligned} \|C_n(V)_k\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^n |c_{ij}| = \max_{1 \leq j \leq n} \{|c_{1j}| + |c_{2j}| + |c_{3j}| + \dots + |c_{nj}|\} \\ &= |c_{1n}| + |c_{2n}| + |c_{3n}| + \dots + |c_{nn}| \\ &= V_{n-1} + V_{n-2} + \dots + V_3 + V_2 + V_1 + V_0 \\ &= -V_n + \sum_{i=0}^n V_i \\ &= -V_n + \frac{1}{9}(V_{n+2} - V_{n+1} + 5V_n - V_2 + 4V_0 + V_1) \\ &= \frac{1}{9}(V_{n+2} - V_{n+1} - 4V_n - V_2 + 4V_0 + V_1) \end{aligned}$$

Similarly, we have

$$\|C_n(V)_k\|_\infty = \frac{1}{9}(V_{n+2} - V_{n+1} - 4V_n - V_2 + 4V_0 + V_1).$$

□

Taking $V_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$ and $V_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$, respectively in the last theorem, we obtain the following corollary.

Corollary 2.8. We have the following results:

(a) If $k \geq 1$ then

$$\|C_n(G)_k\|_1 = \|C_n(G)_k\|_\infty = \frac{1}{9}(kG_{n+2} - kG_{n+1} - 4kG_n - k),$$

and if $k < 1$ then

$$\|C_n(G)_k\|_1 = \|C_n(G)_k\|_\infty = \frac{1}{9}(G_{n+2} - G_{n+1} - 4G_n - 1).$$

(b) If $k \geq 1$ then

$$\|C_n(H)_k\|_1 = \|C_n(H)_k\|_\infty = \frac{1}{9}(kH_{n+2} - kH_{n+1} - 4kH_n + (27 - 23k)),$$

and if $k < 1$ then

$$\|C_n(H)_k\|_1 = \|C_n(H)_k\|_\infty = \frac{1}{9}(H_{n+2} - H_{n+1} - 4H_n + 4).$$

Now, we determine the Euclidean (Frobenius) norm of k -circulant matrix $C_n(V)_k$.

Theorem 2.9. The Euclidean (Frobenius) norm of k -circulant matrix $C_n(V)_k$ is:

$$\|C_n(V)_k\|_F = \sqrt{n(\varphi_1(V)) + \varphi_2(V)}$$

where

$$\varphi_1(V) = \frac{1}{495}(37V_{n+2}^2 + 253V_{n+1}^2 + 430V_n^2 - 182V_{n+1}V_{n+2} - 170V_nV_{n+2} + 260V_nV_{n+1} + 182V_1V_2 + 170V_0V_2 - 260V_0V_1 - 37V_2^2 - 253V_1^2 - 430V_0^2),$$

$$\varphi_2(V) = \frac{1}{245025}(|k|^2 - 1)((18315n - 44989)V_{n+2}^2 + 11(11385n - 32336)V_{n+1}^2 + 50(4257n - 13337)V_n^2 - 2(45045n - 118087)V_{n+2}V_{n+1} - 10(8415n - 19694)V_{n+2}V_n + 10(12870n - 23627)V_{n+1}V_n - 236174V_1V_2 - 196940V_0V_2 + 236270V_0V_1 + 44989V_2^2 + 355696V_1^2 + 666850V_0^2).$$

Proof . From the definition of the Euclidean norm of a matrix, using Theorem 1.3, we obtain

$$\begin{aligned} (\|C_n(V)_k\|_F)^2 &= \sum_{i=1, j=1}^n |c_{ij}|^2 \\ &= \sum_{i=0}^{n-1} (n-i)V_i^2 + |k|^2 \sum_{i=1}^{n-1} iV_i^2 \\ &= n \sum_{i=0}^{n-1} V_i^2 + (|k|^2 - 1) \sum_{i=1}^{n-1} iV_i^2 \\ &= n(\varphi_1(V)) + \varphi_2(V) \end{aligned}$$

where $\varphi_1(V)$ and $\varphi_2(V)$ are as in the statement of the theorem. Now, it follows that

$$\|C_n(V)_k\|_F = \sqrt{n(\varphi_1(V)) + \varphi_2(V)}.$$

□

Note that

$$\varphi_1(V) = \sum_{i=0}^{n-1} V_i^2$$

and

$$\varphi_2(V) = (|k|^2 - 1) \sum_{i=1}^{n-1} iV_i^2.$$

Taking $V_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$ and $V_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$, respectively in the last theorem, we obtain the following corollary.

Corollary 2.10. We have the following results:

(a) The Euclidean (Frobenius) norm of k -circulant matrix $C_n(G)_k$ is:

$$\|C_n(G)_k\|_F = \sqrt{n(\varphi_1(G)) + \varphi_2(G)}$$

where

$$\begin{aligned} \varphi_1(G) &= \frac{1}{495}(37G_{n+2}^2 + 253G_{n+1}^2 + 430G_n^2 - 182G_{n+1}G_{n+2} - 170G_nG_{n+2} + 260G_nG_{n+1} + 182G_1G_2 + 170G_0G_2 - 260G_0G_1 - 37G_2^2 - 253G_1^2 - 430G_0^2), \\ \varphi_2(G) &= \frac{1}{245025}(|k|^2 - 1)((18315n - 44989)G_{n+2}^2 + 11(11385n - 32336)G_{n+1}^2 + 50(4257n - 13337)G_n^2 - 2(45045n - 118087)G_{n+2}G_{n+1} - 10(8415n - 19694)G_{n+2}G_n + 10(12870n - 23627)G_{n+1}G_n - 236174G_1G_2 - 196940G_0G_2 + 236270G_0G_1 + 44989G_2^2 + 355696G_1^2 + 666850G_0^2). \end{aligned}$$

(b) The Euclidean (Frobenius) norm of k -circulant matrix $C_n(H)_k$ is:

$$\|C_n(H)_k\|_F = \sqrt{n(\varphi_1(H)) + \varphi_2(H)}$$

where

$$\begin{aligned} \varphi_1(H) &= \frac{1}{495}(37H_{n+2}^2 + 253H_{n+1}^2 + 430H_n^2 - 182H_{n+1}H_{n+2} - 170H_nH_{n+2} + 260H_nH_{n+1} - 1402), \\ \varphi_2(H) &= \frac{1}{245025}(|k|^2 - 1)((18315n - 44989)H_{n+2}^2 + 11(11385n - 32336)H_{n+1}^2 + 50(4257n - 13337)H_n^2 - 2(45045n - 118087)H_{n+2}H_{n+1} - 10(8415n - 19694)H_{n+2}H_n + 10(12870n - 23627)H_{n+1}H_n + 2709274). \end{aligned}$$

The following theorem gives us the eigenvalues of the matrix in (2.2).

Theorem 2.11. The eigenvalues of $C_n(V)_k$ are

$$\lambda_j(C_n(V)) = \frac{\Phi_j(V)}{5(k^{\frac{1}{n}}\omega^{-j})^3 + 3(k^{\frac{1}{n}}\omega^{-j})^2 + 2(k^{\frac{1}{n}}\omega^{-j}) - 1}$$

where

$$\Phi_j(V) = kV_n - V_0 - k^{\frac{1}{n}}(-kV_{n+1} + 2kV_n + V_1 - 2V_0)\omega^{-j} + k^{\frac{2}{n}}(kV_{n+2} - 2kV_{n+1} - 3kV_n - V_2 + 2V_1 + 3V_0)\omega^{-2j}$$

and

$$\begin{aligned} \omega &= \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}, \\ j &= 0, 1, 2, 3, \dots, n - 1. \end{aligned}$$

Proof . By using Lemma 2.2, we obtain

$$\begin{aligned} \lambda_j(C_n(V)_k) &= \sum_{p=0}^{n-1} k^{\frac{p}{n}} \omega^{-jp} V_p \\ &= -k\omega^{-jn} V_n + \sum_{p=0}^n k^{\frac{p}{n}} \omega^{-jp} V_p \\ &= -k\omega^{-jn} V_n + \sum_{p=0}^n (k^{\frac{1}{n}} \omega^{-j})^p V_p. \end{aligned}$$

Now using Theorem 1.1 (by putting $x = k^{\frac{1}{n}} \omega^{-j}$) and recurrence relation $V_{n+3} = 2V_{n+2} + 3V_{n+1} + 5V_n$, we obtain required result. \square

Taking $V_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$ and $V_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$, respectively in the last theorem, we obtain the following corollary.

Corollary 2.12. We have the following results:

(a) The eigenvalues of $C_n(G)_k$ are

$$\lambda_j(C_n(G)) = \frac{\Phi_j(G)}{5(k^{\frac{1}{n}} \omega^{-j})^3 + 3(k^{\frac{1}{n}} \omega^{-j})^2 + 2(k^{\frac{1}{n}} \omega^{-j}) - 1},$$

(b) The eigenvalues of $C_n(H)_k$ are

$$\lambda_j(C_n(H)) = \frac{\Phi_j(H)}{5(k^{\frac{1}{n}} \omega^{-j})^3 + 3(k^{\frac{1}{n}} \omega^{-j})^2 + 2(k^{\frac{1}{n}} \omega^{-j}) - 1},$$

where

$$\begin{aligned} \Phi_j(G) &= kG_n - k^{\frac{1}{n}}(-kG_{n+1} + 2kG_n + 1)\omega^{-j} + k^{\frac{2}{n}}(kG_{n+2} - 2kG_{n+1} - 3kG_n)\omega^{-2j}, \\ \Phi_j(H) &= kH_n - 3 - k^{\frac{1}{n}}(-kH_{n+1} + 2kH_n - 4)\omega^{-j} + k^{\frac{2}{n}}(kH_{n+2} - 2kH_{n+1} - 3kH_n + 3)\omega^{-2j}, \\ \omega &= \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}, j = 0, 1, 2, 3, \dots, n - 1. \end{aligned}$$

The following theorem presents the upper and lower bounds of the spectral norm of $C_n(V)_k$.

Theorem 2.13. Let $C_n(V)_k = Circ_k(V_0, V_1, \dots, V_{n-1})$ be a k -circulant matrix. Then if $|k| \geq 1$ then

$$\sqrt{\varphi_1(V)} \leq \|C_n(V)_k\|_2 \leq \sqrt{V_0^2 + |k|^2(-V_0^2 + \varphi_1(V))} \sqrt{1 - V_0^2 + \varphi_1(V)},$$

and if $|k| < 1$ then

$$|k| \sqrt{\varphi_1(V)} \leq \|C_n(V)_k\|_2 \leq \sqrt{n(\varphi_1(V))}$$

where $\varphi_1(V)$ is as in Theorem 2.9.

Proof . Note that we can write $\varphi_1(V)$ as in the following forms.

$$\begin{aligned} \varphi_1(V) &= \sum_{i=0}^{n-1} V_i^2 \\ &= \frac{1}{495}(37V_{n+2}^2 + 253V_{n+1}^2 + 430V_n^2 - 182V_{n+1}V_{n+2} - 170V_nV_{n+2} + 260V_nV_{n+1} \\ &\quad + 182V_1V_2 + 170V_0V_2 - 260V_0V_1 - 37V_2^2 - 253V_1^2 - 430V_0^2), \\ \varphi_1(V) &= V_0^2 + \sum_{i=1}^{n-1} V_i^2 \Rightarrow -V_0^2 + \varphi_1(V) = \sum_{i=1}^{n-1} V_i^2. \end{aligned}$$

From Theorem 2.9, we know that the Euclidean (Frobenius) norm of k -circulant matrix $C_n(V)_k$ is

$$\begin{aligned} (\|C_n(V)_k\|_F)^2 &= \sum_{i=0}^{n-1} (n-i)V_i^2 + |k|^2 \sum_{i=1}^{n-1} iV_i^2 \\ &= n \sum_{i=0}^{n-1} V_i^2 + (|k|^2 - 1) \sum_{i=1}^{n-1} iV_i^2. \end{aligned}$$

If $|k| \geq 1$, then we get, using Theorem 1.3,

$$(\|C_n(V)_k\|_F)^2 \geq \sum_{i=0}^{n-1} (n-i)V_i^2 + \sum_{i=1}^{n-1} iV_i^2 = n \sum_{i=0}^{n-1} V_i^2 = n(\varphi_1(V)).$$

That is,

$$\|C_n(V)_k\|_F \geq \sqrt{n(\varphi_1(V))}.$$

It follows that

$$\frac{\|C_n(V)_k\|_F}{\sqrt{n}} \geq \sqrt{\varphi_1(V)}.$$

Then by (2.1), we obtain

$$\|C_n(V)_k\|_2 \geq \sqrt{\varphi_1(V)}.$$

Similarly, if $|k| < 1$, then we obtain

$$\begin{aligned} \|C_n(V)_k\|_F^2 &= \sum_{i=0}^{n-1} (n-i)V_i^2 + |k|^2 \sum_{i=1}^{n-1} iV_i^2 \\ &\geq \sum_{i=0}^{n-1} (n-i)|k|^2 V_i^2 + |k|^2 \sum_{i=1}^{n-1} iV_i^2 = n|k|^2 \sum_{i=0}^{n-1} V_i^2 \\ &= n|k|^2(\varphi_1(V)). \end{aligned}$$

That is,

$$\|C_n(V)_k\|_F \geq \sqrt{n|k|^2(\varphi_1(V))}.$$

It follows that

$$\frac{\|C_n(V)_k\|_F}{\sqrt{n}} \geq |k| \sqrt{\varphi_1(V)}.$$

Then by considering (2.1), we get

$$\|C_n(V)_k\|_2 \geq |k| \sqrt{\varphi_1(V)}.$$

Now, for $|k| \geq 1$, we give the upper bound for the spectral norm of the matrix $C_n(V)_k$ as follows. Let the matrices B and C be as

$$B = \begin{pmatrix} V_0 & 1 & 1 & \cdots & 1 & 1 \\ kV_{n-1} & V_0 & 1 & \cdots & 1 & 1 \\ kV_{n-2} & kV_{n-1} & V_0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ kV_1 & kV_2 & kV_3 & \cdots & kV_{n-1} & V_0 \end{pmatrix}_{n \times n}$$

and

$$C = \begin{pmatrix} 1 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ 1 & 1 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ 1 & 1 & 1 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}_{n \times n}$$

such that $C_n(V)_k = B \circ C$. Then we obtain

$$r_1(B) = \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |b_{ij}|^2 \right)^{1/2} = \sqrt{V_0^2 + |k|^2 \sum_{j=1}^{n-1} V_j^2} = \sqrt{V_0^2 + |k|^2 (-V_0^2 + \varphi_1(V))},$$

$$c_1(C) = \max_{1 \leq j \leq n} \left(\sum_{i=1}^n |c_{ij}|^2 \right)^{1/2} = \sqrt{1 + \sum_{i=1}^{n-1} V_i^2} = \sqrt{1 - V_0^2 + \varphi_1(V)}.$$

By Lemma 2.1, we have

$$\|C_n(V)_k\|_2 \leq r_1(B)c_1(C) = \sqrt{V_0^2 + |k|^2 (-V_0^2 + \varphi_1(V))} \sqrt{1 - V_0^2 + \varphi_1(V)}.$$

For $|k| < 1$, we give the upper bound for the spectral norm of the matrix $C_n(V)_k$ as follows. We define the matrices D and E as

$$D = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ k & 1 & 1 & \cdots & 1 & 1 \\ k & k & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ k & k & k & \cdots & k & 1 \end{pmatrix}_{n \times n}$$

and

$$E = \begin{pmatrix} V_0 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ V_{n-1} & V_0 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-1} & V_0 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ V_1 & V_2 & V_3 & \cdots & V_{n-1} & V_0 \end{pmatrix}_{n \times n}$$

such that $C_n(V)_k = D \circ E$. Then we obtain

$$r_1(D) = \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |d_{ij}|^2 \right)^{1/2} = \sqrt{n},$$

and

$$c_1(E) = \max_{1 \leq j \leq n} \left(\sum_{i=1}^n |e_{ij}|^2 \right)^{1/2} = \sqrt{\sum_{i=0}^{n-1} V_i^2} = \sqrt{\varphi_1(V)}.$$

By Lemma 2.1, we have

$$\|C_n(V)_k\|_2 \leq r_1(D)c_1(E) = \sqrt{n(\varphi_1(V))}.$$

This completes the proof. \square

We consider two special cases of the above theorem: the upper and lower bounds of the spectral norm of 3-primes k -circulant matrix: $C_n(G)_k = \text{Circ}_k(G_0, G_1, \dots, G_{n-1})$ and the upper and lower bounds of the spectral norm of Lucas 3-primes k -circulant matrix: $C_n(H)_k = \text{Circ}_k(H_0, H_1, \dots, H_{n-1})$.

Firstly, the following corollary gives the upper and lower bounds of the spectral norm of $C_n(G)_k$.

Corollary 2.14. Let $C_n(G)_k = Circ_k(G_0, G_1, \dots, G_{n-1})$ be 3-primes k -circulant matrix. Then if $|k| \geq 1$ then

$$\sqrt{\varphi_1(G)} \leq \|C_n(G)_k\|_2 \leq \sqrt{G_0^2 + |k|^2(-G_0^2 + \varphi_1(G))} \sqrt{1 - G_0^2 + \varphi_1(G)},$$

and if $|k| < 1$ then

$$|k| \sqrt{\varphi_1(G)} \leq \|C_n(G)_k\|_2 \leq \sqrt{n(\varphi_1(G))}$$

where $\varphi_1(G)$ is as in Corollary 2.10.

Proof . The proof is complete by considering $V_n = G_n, G_0 = 0, G_1 = 1, G_2 = 2$ in Theorem 2.13. \square

Secondly, the following corollary gives the upper and lower bounds of the spectral norm of $C_n(H)_k$.

Corollary 2.15. Let $C_n(H)_k = Circ_k(H_0, H_1, \dots, H_{n-1})$ be Lucas 3-primes k -circulant matrix. Then if $|k| \geq 1$ then

$$\sqrt{\varphi_1(H)} \leq \|C_n(H)_k\|_2 \leq \sqrt{H_0^2 + |k|^2(-H_0^2 + \varphi_1(H))} \sqrt{1 - H_0^2 + \varphi_1(H)},$$

and if $|k| < 1$ then

$$|k| \sqrt{\varphi_1(H)} \leq \|C_n(H)_k\|_2 \leq \sqrt{n(\varphi_1(H))}$$

where $\varphi_1(H)$ is as in Corollary 2.10.

Proof. The desired result can be obtained by getting $V_n = H_n, H_0 = 3, H_1 = 2, H_2 = 10$ in Theorem 2.13. \square

Next, we present the determinant of $C_n(V)_k$.

Theorem 2.16. The determinant of $C_n(V)_k$ is given by

$$\det(C_n(V)_k) = \frac{\Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1} \right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1} \right)^n + \left(\frac{\Lambda_3}{\Lambda_1} \right)^n \right)}{(-1)^{n+1}(kH_n + (k - H_{-n})k^2 \times 5^n - 1)}$$

where

$$\begin{aligned} \Lambda_1 &= kV_n - V_0, \\ \Lambda_2 &= k^{\frac{1}{n}}(-kV_{n+1} + 2kV_n + V_1 - 2V_0), \\ \Lambda_3 &= k^{\frac{2}{n}}(kV_{n+2} - 2kV_{n+1} - 3kV_n - V_2 + 2V_1 + 3V_0). \end{aligned}$$

Proof . By considering identities

$$\begin{aligned} \prod_{k=0}^{n-1} (x - y\omega^{-k}) &= x^n - y^n \\ \prod_{j=0}^{n-1} (x - y\omega^{-j} + z\omega^{-2j}) &= x^n \left(1 - \left(\frac{y - \sqrt{y^2 - 4xz}}{2x} \right)^n - \left(\frac{y + \sqrt{y^2 - 4xz}}{2x} \right)^n + \left(\frac{z}{x} \right)^n \right) \end{aligned}$$

and

$$\left(5(k^{\frac{1}{n}}\omega^{-j})^3 + 3(k^{\frac{1}{n}}\omega^{-j})^2 + 2(k^{\frac{1}{n}}\omega^{-j}) - 1 \right) = (\alpha k^{\frac{1}{n}}\omega^{-j} - 1)(\beta k^{\frac{1}{n}}\omega^{-j} - 1)(\gamma k^{\frac{1}{n}}\omega^{-j} - 1),$$

we see that

$$\prod_{j=0}^{n-1} \left(5(k^{\frac{1}{n}}\omega^{-j})^3 + 3(k^{\frac{1}{n}}\omega^{-j})^2 + 2(k^{\frac{1}{n}}\omega^{-j}) - 1 \right) = (-1)^{n+1}(kH_n + (k - H_{-n})k^2 \times 5^n - 1)$$

and

$$\prod_{j=0}^{n-1} \Phi_j(V) = \Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1} \right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1} \right)^n + \left(\frac{\Lambda_3}{\Lambda_1} \right)^n \right),$$

where

$$\begin{aligned} \omega &= \exp(2\pi i/n), \\ \Phi_j(V) &= kV_n - V_0 - k^{\frac{1}{n}}(-kV_{n+1} + 2kV_n + V_1 - 2V_0)\omega^{-j} \\ &\quad + k^{\frac{2}{n}}(kV_{n+2} - 2kV_{n+1} - 3kV_n - V_2 + 2V_1 + 3V_0)\omega^{-2j} \end{aligned}$$

and

$$\begin{aligned} \Lambda_1 &= kV_n - V_0, \\ \Lambda_2 &= k^{\frac{1}{n}}(-kV_{n+1} + 2kV_n + V_1 - 2V_0), \\ \Lambda_3 &= k^{\frac{2}{n}}(kV_{n+2} - 2kV_{n+1} - 3kV_n - V_2 + 2V_1 + 3V_0). \end{aligned}$$

From Theorem 2.11, we have

$$\begin{aligned} \det(C_n(V)_k) &= \prod_{j=0}^{n-1} \lambda_j(C_n(V)_k) \\ &= \prod_{j=0}^{n-1} \frac{\Phi_j(V)}{5(k^{\frac{1}{n}}\omega^{-j})^3 + 3(k^{\frac{1}{n}}\omega^{-j})^2 + 2(k^{\frac{1}{n}}\omega^{-j}) - 1} \\ &= \frac{\prod_{j=0}^{n-1} \Phi_j(V)}{\prod_{j=0}^{n-1} 5(k^{\frac{1}{n}}\omega^{-j})^3 + 3(k^{\frac{1}{n}}\omega^{-j})^2 + 2(k^{\frac{1}{n}}\omega^{-j}) - 1} \\ &= \frac{\Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1} \right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1} \right)^n + \left(\frac{\Lambda_3}{\Lambda_1} \right)^n \right)}{(-1)^{n+1}(kH_n + (k - H_{-n})k^2 \times 5^n - 1)} \end{aligned}$$

which completes the proof. \square

We consider two special cases of the above theorem: the determinant of 3-primes k -circulant matrix: $C_n(G)_k = \text{Circ}_k(G_0, G_1, \dots, G_{n-1})$ and the determinant of Lucas 3-primes k -circulant matrix: $C_n(H)_k = \text{Circ}_k(H_0, H_1, \dots, H_{n-1})$.

Firstly, the following corollary gives the determinant of $C_n(G)_k$.

Corollary 2.17. The determinant of $C_n(G)_k$ is given by

$$\det(C_n(G)_k) = \frac{\Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1} \right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1} \right)^n + \left(\frac{\Lambda_3}{\Lambda_1} \right)^n \right)}{(-1)^{n+1}(kH_n + (k - H_{-n})k^2 \times 5^n - 1)}$$

where

$$\begin{aligned} \Lambda_1 &= kG_n, \\ \Lambda_2 &= k^{\frac{1}{n}}(-kG_{n+1} + 2kG_n + 1), \\ \Lambda_3 &= k^{\frac{2}{n}}(kG_{n+2} - 2kG_{n+1} - 3kG_n). \end{aligned}$$

Proof. The proof is complete by taking $V_n = G_n, G_0 = 0, G_1 = 1, G_2 = 2$ in Theorem 2.16. \square

Secondly, the following corollary gives the determinant of $C_n(H)_k$.

Corollary 2.18. The determinant of $C_n(H)_k$ is given by

$$\det(C_n(H)_k) = \frac{\Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1} \right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1} \right)^n + \left(\frac{\Lambda_3}{\Lambda_1} \right)^n \right)}{(-1)^{n+1}(kH_n + (k - H_{-n})k^2 \times 5^n - 1)}$$

where

$$\begin{aligned}\Lambda_1 &= kH_n - 3, \\ \Lambda_2 &= k^{\frac{1}{n}}(-kH_{n+1} + 2kH_n - 4), \\ \Lambda_3 &= k^{\frac{2}{n}}(kH_{n+2} - 2kH_{n+1} - 3kH_n + 3).\end{aligned}$$

Proof . The results can be obtained by taking $V_n = H_n$, $H_0 = 3$, $H_1 = 2$, $H_2 = 10$ in Theorem 2.16. \square

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