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Multiplicative and almost multiplicative maps in probabilistic normed algebras

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Abstract

Our main purpose of this paper is to study the relationship between multiplicative maps and almost multiplicative maps between probabilistic normed algebras. We first derive some properties of invertible elements and their relation with multiplicative maps. Then we show that every complex homomorphism on elements whose probabilistic norm is equal to 1, is bounded. In the following, we give an open problem about the functionally continuous of unital commutative probabilistic Banach algebra. Finally, we prove that every almost multiplicative map that is not a multiplicative map is continuous.

Keywords: Multiplicative map, almost multiplicative map, probabilistic normed algebras
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1 Introduction and preliminaries

Suppose that A and B are two complex algebras and $\varphi : A \to B$ is a mapping. Then φ is multiplicative mapping if

$$\varphi(xy) = \varphi(x)\varphi(y),$$

for all $x, y \in A$. Also, φ is said to be homomorphism, if φ is a linear mapping and if $B = \mathbb{C}$, then homomorphism φ is called complex homomorphism or character. Many properties of homomorphisms between topological algebras such as Banach algebras, Fréchet algebras,... (one of them properties, is automatic continuity of homomorphisms), have been studied by many authors, for example see the monographs [6, 11, 21, 24, 25, 26, 27, 29]. In 2005 Hejazian et al [16] developed the concept of homomorphism and introduced the concept of n-homomorphism. After that, many authors studied many properties of n-homomorphism, see for example [1, 3, 10, 17, 18, 19, 33, 40].

Jarosz in 1985 [24], initiated a class of mappings between normed algebras called almost multiplicative mapping and proved every almost multiplicative mapping on a Banach algebra is bounded.

Suppose that A and B are two normed algebras, $\varphi : A \to B$ is a mapping and $\delta > 0$. Then φ is δ-multiplicative mapping if

$$\| \varphi(xy) - \varphi(x)\varphi(y) \| \leq \delta \| x \| \| y \|,$$

for all $x, y \in A$. In addition, a mapping φ is almost multiplicative mapping if there exists a constant $\delta > 0$ such that φ is δ-multiplicative mapping.

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Later, many authors investigated the properties of almost multiplicative mappings, for example, see, [20, 23, 25, 26, 37, 38].

In the following years, many researchers have given many generalizations of the concept of almost multiplicative mappings, for example, one can see, [2, 15, 39].

Menger in 1942 [28] introduced probabilistic metric space. He used distribution functions instead of nonnegative real numbers as values of the metric. Probabilistic metric spaces are widely used in probabilistic functional analysis, ε^∞ theory, quantum particle physics, nonlinear analysis and applications, for example see [4, 5, 7, 8, 9]. Following that, many researchers like Schweizer and Sklar [36] became interested in studying probabilistic metric spaces.

The concept of probabilistic or fuzzy normed algebras has been introduced by a number of authors, for example see, [12, 30, 34, 41, 42]. Since then, some researchers have introduced the concept of almost multiplicative mapping on probabilistic or fuzzy normed algebras in various ways and studied some of their properties, for example, one can refer to [22, 31, 32].

In the following, we bring some basic definitions and lemmas which we will need in our main result.

Definition 1.1. A distribution function is a function $F : [-\infty, \infty] \to [0, 1]$, that is left continuous on $\mathbb{R}$ and nondecreasing, moreover, $F(-\infty) = 0$ and $F(\infty) = 1$.

The set of all the distribution functions that $F(0) = 0$ and $\lim_{t\to\infty} F(t) = 1$ is denoted by D^+. The space D^+ is partially ordered by the usual pointwise ordering of functions, and has a maximal element ϵ_0, defined by

$$\epsilon_0(x) = \begin{cases} 0 & x \le 0, \\ 1 & x > 0. \end{cases}$$

Definition 1.2. A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied:

(i) $T(a, b) = T(b, a)$,

(ii) $T(a, T(b, c)) = T(T(a, b), c)$,

(iii) $T(a, b) \ge T(c, d)$ whenever $a \ge c$ and $b \ge d$,

(iv) $T(a, 1) = a$,

for every $a, b, c, d \in [0, 1]$.

Some typical examples of continuous t-norm are $T_p(a, b) = ab$, $T_m(a, b) = \min\{a, b\}$ and $T_l(a, b) = \max\{a + b - 1, 0\}$. It is evident that, as regards the pointwise ordering, $T \le T_m$, for each t-norm T.

For $(a_1, \cdots, a_n) \in [0, 1]^n$ $(n \in \mathbb{N})$, the value $T^n(a_1, \cdots, a_n)$ is defined by $T^1(a_1) = a_1$ and $T^n(a_1, \cdots, a_n) = T(T^{n-1}(a_1, \cdots, a_{n-1}), a_n)$. For each $a \in [0, 1]$, the sequence $(T^n(a))$ is defined by $T^n(a) = T^n(a, \cdots, a)$.

Definition 1.3. A t-norm T is said to be of Hadžić type (abbreviated, H-type) if the sequence of functions $(T^n(a))$ is equicontinuous at $a = 1$, that is

$$\forall\, \varepsilon \in (0, 1), \quad \exists\, \delta \in (0, 1) : \; a > 1 - \delta \Rightarrow T^n(a) > 1 - \varepsilon \quad (n \in \mathbb{N}).$$

The t-norm T_m is a trivial example of a t-norm of H-type, but there are t-norms T of H-type with $T \ne T_m$, see [14]. It is easy to see that if T is of H-type, then T satisfies $\sup_{a\in(0,1)} T(a, a) = 1$.

Definition 1.4. A probabilistic normed space (abbreviated, PN-space) over K (where $K = \mathbb{R}$ or $\mathbb{C}$) is a triplet (A, ν, T), where A is a vector space over K, T is a continuous t-norm and ν is a mapping from A into $\mathcal{D}^+$ (ν_x denoted the value of ν at x) such that the following conditions are satisfied:

(i) $\nu_x = \epsilon_0$ iff $x = 0$, 0 being the null vector in A,

(ii) $\nu_{\lambda x}(t) = \nu_x\left(\frac{t}{|\lambda|}\right)$, for all $\lambda \in K \setminus \{0\}$ and $t \ge 0$,

(iii) $\nu_{x+y}(t+s) \geq T(\nu_x(t), \nu_y(s))$ for all $t, s \geq 0$.

Example 1.5. [35] Let $(A, \| \,.\, \|)$ be a real normed space. If $\nu : A \to \mathcal{D}^+$ is defined by

$$\nu_x(t) = \begin{cases} \frac{t}{t+\|x\|} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then (A, ν, T) is a PN-space where T is any t-norm.

Definition 1.6. Let (A, ν, T) be a PN-space and $x \in A$. An open ball with center x and radius λ $(0 < \lambda < 1)$ in A is the set $B_x(\varepsilon, \lambda) = \{y \in A : \nu_{x-y}(\varepsilon) > 1 - \lambda\}$, for all $\varepsilon > 0$. It is easy to see that $\mathfrak{B} = \{B_x(\varepsilon, \lambda) : \ x \in A, \ \varepsilon > 0, \ \lambda \in (0, 1)\}$ determines a Hausdorff topology for A [36].

Definition 1.7. A sequence (x_n) in a PN-space (A, ν, T) is said to be convergent to a point $x \in A$ if and only if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that $\nu_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0(\varepsilon, \lambda)$. In this case we say that the limit of the sequence (x_n) is x.

Definition 1.8. A sequence (x_n) in a PN-space (A, ν, T) is said to be Cauchy sequence if and only if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that $\nu_{x_{n+p} - x_n}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0(\varepsilon, \lambda)$ and every $p \in \mathbb{N}$, or $\lim_{n\to\infty} \nu_{x_{n+p} - x_n}(t) = 1$, for all $t > 0$ and $p \in \mathbb{N}$.

Also, a PN-space (A, ν, T) is said to be complete if and only if every Cauchy sequence in A converges to an element of A.

The concept of the Cauchy sequence is inspired from that of the G-Cauchy sequence (it belongs to Grabiec [13]). It is easy to see that the limit of a convergent sequence in a PN-space is unique.

Definition 1.9. Every complete PN-space is called a probabilistic Banach space (abbreviated, PB-space).

Definition 1.10. Let (A, ν, T) and (B, μ, S) be two PN-spaces, $\varphi : A \to B$ be a mapping. Then the mapping φ is said to be continuous at a point $x \in A$ if for every sequence (x_n) in A, which converges to x, the sequence $(\varphi(x_n))$ in B converges to $\varphi(x)$.

A mapping φ is said to be continuous on A if φ is continuous at every point in A.

Definition 1.11. A probabilistic normed algebra (abbreviated, PN-algebra) over K (where $K = \mathbb{R}$ or $\mathbb{C}$) is a quadruplet (A, ν, T_1, T_2), where A is an algebra over K, T_1 and T_2 are two continuous t-norms such that (A, ν, T_1) is PN-space and for all $x, y \in A$ and for all $t, s \geq 0$,

$$\nu_{xy}(ts) \geq T_2(\nu_x(t), \nu_y(s)).$$

Example 1.12. [30] Let $(A, \| \,.\, \|)$ be a real normed algebra. If $\nu : A \to \mathcal{D}^+$ is defined by

$$\nu_x(t) = \begin{cases} \frac{t}{t+\|x\|} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then (A, ν, T_m, T_p) is a PN-algebra if and only if for all $x, y \in A$ and for all $t, s > 0$,

$$\| xy \| \leq \| x \| \| y \| + t \| x \| + s \| y \| .$$

Example 1.13. [30] If $\| \,.\, \|_n$ denote supnorm on $[-n, n]$, then $(C(\mathbb{R}), \nu, T_m, T_l)$ is a PN-algebra where

$$\nu_f(t) = \begin{cases} \sup\{\frac{n}{n+1} : \| f \|_n \leq t\} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Definition 1.14. A PN-algebra (A, ν, T_1, T_2) is called a probabilistic Banach algebra (abbreviated, PB-algebra), if (A, ν, T_1) is a PB-space.

Lemma 1.15. [30] If (A, ν, T_1, T_2) is a PN-algebra, then the algebraic operations are continuous.

Definition 1.16. Let (A, ν, T_1, T_2) and (B, μ, S_1, S_2) be two PN-algebras, $\varphi : A \to B$ be a map and $\delta > 0$. If

$$\mu_{\varphi(xy)-\varphi(x)\varphi(y)}(\delta t) \geq T_2(\nu_x(t), \nu_y(t)),$$

for all $x, y \in A$ and for all $t > 0$, then the map φ is called a δ-multiplicative map. Furthermore, a map φ is said to be a δ-homomorphism, if φ is linear and δ-multiplicative map.

Also we say that φ is almost multiplicative map (almost homomorphism) if there exists a constant $\delta > 0$ such that φ is δ-multiplicative map (δ-homomorphism).

Definition 1.17. A PN-algebra (A, ν, T_1, T_2) is said to be functionally continuous, if every complex homomorphism (or character) on (A, ν, T_1, T_2) is continuous.

Let (A, ν, T_1, T_2) be a unital PB-algebra, in this paper we show that if $x \in A$, $\nu_x(\lambda) = 1$ and $\varphi : A \to \mathbb{C}$ is a complex homomorphism, then $\lambda e_A - x$ is invertible and $\mid \varphi(x) \mid < \lambda$. In the following, we give an open problem that if A is a unital commutative PB-algebra such that T_1 and T_2 are two t-norms of H-type. Is A functionally continuous? Finally, we prove that if (A, ν, T_m, T_m) is a PN-algebra and $\varphi : A \to \mathbb{C}$ is a δ-multiplicative map, then φ is a multiplicative map or $\Xi_{\varphi(x)}((1+\delta)t) \geq \nu_x(t)$ for all $x \in A$ and for all $t > 0$.

2 Main results

In this section, we present our main results. Let (A, ν, T_1, T_2) be a unital PN-algebra, with the unit e_A. The set invertible elements of A is denoted by $InvA$.

Theorem 2.1. Suppose that (A, ν, T_1, T_2) is a unital PB-algebra with the unit e_A such that T_1 is a t-norm of H-type, $x \in A$ and $\lim_{n\to\infty} x^n = 0$. Then $e_A - x \in InvA$ and

$$(e_A - x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

Proof . Suppose that $\varepsilon > 0$ and $\lambda \in (0, 1)$, since T_1 is a t-norm of H-type, so there exists $\eta \in (0, 1)$ such that for all $a > 1 - \eta$ and for all $p \in \mathbb{N}$ we conclude that $T_1^p(a) > 1 - \lambda$, also since $\lim_{n\to\infty} x^n = 0$, then there exists $n_0 = n_0(\varepsilon, \eta) \in \mathbb{N}$ such that $\nu_{x^n}(\frac{\varepsilon}{2^p}) > 1 - \frac{\eta}{2}$ for all $n \geq n_0$. Let $S_n = \sum_{k=0}^{n} x^k$, clearly, for every $n, p \in \mathbb{N}$, we have $S_{n+p} - S_n = \sum_{k=n+1}^{n+p} x^k$. Therefore for all $n \geq n_0$ and for every $p \in \mathbb{N}$, we obtain

$$\begin{aligned}
\nu_{S_{n+p}-S_n}(\varepsilon) =& \nu_{\sum_{k=n+1}^{n+p} x^k}(\varepsilon) \geq T_1(\nu_{x^{n+1}}(\frac{\varepsilon}{2}), T_1(\nu_{x^{n+2}}(\frac{\varepsilon}{4}), \cdots, T_1(\nu_{x^{n+p-1}}(\frac{\varepsilon}{2^{p-1}}), \nu_{x^{n+p}}(\frac{\varepsilon}{2^{p-1}}))))\\
\geq& T_1(\nu_{x^{n+1}}(\frac{\varepsilon}{2^p}), T_1(\nu_{x^{n+2}}(\frac{\varepsilon}{2^p}), \cdots, T_1(\nu_{x^{n+p-1}}(\frac{\varepsilon}{2^p}), \nu_{x^{n+p}}(\frac{\varepsilon}{2^p}))))\\
\geq& T_1^{p-1}(1 - \frac{\eta}{2})\\
>& 1 - \lambda,
\end{aligned}$$

so the sequence (S_n) is a Cauchy sequence converges to some element $a \in A$. It is easy to check that, $(e_A - x)S_n = S_n(e_A - x) = e_A - x^{n+1}$, now letting $n \to \infty$, we get $(e_A - x)a = a(e_A - x) = e_A$, hence the result follows. □

In the following proposition, we present a sufficient condition for $\lim_{n\to\infty} x^n = 0$.

Proposition 2.2. Let (A, ν, T_1, T_2) be a PN-algebra such that T_2 is a t-norm of H-type and $x \in A$. If $\nu_x(1) = 1$, then $\lim_{n\to\infty} x^n = 0$.

Proof . Let $\varepsilon > 0$ and $\lambda \in (0, 1)$ be arbitrary. Since T_2 is a t-norm of H-type, so there exists $\eta \in (0, 1)$ such that for all $a > 1 - \eta$ and for all $n \in \mathbb{N}$ we conclude that $T_2^n(a) > 1 - \lambda$. Also, since $\lim_{t\to 1^-} \nu_x(t) = \nu_x(1) = 1$, so there exists

$\delta > 0$ such that for every $t > 1-\delta$ we conclude that $\nu_x(t) > 1-\frac{\eta}{2}$. Whereas $\lim_{n\to\infty} \varepsilon^{\frac{1}{n}} = 1$, there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ we have $\varepsilon^{\frac{1}{n}} > 1-\frac{\delta}{2}$. So for every $n > n_0$ we get

$$\begin{aligned}\nu_{x^n}(\varepsilon) \geq & T_2(\nu_x(\varepsilon^{\frac{1}{n}}), T_2(\nu_x(\varepsilon^{\frac{1}{n}}), \cdots, T_2(\nu_x(\varepsilon^{\frac{1}{n}}), \nu_x(\varepsilon^{\frac{1}{n}})))) \\ = & T_2^{n-1}(\nu_x(\varepsilon^{\frac{1}{n}})) \\ \geq & T_2^{n-1}(\nu_x(1-\frac{\delta}{2})) \\ \geq & T_2^{n-1}(1-\frac{\eta}{2}) \\ > & 1-\lambda.\end{aligned}$$

This shows that $\lim_{n\to\infty} x^n = 0$. □

Now we can ask the following significant question.

Question. Let (A, ν, T_1, T_2) be a PN-algebra and $x \in A$. Under what other conditions the sequence (x^n) converges to zero?

The following corollary is a direct consequence of Theorem 2.1 and Proposition 2.2.

Corollary 2.3. Suppose that (A, ν, T_1, T_2) is a unital PB-algebra with the unit e_A such that T_1 and T_2 are two t-norms of H-type, $x \in A$ and $\nu_x(1) = 1$. Then $e_A - x \in InvA$ and

$$(e_A - x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

Theorem 2.4. Let (A, ν, T_1, T_2) be a unital PB-algebra with the unit e_A such that T_1 and T_2 are two t-norms of H-type, $x \in A$ and $\nu_x(1) = 1$. If $\varphi : A \to \mathbb{C}$ is a homomorphism, then $\mid \varphi(x) \mid < 1$.

Proof . If $\mid \varphi(x) \mid \geq 1$, then we replace x by $\frac{x}{\varphi(x)}$ and by hypotheses,

$$1 \geq \nu_{\frac{x}{\varphi(x)}}(1) = \nu_x(\mid \varphi(x) \mid) \geq \nu_x(1) = 1.$$

Therefore we assume that $\varphi(x) = 1$. By Corollary 2.3 there exists $y \in A$ such that $y = \sum_{n=1}^{\infty} x^n$. Clearly $x + xy = y$ and so

$$1 + \varphi(y) = \varphi(x) + \varphi(x)\varphi(y) = \varphi(x + xy) = \varphi(y),$$

which is absurd, thus $\mid \varphi(x) \mid < 1$, as desired. □

Corollary 2.5. Let (A, ν, T_1, T_2) be a unital PB-algebra with the unit e_A such that T_1 and T_2 are two t-norms of H-type, $x \in A$, $\nu_x(\lambda) = 1$ $(\lambda > 0)$ and $\varphi : A \to \mathbb{C}$ be a complex homomorphism. Then $\lambda e_A - x \in InvA$ and $\mid \varphi(x) \mid < \lambda$.

Proof . In Corollary 2.3 and Theorem 2.4, replace x by $\frac{x}{\lambda}$. □

Example 2.6. It is easy to check that, $C([0,1])$ (the space of continuous functions on [0,1]) with the L^1-norm $\| \, . \, \|_1$ and the usual pointwise operations of functions, is a unital commutative normed algebra. Now it is again easy to see that, $(C([0,1]), \nu, T_m, T_p)$ is a unital commutative PN-algebra where

$$\nu_f(t) = \begin{cases} \frac{t}{t+\|f\|_1} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Define $\varphi : C([0,1]) \to \mathbb{C}$ with $\varphi(f) = f(0)$. Obviously, φ is a complex homomorphism on $C([0,1])$. If $f_n(x) = \frac{1}{(1+x)^n}$, then clearly $f_n \in C([0,1]$, $f_n \to 0$ and $\varphi(f_n) \nrightarrow 0$ as n tends infinity so φ is not continues, hence $(C([0,1]), \nu, T_m, T_p)$ is not functionally continuous.

The above example show that every PN-algebras may not be functionally continuous. So we ask the following significant question.

Question. Let (A,ν,T_1,T_2) be a unital commutative PB-algebra such that T_1 and T_2 are two t-norms of H-type. Is A functionally continuous?

Let (A,ν,T_1,T_2) be a PN-algebra. For each $\alpha\in(0,1)$, define the function $p_\alpha : A\to\mathbb{R}$, by

$$p_\alpha(x)=\sup_{t\in\mathbb{R}}\{t\in\mathbb{R}\ :\ \nu_x(t)\le 1-\alpha\},$$

since $\nu_x\in D^+$, $p_\alpha(x)$ is finite.

Proposition 2.7. Let (A,ν,T_m,T_m) be a PN-algebra over K (where $K=\mathbb{R}$ or $\mathbb{C}$). Then every p_α ($\alpha\in(0,1)$) is a multiplicative seminorm (m-seminorm). Furthermore $p_\alpha(x)=0$ for all $\alpha\in(0,1)$ if and only if $x=0$.

Proof . Clearly $p_\alpha(x)\ge 0$ (since $\nu_x\in\mathcal{D}^+$). Assume that $\lambda\in K$, then for all $\alpha\in(0,1)$ and for all $x\in A$ we have

$$\begin{aligned}p_\alpha(\lambda x)=&\sup_{t\in\mathbb{R}}\{t\in\mathbb{R}\ :\ \nu_{\lambda x}(t)\le 1-\alpha\}=\sup_{t\in\mathbb{R}}\{t\in\mathbb{R}\ :\ \nu_x(\frac{t}{\mid\lambda\mid})\le 1-\alpha\}\\ =&\sup_{t\in\mathbb{R}}\{\mid\lambda\mid t\in\mathbb{R}\ :\ \nu_x(t)\le 1-\alpha\}=\mid\lambda\mid\sup_{t\in\mathbb{R}}\{t\in\mathbb{R}\ :\ \nu_x(t)\le 1-\alpha\}\\ =&\mid\lambda\mid p_\alpha(x).\end{aligned}$$

Now we show that $p_\alpha(xy)\le p_\alpha(x)p_\alpha(y)$, for all $\alpha\in(0,1)$ and for all $x,y\in A$. To this end, suppose that for some $x,y\in A$, $p_\alpha(xy)>p_\alpha(x)p_\alpha(y)$, we can choose $t>p_\alpha(x)$, $s>p_\alpha(y)$ and $ts<p_\alpha(xy)$, thus $\nu_x(t)>1-\alpha$, $\nu_y(s)>1-\alpha$ and $\nu_{xy}(ts)\le 1-\alpha$. If $\beta=\min\{\nu_x(t),\nu_y(s)\}$, then we have

$$1-\alpha\ge\nu_{xy}(ts)\ge T_m(\nu_x(t),\nu_y(s))\ge T_m(\beta,\beta)=\beta,$$

hence $\nu_x(t)\le 1-\alpha$ or $\nu_y(s)\le 1-\alpha$, which is a contradiction. Finally, by using a similar argument, we can show that $p_\alpha(x+y)\le p_\alpha(x)+p_\alpha(y)$, for all $\alpha\in(0,1)$ and for all $x,y\in A$. If $x=0$, then $\nu_x=\epsilon_0$ and for all $\alpha\in(0,1)$ obviously $\{t\in\mathbb{R}\ :\ \nu_x(t)\le 1-\alpha\}=(-\infty,0]$, so $p_\alpha(x)=0$ for all $\alpha\in(0,1)$. Conversely, if $p_\alpha(x)=0$ for all $\alpha\in(0,1)$, then for $s>0$ we have $s\notin\{t\in\mathbb{R}\ :\ \nu_x(t)\le 1-\alpha\}$, so $1-\alpha<\nu_x(s)\le 1$ for all $\alpha\in(0,1)$, thus $\nu_x=\epsilon_0$, as desired. □

Theorem 2.8. Let (A,ν,T_m,T_m) be a PN-algebra. Then the topology on A generated by the family of m-seminorms $\{p_\alpha :\ \alpha\in(0,1)\}$ is the same as the topology generated by the system of (α,ε)-neighborhoods.

Proof . It is easy to check that (for all $x\in A$ and for all $\alpha\in(0,1)$) $p_\alpha(x)<\varepsilon$ if and only if $\nu_x(\varepsilon)>1-\alpha$, so

$$B_0(\alpha,\varepsilon)=\{x\in A:\ \nu_x(\varepsilon)>1-\alpha\}=\{x\in A:\ p_\alpha(x)<\varepsilon\}=B_0(p_\alpha,\varepsilon).$$

Therefore the topology generated by the system of (α,ε)-neighborhoods coincides with the topology generated by the family of m-seminorms $\{p_\alpha :\ \alpha\in(0,1)\}$. □

Lemma 2.9. Let $A=\mathbb{R}$ or $\mathbb{C}$. If $\Xi : A\to\mathcal{D}^+$ is defined by

$$\Xi_x(t)=\begin{cases}\frac{t}{t+|x|} & t>0,\\ 0 & t\le 0.\end{cases}$$

Then the quadruple (A,Ξ,T_m,T_p) is PN-algebra and $\xi_\alpha(x)=\left(\frac{1}{\alpha}-1\right)\mid x\mid$, for all $\alpha\in(0,1)$, where ξ_α is the m-seminorm associated with the probabilistic norm Ξ.

Proof . By Example 1.12, (A,Ξ,T_m,T_p) is PN-algebra and it is easy to see that $\xi_\alpha(x)=\left(\frac{1}{\alpha}-1\right)\mid x\mid$. □

Remark 2.10. Throughout this paper, we denote the usual probabilistic norm on $\mathbb{R}$ (or $\mathbb{C}$) by Ξ and the m-seminorms associated with the probabilistic norm Ξ by ξ_α, which is defined in the above lemma, also we consider $\mathbb{R}$ or $\mathbb{C}$ with quadruplet $(\mathbb{R}\ (\ or\ \mathbb{C}),\Xi,T_m,T_p)$ as a PN-algebra.

Proposition 2.11. Let (A,ν,T_m,T_m) be a PN-algebra and (p_α) be the m-seminorms associated with the probabilistic norm ν. If $\varphi : A\to\mathbb{C}$ is a δ-multiplicative map, then

$$\xi_\alpha(\varphi(xy)-\varphi(x)\varphi(y))\le\delta p_\alpha(x)p_\alpha(y),$$

for all $\alpha\in(0,1)$ and for all $x,y\in A$.

Proof . Assume towards a contradiction that there exist $\alpha \in (0,1)$ and $x, y \in A$ such that

$$\delta p_\alpha(x)p_\alpha(y) < \xi_\alpha(\varphi(xy) - \varphi(x)\varphi(y)).$$

So there exist $t, s > 0$ such that $t > p_\alpha(x)$, $s > p_\alpha(y)$ and $\delta ts < \xi_\alpha(\varphi(xy)-\varphi(x)\varphi(y))$, thus $\nu_x(t) > 1-\alpha$, $\nu_y(s) > 1-\alpha$ and $\Xi_{\varphi(xy)-\varphi(x)\varphi(y)}(\delta ts) \leq 1-\alpha$. If $\beta = \min\{\nu_x(t), \nu_y(s)\}$, then we have

$$1-\alpha \geq \Xi_{\varphi(xy)-\varphi(x)\varphi(y)}(\delta ts) \geq T_m(\nu_x(t), \nu_y(s)) \geq T_m(\beta, \beta) = \beta,$$

hence $\nu_x(t) \leq 1-\alpha$ or $\nu_y(s) \leq 1-\alpha$, which is a contradiction. □

Lemma 2.12. Let (A, ν, T_m, T_m) be a PN-algebra and (p_α) be the m-seminorms associated with the probabilistic norm ν. If $x \in A$ and $\varphi : A \to \mathbb{C}$ is a map such that for all $\alpha \in (0,1)$,

$$\big(\frac{1}{\alpha} - 1\big) \mid \varphi(x) \mid \leq kp_\alpha(x).$$

Then

$$\Xi_{\varphi(x)}(kt) \geq \nu_x(t),$$

for all $t > 0$.

Proof . Assume towards a contradiction that there exists $t > 0$ such that

$$\Xi_{\varphi(x)}(kt) < \nu_x(t).$$

Choose $\alpha \in (0,1)$ such that $\Xi_{\varphi(x)}(kt) < 1-\alpha < \nu_x(t)$, since ν_x is left continuous at t, there exists $s < t$ such that $1-\alpha < \nu_x(s)$. Now it is easy to see that

$$p_\alpha(x) \leq s < t \leq \xi_\alpha(\frac{\varphi(x)}{k}) = \big(\frac{1}{\alpha} - 1\big) \mid \frac{\varphi(x)}{k} \mid,$$

which is a contradiction. □

Theorem 2.13. Let (A, ν, T_m, T_m) be a PN-algebra and $\varphi : A \to \mathbb{C}$ be a δ-multiplicative map. Then φ is a multiplicative map or for all $\alpha \in (0,1)$ there exists a constant $k_\alpha > 0$ such that $\xi_\alpha(\varphi(x)) \leq k_\alpha p_\alpha(x)$ for all $x \in A$, where (p_α) $(\alpha \in (0,1))$ is the m-seminorms associated with the probabilistic norm ν.

Proof . Suppose that φ is not multiplicative thus there exist $a, b \in A$ such that $\varphi(ab) \neq \varphi(a)\varphi(b)$. By Proposition 2.11 and Lemma 2.9, for all $\alpha \in (0,1)$ and for all $x, y \in A$, we have

$$\big(\frac{1}{\alpha} - 1\big) \mid \varphi(xy) - \varphi(x)\varphi(y) \mid \leq \delta p_\alpha(x)p_\alpha(y).$$

Note that for all $\alpha \in (0,1)$ $p_\alpha(a) \neq 0 \neq p_\alpha(b)$. Now for all $\alpha \in (0,1)$ and for all $x \in A$, we get

$$\begin{aligned}
\big(\frac{1}{\alpha} - 1\big) \mid \varphi(x) \mid\mid \varphi(ab) - \varphi(a)\varphi(b) \mid =& \big(\frac{1}{\alpha} - 1\big) \mid \varphi(x)\varphi(ab) - \varphi(x)\varphi(a)\varphi(b) \mid \\
=& \big(\frac{1}{\alpha} - 1\big) \mid \varphi(x)\varphi(ab) - \varphi(x)\varphi(a)\varphi(b) \pm \varphi(xab) \pm \varphi(xa)\varphi(b) \mid \\
\leq& \big(\frac{1}{\alpha} - 1\big)\Big(\mid \varphi(x)\varphi(ab) - \varphi(xab) \mid + \mid \varphi(xab) - \varphi(xa)\varphi(b) \mid \\
&+ \mid \varphi(xa)\varphi(b) - \varphi(x)\varphi(a)\varphi(b) \mid \Big) \\
\leq& \delta p_\alpha(x)p_\alpha(ab) + \delta p_\alpha(xa)p_\alpha(b) + \delta p_\alpha(x)p_\alpha(a) \mid \varphi(b) \mid \\
\leq& 2\delta p_\alpha(x)p_\alpha(a)p_\alpha(b) + \delta p_\alpha(x)p_\alpha(a) \mid \varphi(b) \mid \\
=& p_\alpha(x)\Big(2\delta p_\alpha(a)p_\alpha(b) + \delta p_\alpha(a) \mid \varphi(b) \mid \Big).
\end{aligned}$$

If

$$k_\alpha = \frac{2\delta p_\alpha(a)p_\alpha(b) + \delta p_\alpha(a) \mid \varphi(b) \mid}{\mid \varphi(ab) - \varphi(a)\varphi(b) \mid},$$

then for all $\alpha \in (0,1)$ and for all $x \in A$ we obtain

$$\xi_\alpha(\varphi(x)) = \big(\frac{1}{\alpha} - 1\big) \mid \varphi(x) \mid \leq k_\alpha p_\alpha(x),$$

as desired. $\square$

Now, we state the most important theorem of this paper as follows.

Theorem 2.14. Let (A, ν, T_m, T_m) be a PN-algebra and $\varphi : A \to \mathbb{C}$ be a δ-multiplicative map. Then φ is a multiplicative map or $\Xi_{\varphi(x)}((1+\delta)t) \geq \nu_x(t)$ for all $x \in A$ and for all $t > 0$.

Proof . Suppose that φ is not multiplicative, we show that $\Xi_{\varphi(x)}((1+\delta)t) \geq \nu_x(t)$ for all $x \in A$ and for all $t > 0$. To this end, assume towards a contradiction that there exist $x_0 \in A$ and $t_0 > 0$ such that $\Xi_{\varphi(x_0)}((1+\delta)t_0) < \nu_{x_0}(t_0)$. By Theorem 2.13, for all $\alpha \in (0,1)$ there exists $k_\alpha > 0$ such that

$$\big(\frac{1}{\alpha} - 1\big) \mid \varphi(x) \mid \leq k_\alpha p_\alpha(x), \tag{2.1}$$

for all $x \in A$. It is easy to check that there exists $\alpha_0 \in (0,1)$ such that

$$\big(\frac{1}{\alpha_0} - 1\big) \mid \varphi(x_0) \mid > (1+\delta) p_{\alpha_0}(x_0), \tag{2.2}$$

by Lemma 2.12. By (2.1) and (2.2) we get $p_{\alpha_0}(x_0) \neq 0$. Also, there exists $q > 0$ such that $\big(\frac{1}{\alpha_0} - 1\big) \mid \varphi(x_0) \mid = (1+\delta+q)p_{\alpha_0}(x_0)$. By Proposition 2.11 and Lemma 2.9, for all $\alpha \in (0,1)$ and for all $x, y \in A$, we have

$$\big(\frac{1}{\alpha} - 1\big) \mid \varphi(xy) - \varphi(x)\varphi(y) \mid \leq \delta p_\alpha(x) p_\alpha(y).$$

Now we show by induction that

$$\big(\frac{1}{\alpha_0} - 1\big) \mid \varphi(x_0^{2^m}) \mid \geq (1+\delta+mq)(p_{\alpha_0}(x_0))^{2^m}. \tag{2.3}$$

for every $m \in \mathbb{N}$. Using the above results, we have

$$\begin{aligned}
\big(\frac{1}{\alpha_0} - 1\big) \mid \varphi(x_0^2) \mid \geq & \big(\frac{1}{\alpha_0} - 1\big) \mid \varphi(x_0)^2 \mid - \big(\frac{1}{\alpha_0} - 1\big) \mid \varphi(x_0^2) - \varphi(x_0)^2 \mid \\
\geq & (1+\delta+q)^2 (p_{\alpha_0}(x_0))^2 - \delta (p_{\alpha_0}(x_0))^2 \\
\geq & (1+\delta+q)(p_{\alpha_0}(x_0))^2,
\end{aligned}$$

thus (2.3) is true for $m = 1$. Now suppose that (2.3) for m holds, so we obtain

$$\begin{aligned}
\big(\frac{1}{\alpha_0} - 1\big) \mid \varphi(x_0^{2^{m+1}}) \mid \geq & \big(\frac{1}{\alpha_0} - 1\big) \mid \varphi(x_0^{2^m})^2 \mid - \big(\frac{1}{\alpha_0} - 1\big) \mid \varphi(x_0^{2^{m+1}}) - \varphi(x_0^{2^m})^2 \mid \\
\geq & (1+\delta+mq)^2 (p_{\alpha_0}(x_0))^{2^{m+1}} - \delta (p_{\alpha_0}(x_0^{2^m}))^2 \\
\geq & (1+\delta+mq)^2 (p_{\alpha_0}(x_0))^{2^{m+1}} - \delta (p_{\alpha_0}(x_0))^{2^{m+1}} \\
\geq & (1+\delta+(m+1)q)(p_{\alpha_0}(x_0))^{2^{m+1}}.
\end{aligned}$$

Hence (2.3) holds for all $m \in \mathbb{N}$. Now for each $x, y \in A$, we have

$$\begin{aligned}
(1+\delta+mq)(p_{\alpha_0}(x_0))^{2^m} \big(\frac{1}{\alpha_0} - 1\big) \mid \varphi(xy) - \varphi(x)\varphi(y) \mid \leq & \big(\frac{1}{\alpha_0} - 1\big)^2 \mid \varphi(x_0^{2^m}) \mid\mid \varphi(xy) - \varphi(x)\varphi(y) \mid \\
\leq & k_{\alpha_0} \delta p_{\alpha_0}(x) p_{\alpha_0}(y) p_{\alpha_0}(x_0^{2^m}) \\
\leq & k_{\alpha_0} \delta p_{\alpha_0}(x) p_{\alpha_0}(y) (p_{\alpha_0}(x_0))^{2^m},
\end{aligned}$$

so

$$\mid \varphi(xy) - \varphi(x)\varphi(y) \mid \leq \frac{\alpha_0 k_{\alpha_0} \delta p_{\alpha_0}(x) p_{\alpha_0}(y)}{(1-\alpha_0)(1+\delta+mq)}.$$

We let m go to infinity, we obtain φ is multiplicative, which is a contradiction, therefore

$$\Xi_{\varphi(x)}((1+\delta)t) \geq \nu_x(t),$$

for all $x \in A$ and for all $t > 0$. $\square$

Corollary 2.15. Let (A, ν, T_m, T_m) be a functionally continuous PN-algebra and $\varphi : A \to \mathbb{C}$ be a almost homomorphism. Then φ is automatically continuous.

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