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Proposing a portfolio optimization model based on the GARCH-EVT-Copula combined approach

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Abstract

This study aims at optimizing the portfolio of financial assets and in particular focuses on the stock market with conditional value at risk (CVaR) as the portfolio risk measure. This study uses generalized conditional heterogeneity variance methods, the dependency structure, the extreme value theory, and with the GARCH-EVT-Vine-Copula approach to optimize the portfolio and minimize the CVaR of a stock portfolio during a certain period by the re-weighting method. Modeling is based on the performance data of 7 companies among the top 50 listed companies during the period 2015 to 2021. The results show that considering the extreme values and structural dependence between the examined time series improves the risk identification between these markets. In addition, among the studied models, the out-of-sample results for the accumulated wealth function of different models show that when considering the dependence structure, the EGARCH-EVT model based on the Coppola Vine function results outperforms other models.

Keywords: Portfolio optimization, Extreme value theory, Copulas functions, GARCH, Conditional value at risk
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1 Introduction

In recent decades, many efforts have been made to guide investors for proper investment and countless models have been offered. Concepts such as stock portfolio optimization and diversification have become important tools for developing and understanding financial markets and decision-making. The model proposed by Markowitz [19] has led to many changes and improvements in people's attitudes to investment and the portfolio, and it has been used as an efficient tool in portfolio optimization. Although minimizing risk and maximizing return on investment may seem simple, there are several methods in practice to build an optimal portfolio. Many people have tried to develop and modify his model; therefore, a lot of research has been done to improve his basic model both computationally and theoretically.

The concept of dependence between different stock returns is an important issue in the financial literature on portfolio optimization. Previous approaches are focused on the common distribution between returns. In the financial

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economics literature, an alternative approach to modeling the dependency structure between multivariate data, without imposing any assumptions on the marginal distributions, is proposed based on copula functions, with shortcomings such as linear correlation coefficient, asymmetry and sequential dependence between distributions of financial returns. Thus, the copula parameter more than the linear correlation coefficient has been used in the financial literature to calculate the portfolio value-at-risk in the last decade [21].

Calculations to examine portfolio risk have led to the emergence of new and broad issues in the field of stock portfolio risk calculation. For example, studies such as Karmakar [15], Han et al. [14] and Sahamkhadam et al. [25] used a combinatory approach to simultaneously consider conditional heteroscedasticity between financial returns, the nonlinear structural dependence, and extreme value in the returns distribution sequences, generally called GARCH-EVT-Copula. Consideration of all the three features in returns distribution simultaneously leads to the optimal calculation of portfolio risk values by the conditional value at risk method so that results of optimizing this function in these studies improved the stock portfolio value.

As a result, reviewing and optimizing the asset portfolio is one of the most important concerns of risk management to find the optimal stock portfolio with the highest profitability and lowest risk that is investigated in this study using a new approach. This study examines the Copula-Vine dependency structure model with the extreme value theory approach as well as the GARCH-EVT-Vine-Copula model.

In the following, the research methodology is explained. Then, the theoretical and experimental basis of the above topics as well as the experimental findings of the present research are presented. At the end, the conclusions and future research are discussed.

2 Theoretical foundations of research

Stock portfolio optimization and diversification are considered tools for developing and understanding financial markets as well as decision making. Markowitz's model published in 1952 resulted in great changes and improvements in the way people view investment and portfolios and was used as an efficient tool for portfolio optimization. Minimizing risk and maximizing return on investment may seem simple, but in practice, there are several methods to build an optimal portfolio. Many people have tried to improve and modify Markowitz's model, and so there is much research on the improvement of this basic model computationally and theoretically.

Research on financial markets shows that the distribution of returns in these markets is not normal. As a result, the theory of determining the optimal portfolio based on adverse risk (postmodern portfolio theory) is proposed. This theory is based on considering various risk criteria such as the semi-variance model, mean absolute deviation model, variance model with skewness and the VaR criterion. In other words, this theory is based on the relationship between return and adverse risk and explains the investor's behavior and the criteria for selecting the optimal portfolio. The model for determining the optimal portfolio in the context of the VaR is the most important model in this area [7].

The VaR's widespread use as a tool for risk assessment has provided extensive literature in the field of financial economics and risk. These methods seek to improve the performance of this risk assessment criterion. In this setting, the CVaR has attracted the most attention for stock portfolio optimization so most stock portfolio optimization functions seek to minimize the CVaR [15].

The previous section showed that calculations to examine portfolio risk have led to the emergence of new and broad issues in the field of stock portfolio risk calculation so that studies such as Karmakar [15], Han et al. [14] and Sahamkhadam et al. [25] have used a combinatory approach to simultaneously considering conditional heteroscedasticity between financial returns, the nonlinear structural dependence, and extreme value in the returns distribution sequences, which is generally called GARCH-EVT-Copula. Consideration of all the three features in returns distribution simultaneously leads to the optimal calculation of portfolio risk values using the conditional value at risk method so that the results of optimizing this function in these studies improved the stock portfolio value.

Despite the widespread use of Copula functions in recent years in financial discussions, the development of these functions in most studies with the GARCH-EVT-Copula method is less considered and most of them have used pair and multidimensional Copula functions with a type of copula dependence. The study of N-dimensional Copula functions is often more difficult so its complex calculations need the use of elliptical copula functions, including the normal distribution and t-test at all levels. Aas et al. [1] to develop and solve the fixed approach problems in Copula functions linked both of these variables to each other by two-dimensional copula functions and used a structure called vine. The vine approach is hierarchical so that in the first stage, the pair-copula functions between the variables are determined and in the second stage, the pair-copula functions between the conditioned variables are used with respect to the variables between the two variables and this process continues until the last stage. Therefore, for N variables exist there is a $N(N-1)/2$ two-dimensional copula function, and the combined distribution of these N variables can be obtained. Thus, the estimation problem is eliminated by considering the same in the Copula function as the Copula elliptic functions, and this issue can be covered [1]. Accordingly, different copula functions can be used simultaneously according to the dependency type, their suitability, and with a combination of elliptical and Archimedean functions using the vine structure, so that the dependency structure between financial markets can be well calculated and included in the CVaR.

Thus, as mentioned, the defect of portfolio optimization papers with the GARCH-EVT-Copula approach is the consideration of the structure of pair or multivariate copula functions in the same way (i.e. using a copula function to measure the dependency structure between the portfolio stocks). However, as mentioned by [1] and [5], this approach cannot represent the dependency structure well, but using the vine structure can overcome these problems. Therefore, the present study to solve the problem of considering the same copula functions for all levels of the dependency structure between financial returns in the stock portfolio uses the vine copula functions with consideration of conditional dependency structure and a combination of copula functions. In general, according to the above, some reasons for the superiority of this type of function are as follows:

1. Using a combination of copula functions
2. Consideration of the conditional dependence structure between variables
3. Using more variables in examining the dependence structure

For example, when considering N different variables, the use of this approach for each pair of variables causes a different copula function (i.e. a total of $N(N-1)/2$ different copula functions) to determine the dependency structure. In addition, the dependence structures of variables are also measured in presence of other variables [5]. Thus, the basis and the main difference between this approach and its combination with the extreme value theory and modeling of conditional variance has made this research different from the similar research in terms of advanced modeling and the reviewing approach.

3 Empirical foundations of the research

Research on optimization of the financial asset portfolio is mainly suggested in two main steps over the past few decades. The first step involves predicting the future returns of financial assets. In the second step, we examine the optimization function of the intended problem and solve it to determine the optimal weight of each financial asset in the portfolio. In both steps, different methods have been used over the past decades. However, the growing trend of mathematical and statistical theories has led to the application of various development optimization approaches in each of these steps. Patton [21] examined monthly data of the Center for Research in Security Prices (CRSP) from January 1954 to December 1999 and used copula's theory to determine the skewness and asymmetry in financial returns to build an investor's portfolio.

Poon et al. [23] used the extreme value theory and two nonparametric methods in five international stock markets to describe the portfolio return sequence dependence and showed that the return volatility correlation between international markets increased over time so that this dependence was greater in European markets (e.g. the UK, France and Germany) than in Asian markets (e.g., Japan) and North America (e.g., the US). However, these markets (including European, North American and Asian countries) were not asymptotically dependent, which is an important point to be considered in creating the optimal asset portfolio.

Boubaker and Sghaier [2] used Archimedean copula functions and optimized the portfolio consisting of a pair of exchange rates and two stocks in the case of their long-term memory and dependency. They used the euro and the Japanese yen against the dollar as well as the Dow Jones and CAC 40 stock indexes from July 1995 to July 2008. Their results show that the dependence between the two pairs of exchange rates and the two stock indexes is asymmetric. They also found that the use of the value at risk in the case of dependence improves the calculations.

Christoffersen et al. [3] investigated the effect of dependency structure on explaining the optimal stock portfolio based on copula functions. Their results showed that the use of Archimedean functions and consideration of structural dependence in the return of different currencies significantly improves the stock portfolio optimization performance.

Luo et al. [18] used the OGARCH method and Markov Switching to optimize the portfolio for mutual funds from 2000 to 2014. The researchers also tested the model capacity using the variance-covariance approach. They used Sharpe criteria and calculated boundaries of full efficiency in a suitable time period.

Considering the structural dependence, Karmakar [15] used the AR-GARCH-EVT approach and tried to optimize different pairs of currencies. Their results show that the use of extreme value theory and consideration of structural dependence compared to the variance-covariance approach improves the ratio of the value at risk to return in optimizing the stock portfolio.

Krzemienowski and Szymczyk [17] used Archimedean copula functions to optimize the stock portfolio by the structural-dependence-based value at risk method and the copula functions. Their results showed that the mentioned method provides better results for the wealth accumulation function using the value at risk and consideration of the variance-covariance approach compared to other common approaches.

Han et al. [14] used the DCC-GARCH-Copula approach for portfolio optimization. Their results showed that the use of the copula functions approaches for optimization and the use of the GARCH method in conditional dynamic dependence for the stock portfolio optimization with consideration of CVaR lead to better results than conventional approaches.

Sahamkhadam et al. [25] investigated the optimization of major stock indexes in different countries using the CVaR method and the extreme value theory with conditional heteroscedasticity and consideration of structural dependence (i.e. the use of Gaussian Copulas and t-test). Their results show that the use of the reweighting approach and one-day forecast provide better results in terms of reducing portfolio risk compared to several other common models.

Fallahpour and Baghban [10] considered the normal copula function in calculating the CVaR of a two-variable portfolio. Fallah and Alizadeh [8] estimated the value at risk and the expected fall of the portfolio containing four investment companies on the Tehran Stock Exchange with an emphasis on the ARIMA-GARCH-COPULA approach and compared it with a performance of generalized extreme value (GEV) approaches. Many studies have used models to calculate the risk of portfolios containing two stocks (e.g., [9, 12, 16, 22, 24]). Many other studies have considered approaches like the GARCH method, copula functions, and the extreme value theory (EVT) separately.

4 The research methodology

4.1 Conditional variance modeling

The ARCH family models and its generalized form, i.e. the GARCH family models, are suitable analytical tools to consider the nonlinear information contained in the regression residues when estimating the ARMA model parameters.

$$\begin{aligned} r_t &= \Phi_0 + \sum_{i=1}^{p} \Phi_i r_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i} + a_t \qquad a_t = \sigma_t \varepsilon_t \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^2 \end{aligned} \tag{4.1}$$

where $\{\varepsilon_t\}$ is a sequence of uncorrected random variables with the same distribution and the mean 0 and variance 1. The expressions $\sum_{i=1}^{m} \alpha_i a_{t-i}^2$ and $\sum_{j=1}^{n} \beta_j \sigma_{t-j}^2$ correspond to the ARCH and the GARCH sections.

The exponential GARCH (EGARCH) model was proposed by Nelson [20]. This model is a new formulation of the conditional variances and is calculated as follows [20]:

$$\ln \sigma_t^2 = \omega + \beta \ln \sigma_{t-1}^2 + \gamma \frac{u_{t-1}}{\sqrt{\sigma_{t-1}^2}} + \alpha \left[\frac{|u_{t-1}|}{\sqrt{\sigma_{t-1}^2}} - \sqrt{\frac{2}{\pi}} \right] \tag{4.2}$$

This model has several advantages. First, the dependent variable, i.e. σ_t^2, is logarithmic, and therefore the coefficients of the right variables can be positive or negative and independent of that σ_t^2 is positive. Thus it is not necessary to impose this constraint when the coefficients are non-negative. Second, it can consider even effect of asymmetric shocks, because γ is coefficient of u_{t-1} and u_{t-1} can be positive or negative. When γ is zero, there is a state of symmetry, and otherwise asymmetry is not confirmed.

4.2 Sequential behavior investigating

The extreme value theory is a powerful framework for studying the sequential behaviors of distribution. The extreme value theory examines the maximum variations of the sample distribution. This theory is very useful in modeling the probabilistic distribution of sequences that their oscillation is greater than the observed limit values.

When using a ARMA-GARCH hybrid model, the distribution function of values greater than $F_u(y)$ threshold can be approximated by the generalized beam distribution as follows:

$$G_{\xi,\mu,\sigma}(x_{\max}) = 1 - \left[1 + \xi_{\max}\left(\frac{x_{\max} - \mu_{\max}}{\sigma_{\max}}\right)\right]^{\frac{-1}{\xi_{\max}}} \tag{4.3}$$

A logical value for the u threshold must be chosen to estimate the parameters. This threshold determines the number of observations beyond the threshold. The probability function and the logarithmic probability function of this distribution are as follows:

$$G_j(z_j) = \begin{cases} \frac{N_{uL}}{N}\{1 + \xi^L \frac{u^L - z_j}{\beta^L}\}^{-\frac{1}{\xi^L}}, & z_j < -u^L \\ \phi(z_j), & u^L < z_j < -u^R \\ 1 - \frac{N_{uL}}{N}\{1 + \xi^L \frac{u^L - z_j}{\beta^L}\}^{-\frac{1}{\xi^L}}, & z_j > -u^R \end{cases} \tag{4.4}$$

where u^R is the upper extreme, u^L is the lower extreme and values of ξ and β are estimated using the maximum likelihood method.

4.3 Copula functions

The concept of sequence dependence can be expressed in copula theory, which was first introduced by Sklar in 1959 [26]. The theory states that any multivariate distribution can have a marginal cumulative distribution factor and the copula functions describe the dependencies between its components.

Let $x = (x_1, x_2)$ is a two-dimensional vector with the joint distribution of $F(x_1, x_2)$ and the marginal distribution of $F_i(x_i)$ for $i = 1, 2$. Hence there will be a copula $C(u_1, u_2)$ as follows:

$$F(x_1, x_2) = P(X_1 < x_1, X_2 < x_2) = C(F_1(x_1), F_2(x_2)) \tag{4.5}$$

If F_i is continuous then $C(u_1, u_2)$ will be unique. The n-dimensional distribution function $C : [0, 1]^n \longrightarrow [0, 1]$ is called copula, if its one-dimensional marginal functions have uniform distribution over the interval $[0, 1]$. Copula functions are consisted of interconnection of n-dimensional joint distribution functions. The F function corresponds to the one-dimensional marginal distribution functions F_i as follows:

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)) \tag{4.6}$$

4.3.1 Copula function types

Normal copula

Xue-Kun Song [27] described the normal copula function as follows:

$$C^{Ga}(u_1, u_2; \rho) = \psi_\rho(\psi^{-1}(u_1), \psi^{-1}(u_2)) \tag{4.7}$$

where ψ_ρ is bivariate standard normal distribution function with the correlation coefficient $\rho \in (0, 1)$. The upper and lower dependencies of this function is zero.

T-copula

Embrechts et al. [6] described the distribution function of the t-copula function as follows:

$$T_{v,\rho}\left(t_v^{-1}(u_1), t_v^{-1}(u_2)\right) \tag{4.8}$$

This function is symmetric like the normal function, except that it measures the upper and lower dependencies in the distribution symmetrically and provides a dependency criterion for the sequence.

Archimedean copula

Archimedean copula is an important class of copula functions with simple structure and many analytical properties. The bivariate Archimedean copula $C(u_1, u_2) = \varphi^{[-1]}\{\varphi(u_1) + \varphi(u_2)\}$ is continuous, strictly decreasing, and with generating function $\varphi : [0, 1] \longrightarrow [0, \infty]$, $\varphi(1) = 0$, and its pseudo-inverse function $\varphi^{[-1]}$ is as follows:

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & 0 \leq t \leq \varphi(0) \\ 0 & \varphi(0) \leq t \leq \infty \end{cases} \tag{4.9}$$

The three most commonly used types of Archimedean copula are: Clayton copula [4], Frank copula [11], and Gumbel copula [13].

Vine-Copula

Although the n-dimensional copula function is widely used in determining the joint distribution of N variables, its complex calculations make it inappropriate in some cases. However, this problem can be solved by linking these two variables to each other by two-dimensional copula functions between them and using a tree structure called vine or pair copula functions.

This model is hierarchical so that in the first stage, the pair copula functions between the variables are determined. In the second stage, the pair copula functions between the conditioned variables are used with the condition of variables between the two variables and this process continues until the last stage. Therefore, if N variables exist, the joint distribution of these N variables can be presented by the $N(N-1)/2$ two-dimensional copula function. The two main structures in this approach are called Canonical Vine (C-vine) and Drawable Vine (D-vine) models.

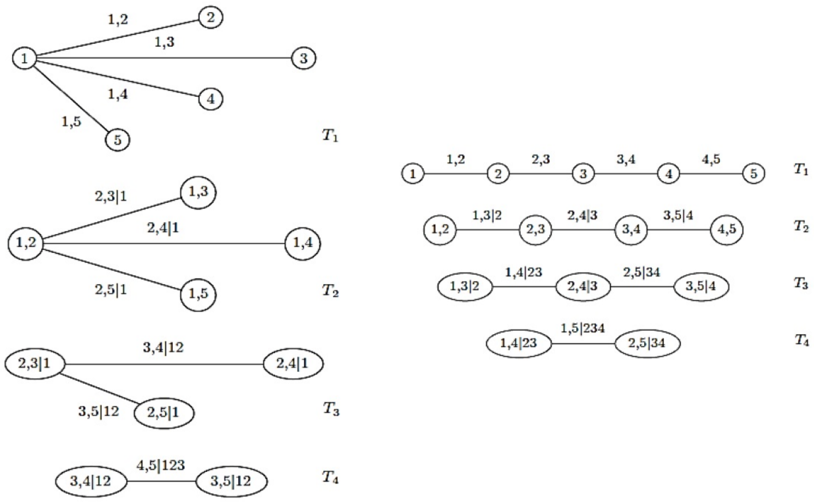

Figure 1: D-Vine (right) and C-Vine (left) structures for a 5-variable distribution

Present research uses CVaR to properly consider the risk. The CVaR is defined as follows:

$$CVaR_\beta(W_t) = \frac{1}{1-\beta} \int f(W_t, r_t) \geq \alpha_\beta(W_t) f(W_t, r_t) p(r_t) dr_t \tag{4.10}$$

where $f(W_t, r_t)$ is the loss function of the financial asset portfolio with specified weight w and $p(r_t)$ is the probability of return r during the time period t.

The optimization model aims to minimize the CVaR and it considers the dependency structures between the financial returns modeled by the copula parameter:

$$\begin{aligned} \min_{(w_L,\alpha)} f_\alpha(W_t, \beta) &= \alpha + \frac{1}{(1-\beta)} \times \sum_{k=1}^{q} [-W_t^T r_{kt} - \alpha]^+ CVaR \\ S.t \;\; & W_t^T 1 = 1 \\ & w_{t,j} \geq 0 \;\; \forall j \in \{1, 2, ..., d\} \mu(w_t) \leq -R \end{aligned} \tag{4.11}$$

This study examines stock portfolio optimization over time using the rebalancing method, so it is necessary to fit data in two separate periods, i.e. in-sample and out-of-sample periods. It should be noted that in this study, rebalancing is done on a weekly basis and based on the rolling window method; in other words, the steps will be repeated until the end of the out-of-sample period.

4.4 The research statistical population and sample

The statistical population of the study includes all companies listed on the Tehran Stock Exchange during the period 2015 to 2021. The statistical sample is selected through several steps as follows:

1. The company's shares must be among the top 50 listed companies during the study period,
2. The shares of the companies in the stock portfolio must be among the top 50 listed companies during whole the study period,
3. The average return during the period for this group of companies must be either zero or positive,
4. The number of shares in the stock portfolio must be more than 4 and less than 11 shares.

4.5 The research question

The main research question is: is the GARCH-EVT-Copula model more efficient in stock portfolio optimization than conventional models?

5 The research findings

5.1 Descriptive statistics of the research data

The data used in the present study for modeling are collected from the return data of 7 companies among the top 50 listed companies on the Tehran Stock Exchange during the period 2015 to 2021 that meet several conditions, mentioned above. The study period includes daily data with 5 working days per week in a continuous manner. It should be noted that the days that their data were available were used in the research. The actual return index was calculated as follows:

$$r_t = 100 \times (p_t - p_{t-1})/p_{t-1}$$

where p_t is equal to the daily closing price of the stock. In addition, we use the R software in this research.

The statistical properties of the studied variables, i.e. the mean, variance, and maximum and minimum of the time series are summarized in Table 1. The augmented Dickey-Fuller (ADF) method was used to test whether the variables are stationary or not and the results show that these time series are stationary and the null hypothesis (i.e., the lack of stationary of variables) is rejected at the level of 99%. In addition, the Jarque–Bera test was used to test the normality of the return series of all 7 stocks.

Table 1: Descriptive statistics of daily return data

Statistical criteria		**Khodro**	**Shebandar**	**Shapna**	**Fakhooz**	**Femeli**	**Foolad**	**Vaghadir**
No. observations		1526	1481	1565	1523	1544	1550	1595
Average		0.0026	0.0028	0.0028	0.0026	0.0028	0.0027	0.0021
SD		0.0391	0.0311	0.0272	0.0240	0.0244	0.0229	0.0229
Mean		−0.0003	0.0008	−0.0002	−0.0004	0.0004	–	−0.0009
Min		−0.0959	−0.1752	−0.0946	−0.2184	−0.1179	–	−0.0500
Max		1.0000	0.5255	0.1889	0.1269	0.1991	0.1184	0.1885
Skewness		1.0959	0.7007	0.2835	0.3453	0.3170	0.2675	0.2385
Kurtosis		11.1849	3.2725	0.5057	−0.1773	0.5983	0.1978	0.5723
Augmented Dickey-Fuller Test	statistic	−32.567	−29.383	−28.813	−28.151	−28.827	–	−27.778
	p-value	0.01	0.01	0.01	0.01	0.01	0.01	0.01
Phillips-Perron Unit Root Test	statistic	−1356.4	−1167.8	−1238.7	−1053.7	−1150.4	–	−1075.9
	p-value	0.01	0.01	0.01	0.01	0.01	0.01	0.01
ARCH LM-test	statistic	12.899	8.5239	10.9	73.743	18.576	28.296	17.306
	p-value	0.00032	0.004	0.0009	0	0.00002	0	0.00003
Jarque.Bera test	statistic	4943176	190231	286	1678	1101	223	669
	p-value	0	0	0	0	0	0	0

The intended data are not normally distributed and the ARCH effects show that the intended series require analysis of variance modeling. In addition, shocks in financial markets are often asymmetric. Therefore, different approaches to GARCH models are used in a simple and asymmetric manner and the most appropriate form of the model is presented.

In order to select the best ARMA-GARCH model for each time series, the following options for the SGARCH and EGARCH models were investigated:

1. The AR model order is 0 or 1,
2. The MA model order is 0 or 1,
3. Whether the model has an average or not,
4. The ARCH order is 0 or 1 (p in {0 or 1})
5. The GARCH order is 0 or 1 (q in {0 or 1})
6. The residual distribution should be one of the following:
   (a) norm: normal distribution
   (b) snorm: skew-normal distribution
   (c) std: t-student
   (d) sstd: skew-student
   (e) ged: generalized error distribution
   (f) sged: skew-generalized error distribution
   (g) nig: normal inverse gaussian distribution
   (h) ghyp: Generalized Hyperbolic

Therefore, 256 different models for each time series are examined and the Akaike criterion for each model is calculated the selected model has the best Akaike criterion, which is shown in the table below for the SGARCH model.

Table 2: Characteristics of the best model selected based on the Akaike criterion in the SGARCH structure

Name	**Models**	**Distribution**	**garchOrder1**	**garchOrder2**	**armaOrder1**	**armaOrder2**	**Include Mean**	**Akaike**
Khodro	sGARCH	ghyp	1	1	1	1	FALSE	-4.357
Shebandar	sGARCH	sstd	1	1	0	1	TRUE	-4.575
Shapna	sGARCH	sstd	1	1	1	0	TRUE	-4.667
Fakhooz	sGARCH	sstd	1	1	1	0	TRUE	-5.100
Femeli	sGARCH	sstd	1	1	0	1	TRUE	-4.975
Foolad	sGARCH	sstd	1	1	1	1	TRUE	-5.089
Vaghadir	sGARCH	sstd	1	1	1	0	TRUE	-5.335

Results of these estimates are summarized for the studied variables in Table 2 and Table 3.

Table 3: The ARMA-SGARCH model parameters for time series

Model parameters	**Vaghadir**	**Foolad**	**Femeli**	**Fakhooz**	**Shapna**	**Shebandar**	**Khodro**
mu	0.0010	0.0018	0.0011	0.0009	0.0018	0.0018	
ar1	0.291	-0.151		0.287	0.247		-0.278
ma1		0.440	0.252			0.260	0.513
alpha1	0.098	0.159	0.100	0.258	0.114	0.146	0.112
beta1	0.902	0.841	0.900	0.742	0.877	0.854	0.875
omega	0.0000015	0.0000056	0.0000024	0.0000119	0.0000102	0.0000091	0.0000161
skew	1.214	6.768	5.445	4.558	9.171	7.323	0.250
ghlambda							-6

The asymmetric effects of shocks in the EGARCH model are shown with the parameter gamma1, which its positive and significant value implies the lack of asymmetric effects and also shows that negative shocks will have a greater effect on the return turbulence than the positive shocks.

Table 4: Characteristics of the best model selected based on the Akaike criterion in the EGARCH structure

Name	Models	Distribution	garchOrder1	garchOrder2	armaOrder1	armaOrder2	Include Mean	Akaike
Khodro	eGARCH	ghyp	1	1	1	1	FALSE	−4.369
Shebandar	eGARCH	sstd	1	1	0	1	TRUE	−4.585
Shapna	eGARCH	sstd	1	1	1	0	TRUE	−4.669
Fakhooz	eGARCH	sstd	1	1	1	0	TRUE	−5.094
Femeli	eGARCH	sstd	1	1	0	1	FALSE	−4.979
Foolad	eGARCH	sstd	1	1	1	1	TRUE	−5.095
Vaghadir	eGARCH	sged	1	1	1	0	TRUE	−5.346

Table 5: Parameters of the ARMA-EGARCH model for time series

Model parameters	Vaghadir	Foolad	Femeli	Fakhooz	Shapna	Shebandar	Khodro
mu	0.021	0.002		0.0014	0.0020	0.0024	
ar1	0.518			0.331	0.601		−0.228
ma1		0.300	−0.228		−0.372	0.251	0.478
alpha1	7.087	0.048	0.026	1.546	0.050	−0.001	0.010
beta1	0.900	0.976	0.927	0.948	0.912	0.997	0.954
omega	0.0022	−0.1930	−0.5762	−0.1621	−0.6613	−0.0274	−0.3358
skew	1.175	1.190	1.113	1.123	1.193	1.148	0.987
shape	0.100	6.356	5.235	2.013	9.720	6.710	0.250
gamma1	−0.079	0.231	0.391	2.689	0.324	0.174	0.146
ghlambda							−6

As stated earlier, the extreme value theory is a powerful framework for studying the behavior of distribution sequences. The extreme value theory examines the maximum fluctuations of the sample distribution. This theory is very useful in modeling the probabilistic distribution of sequences that extend beyond the observed limit oscillation values. There are two methods to determine the extreme data: block maxima, and peak over the threshold. The present study uses the peak over threshold approach. Therefore, the results estimated by Generalized Pareto Distribution (GPD) method with appropriate thresholds selected based on the maximum likelihood approach are summarized in the next two tables.

The Beta parameters for the two sequences on the right and left are positive and significant for the performance of all series, which shows that all of these time series have wide sequences and as a result effective sequences. Thus, extreme events should be considered in the assessment and measurement of return risk for the stock portfolio.

The results estimated by the GPD method with suitable thresholds selected based on the maximum likelihood approach for the ARMA-SGARCH and the ARMA-EGARCH models are summarized in tables 6 and 7.

After estimating the extreme values in the previous section and with given necessary parameters of the F_i marginal distribution of the data, the copula functions including the normal copula, the t-copula, and the Archimedean copula family (e.g. Clayton, Gamble, and Frank) were estimated. In addition, the best copula functions for estimating the dependency structure for both vine copula structures were determined and finally, the portfolio optimization process was actively started.

It should be noted that data of the first 850 days were considered in-sample and used in time series and the rest of the data were considered as out-of-sample data ad used in conditional heteroscedasticity prediction. Weekly accumulated wealth was used to evaluate the accumulated wealth function. The rolling window approach was used to calculate the weekly accumulation return and accordingly, the ability of different models was compared based on the accumulated wealth function.

According to the previous section, it should be noted that for the vine structure, the Archimedean, normal and t-copula functions are considered optimal in the paired functions. The t-copula model, which provides optimal results, and traditional approaches like variance minimization and the CVaR minimization were used to compare the following

Table 6: Estimating the extreme value parameters of the SGARCH model using the GPD method

Model	**Parameter**	**Symbol**	**Threshold upper**	**upperFit Par ests**	**upperFit Par se**	**Threshold lower**	**lowerFit Par ests**	**lowerFit Par se**
sGARCH	xi	Khodro	1.3496	0.2963	0.1051	−1.2272	0.0535	0.0857
sGARCH	beta			0.3281	0.0429		0.3343	0.0395
sGARCH	xi	Shebandar	1.2457	0.2919	0.1062	−1.1687	0.1855	0.0978
sGARCH	beta			0.4237	0.0560		0.4016	0.0510
sGARCH	xi	Shapna	1.2849	0.0871	0.0870	−1.1677	0.1314	0.0909
sGARCH	beta			0.5609	0.0662		0.3798	0.0459
sGARCH	xi	Fakhooz	1.2624	0.1160	0.0905	−1.0985	0.2756	0.1038
sGARCH	beta			0.7315	0.0886		0.4668	0.0607
sGARCH	xi	Femeli	1.3049	0.3267	0.1069	−1.0996	0.2162	0.0983
sGARCH	beta			0.4991	0.0655		0.4077	0.0514
sGARCH	xi	Foolad	1.2839	0.0377	0.0836	−1.1142	0.2289	0.0990
sGARCH	beta			0.6459	0.0750		0.4128	0.0522
sGARCH	xi	Vaghadir	1.3423	0.0888	0.0864	−1.1112	0.1099	0.0883
sGARCH	beta			0.6548	0.0766		0.4136	0.0490

Table 7: Estimating the extreme value parameters of the EGARCH model using GPD method

Model	**Parameter**	**Symbol**	**Threshold upper**	**upperFit Par ests**	**upperFit Par se**	**Threshold lower**	**lowerFit Par ests**	**lowerFit Par se**
EGARCH	xi	Khodro	1.335	0.275	0.103	−1.212	0.011	0.082
EGARCH	beta			0.340	0.044		0.349	0.040
EGARCH	xi	Shebandar	1.197	0.261	0.104	−1.127	0.164	0.096
EGARCH	beta			0.462	0.060		0.426	0.054
EGARCH	xi	Shapna	1.278	0.073	0.086	−1.161	0.073	0.086
EGARCH	beta			0.580	0.068		0.421	0.050
EGARCH	xi	Fakhooz	0.162	0.133	0.092	−0.142	0.356	0.110
EGARCH	beta			0.096	0.012		0.056	0.008
EGARCH	xi	Femeli	1.315	0.213	0.098	−1.055	0.163	0.094
EGARCH	beta			0.552	0.069		0.447	0.055
EGARCH	xi	Foolad	1.242	0.002	0.081	−1.101	0.278	0.103
EGARCH	beta			0.651	0.074		0.383	0.049
EGARCH	xi	Vaghadir	0.027	0.375	0.109	−0.048	−0.147	0.068
EGARCH	beta			0.021	0.003		0.024	0.002

models:

- The ARMA-SGARCH-EVT-CVine Copula model
- The ARMA-SGARCH-EVT-DVine Copula model
- The ARMA-SGARCH-EVT-t Copula model
- The ARMA-EGARCH-EVT-CVine Copula model
- The ARMA-EGARCH-EVT-DVine Copula model
- The ARMA-EGARCH-EVT-t Copula model
- The variance minimization model
- The historical CVaR minimization

Figure 2 shows the accumulated wealth function of various models estimated for the out-of-sample period. According to this figure, the model with the vine structure compared to traditional approaches that do not consider the dependency structure and extreme events provide better results and accumulated wealth function.

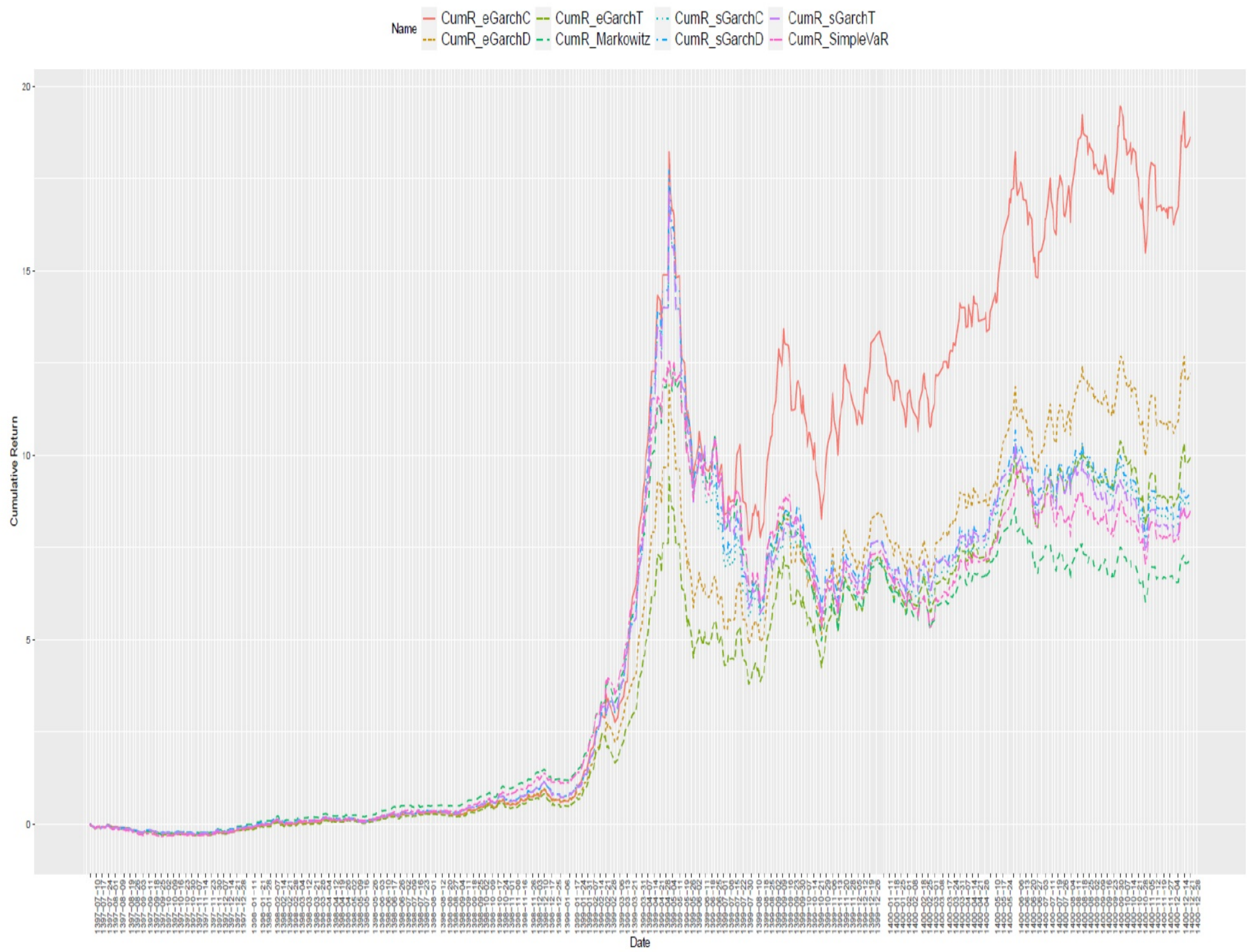

Figure 2: Accumulated wealth function of all the studied models

According to the ratios in Table 8, it is clear that the rebalancing return of the out-of-sample period for different models, taking into account the ARMA-EGARCH-EVT-CVine copula structure and using the normal, t-copula and Archimedean copula functions provide better combinatory structure and the optimization has resulted in higher annual returns.

Table 8: Comparison of different portfolios criteria

Model	**Accumulated annual return for the out-of-sample period**	**Standard deviation for the out-of-sample period**	**The ratio of return to standard deviation**
EGARCH-EVT-CVine copula	18.64	6.97	2.67
EGARCH-EVT-DVine copula	12.22	4.46	2.74
EGARCH-EVT-t copula	9.94	3.70	2.68
SGARCH-EVT-DVine copula	9.00	4.34	2.08
SGARCH -EVT-CVine copula	8.79	4.17	2.10
SGARCH -EVT-t copula	8.49	4.22	2.01
min CVaR	8.48	3.94	2.15
min StdDev	7.20	3.62	1.99

Comparing various GARCH-EVT models based on the Copula function in the vine structure as well as traditional models in Figure 2 well indicates the efficiency of these models in stock portfolio optimization. Thus, the main question of the research can be answered by the proposed GARCH-EVT-Vine copula model can be well used in stock portfolio optimization.

For the last period of fitting the model to the data and based on the fact that the EGARCH-CVine model has the highest amount of wealth accumulation, the efficient boundary diagram for the last period of the model is as presented in Figure 3. In order to compare the optimal point based on different models, each relevant optimal point is also displayed.

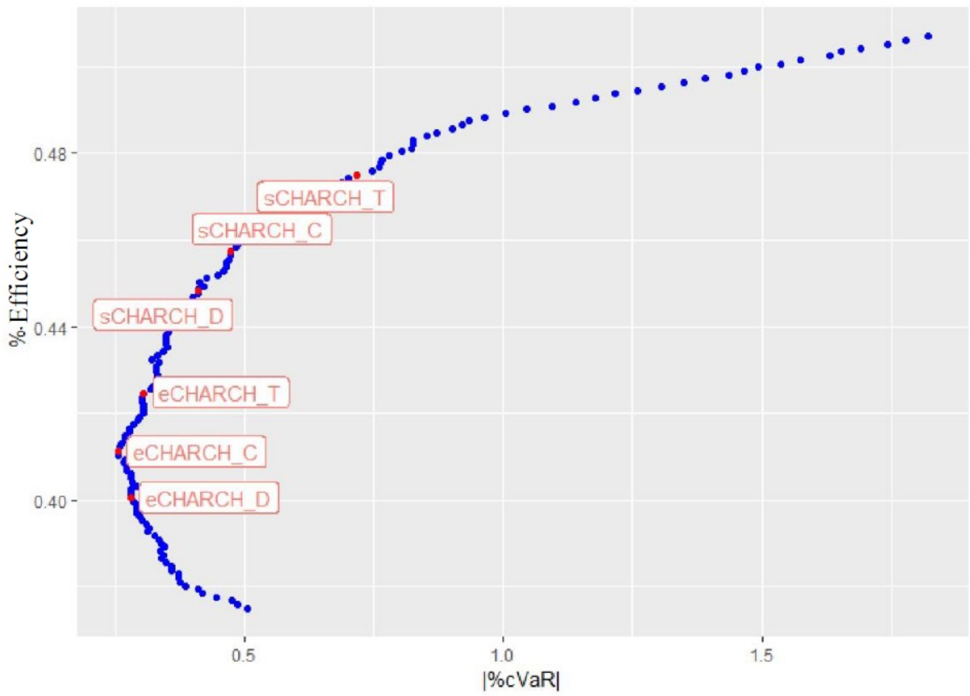

Figure 3: Efficient boundary chart based on conditional risk value for the last data fitting period

The optimal weights related to the last period are presented in Table 9.

Table 9: Estimating the extreme value parameters of the EGARCH model using GPD method

	EGARCH_ CVine	**SGARCH_ CVine**	**EGARCH_ T**	**EGARCH_ DVine**	**SGARCH_ DVine**	**SGARCH_ T**	**Simple_ Markowitz**	**SimpVaR**
Khodro	12.63%	10.75%	11.41%	21.37%	21.45%	5.00%	14.62%	21.65%
Shebandar	6.00%	9.00%	13.75%	1.35%	9.00%	7.00%	6.76%	0.99%
Shapna	31.37%	10.00%	16.99%	28.50%	11.01%	3.00%	6.42%	16.41%
Fakhooz	24.10%	11.02%	14.71%	2.94%	9.00%	27.03%	16.94%	24.48%
Femeli	6.00%	10.04%	13.84%	33.34%	33.53%	11.07%	30.16%	11.74%
Foolad	13.89%	39.18%	12.25%	2.81%	9.01%	41.89%	9.09%	13.06%
Vaghadir	6.00%	10.01%	17.04%	9.70%	7.00%	5.00%	16.01%	11.67%

6 Conclusions

This study aims to optimization of the stock portfolio using copula functions with respect to the extreme values. The generalized conditional heteroscedasticity variance method and the extreme value theory with the GARCH-EVT-Vine-Copula approach were used for the portfolio optimization. In addition, the model was measured using traditional portfolio optimization methods. The return data of 7 companies among the top 50 companies listed on the Tehran Stock Exchange were collected during the period 2015 to 2021.

The results show that consideration of the extreme values and structural dependence between studied time series improves the risk identification in these markets with respect to their dependence structure. Thus, investors in examining the asset portfolio should examine these models to accurately identify the dependence structure and avoid great losses.

The results also indicate that the proposed model can be well used in stock portfolio optimization. In addition, the EGARCH-EVT model considering the dependency structure based on the vine copula function and using the pair conditional structural dependence has a better performance compared to other models.

The research results introduce new products to the capital market, meet some existing needs, and result in more efficient management in this type of investment. In fact, this study using a pragmatic approach and presenting practical models can help to form active funds in the capital market and thus cover part of the market tastes. Since this study emphasizes both optimization and risk management, its results can be used to better allocate limited resources in the capital market, which is an important necessity for any business.

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