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On left ϕ-Connes biprojectivity of dual Banach algebras

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Abstract

We introduce the notion of left (right) ϕ-Connes biprojective for a dual Banach algebra $\mathcal{A}$, where ϕ is a non-zero wk^*-continuous multiplicative linear functional on $\mathcal{A}$. We discuss the relationship of left ϕ-Connes biprojectivity with ϕ-Connes amenability and Connes biprojectivity. For a unital weakly cancellative semigroup S, we show that $\ell^1(S)$ is left ϕ_S-Connes biprojective if and only if S is a finite group, where $\phi_S \in \Delta_{w^*}(\ell^1(S))$. We prove that for a non-empty totally ordered set I with the smallest element, the upper triangular $I \times I$-matrix algebra $UP(I,\mathcal{A})$ is right ψ_ϕ-Connes biprojective if and only if $\mathcal{A}$ is right ϕ-Connes biprojective and I is a singleton, provided that $\mathcal{A}$ has a right identity and $\phi \in \Delta_{w^*}(\mathcal{A})$. Also for a finite set I, if $Z(\mathcal{A}) \cap (\mathcal{A} - \ker\phi) \neq \emptyset$, then the dual Banach algebra $UP(I,\mathcal{A})$ under this new notion forced to have a singleton index.

Keywords: Semigroup algebras, Matrix algebras, Connes amenability, Left ϕ-Connes biprojectivity
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1 Introduction and Preliminaries

Johnson introduced and studied the class of amenable Banach algebras. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \to E$ is called a bounded derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$ for every $a, b \in \mathcal{A}$. A bounded derivation $D : \mathcal{A} \to E$ is called inner if there exists an element x in E such that $D(a) = a \cdot x - x \cdot a$ for every $a \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is amenable if for every Banach $\mathcal{A}$-bimodule E, every bounded derivation $D : \mathcal{A} \to E^*$ is inner. Johnson also showed that a Banach algebra $\mathcal{A}$ is amenable if and only if there exists an element $M \in (\mathcal{A} \otimes_p \mathcal{A})^{**}$ such that $a \cdot M = M \cdot a$ and $\pi^{**}_{\mathcal{A}}(M)a = a$, for each $a \in \mathcal{A}$, where $\pi_{\mathcal{A}} : \mathcal{A} \otimes_p \mathcal{A} \to \mathcal{A}$ is defined by $\pi_{\mathcal{A}}(a \otimes b) = ab$ for every $a, b \in \mathcal{A}$. For more information about amenability, see [11].

The category of dual Banach algebras defined by Runde [9]. For a given Banach algebra $\mathcal{A}$, a Banach $\mathcal{A}$-bimodule E is called dual if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. The Banach algebra $\mathcal{A}$ is called dual if it is dual as a Banach $\mathcal{A}$-bimodule. Equivalently a Banach algebra $\mathcal{A}$ is called dual, if it is a dual Banach space with predual $\mathcal{A}_*$ such that the multiplication in $\mathcal{A}$ is separately wk^*-continuous. A dual Banach $\mathcal{A}$-bimodule E is normal, if for every $x \in E$ the module maps $\mathcal{A} \longrightarrow E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are wk^*-wk^* continuous. The suitable concept

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of amenability for the class of dual Banach algebras is Connes amenability. A dual Banach algebra $\mathcal{A}$ is called Connes amenable if for every normal dual Banach $\mathcal{A}$-bimodule E, every wk^*-continuous derivation $D : \mathcal{A} \longrightarrow E$ is inner. For a given dual Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule E, $\sigma wc(E)$ denote the set of all elements $x \in E$ such that the module maps $\mathcal{A} \to E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are wk^*-wk-continuous, one can see that, it is a closed submodule of E, see [9] and [10] for more details. Since $\sigma wc(\mathcal{A}_*) = \mathcal{A}_*$, the adjoint of $\pi_{\mathcal{A}}$ maps $\mathcal{A}_*$ into $\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*$. Therefore, $\pi^{**}_{\mathcal{A}}$ drops to an $\mathcal{A}$-bimodule morphism $\pi_{\sigma wc} : (\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*)^* \longrightarrow \mathcal{A}$. An element $M \in (\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*)^*$ satisfying

$$a \cdot M = M \cdot a \quad \text{and} \quad a\pi_{\sigma wc}M = a \quad (a \in \mathcal{A}),$$

is called a σwc-virtual diagonal for $\mathcal{A}$. Runde showed that a dual Banach algebra $\mathcal{A}$ is Connes amenable if and only if there exists a σwc-virtual diagonal for $\mathcal{A}$ [10, Theorem 4.8].

The notion of left ϕ-Connes amenability for a dual Banach algebra $\mathcal{A}$ introduced by Mahmoodi and some characterizations were given [5] and [7], where ϕ is a wk^*-continuous character on $\mathcal{A}$. We say that $\mathcal{A}$ is left ϕ-Connes amenable if there exists a bounded linear functional m on $\sigma wc(\mathcal{A}^*)$ satisfying $m(\phi) = 1$ and $m(f \cdot a) = \phi(a)m(f)$ for every $a \in \mathcal{A}$ and $f \in \sigma wc(\mathcal{A}^*)$. The set of all non-zero wk^*-continuous characters on $\mathcal{A}$ is denoted by $\Delta_{wk^*}(\mathcal{A})$. We recall that a Banach algebra $\mathcal{A}$ is left ϕ-contractible, where ϕ is a linear multiplication functional on $\mathcal{A}$, if there exists $m \in \mathcal{A}$ such that $am = \phi(a)m$ and $\phi(m) = 1$, for every $a \in \mathcal{A}$, see [4] and [6]. Ramezanpour showed that the concept of left ϕ-Connes amenability is equivalent with left ϕ-contractible for a dual Banach algebra, where ϕ is a wk^*-continuous character [7, Proposition 2.3].

Helemskii introduced some important homological notions such as biflatness and biprojectivity for Banach algebras [3]. Indeed a Banach algebra $\mathcal{A}$ is called biprojective (biflat), if there exists a bounded $\mathcal{A}$-bimodule $\rho : \mathcal{A} \to \mathcal{A} \otimes_p \mathcal{A}$ ($\rho : \mathcal{A} \to (\mathcal{A} \otimes_p \mathcal{A})^{**}$) such that ρ is a right inverse for $\pi_{\mathcal{A}}$ ($\pi^{**}_{\mathcal{A}} \circ \rho(a) = i(a)$ for every $a \in \mathcal{A}$, where $i : \mathcal{A} \hookrightarrow \mathcal{A}^{**}$ is the canonical embedding). In fact a Banach algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is biflat and has a bounded approximate identity [11]. Helemskii also showed that $L^1(G)$ is biflat if and only if G is amenable. It is well-known that the measure algebra $M(G)$ on a locally compact group G is biprojective if and only if G is finite, for more discussions refer to [11]. Ramsden characterized the biflatness of semigroup algebras associated to a locally finite inverse semigroup [8]. He showed that for a locally finite inverse semigroup S, $\ell^1(S)$ is biflat if and only if each G_p is an amenable group, where p is an idempotent element of S and G_p is a maximal subgroup of S.

The second author with A. Pourabbas introduced some generalizations of Helemskii's concepts like ϕ-biflatness and ϕ-biprojectivity, where ϕ is a character on $\mathcal{A}$. Indeed a Banach algebra $\mathcal{A}$ is called ϕ-biflat (ϕ-biprojective) if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to (\mathcal{A} \otimes_p \mathcal{A})^{**}$ ($\rho : \mathcal{A} \to \mathcal{A} \otimes_p \mathcal{A}$) such that $\tilde{\phi} \circ \pi^{**}_{\mathcal{A}} \circ \rho(a) = \phi(a)$ ($\phi \circ \pi_{\mathcal{A}} \circ \rho(a) = \phi(a)$) for every $a \in \mathcal{A}$. They showed for a locally compact group G, $L^1(G)$ is ϕ-biflat if and only if G is amenable. Furthermore they proved that $A(G)$ is ϕ-biprojective if and only if G is a discrete group. For more details see [14].

A dual Banach algebra $\mathcal{A}$ is called Connes-biprojective if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*)^*$ such that $\pi_{\sigma wc} \circ \rho = id_{\mathcal{A}}$. Shirinkalam and Pourabbas showed that a dual Banach algebra $\mathcal{A}$ is Connes amenable if and only if $\mathcal{A}$ is Connes-biprojective and it has an identity [17]. Motivated by the definitions of ϕ-biprojectivity and Connes-biprojectivity, we introduce a new notion for a class of dual Banach algebras.

Definition 1.1. Let $\mathcal{A}$ be a dual Banach algebra and let $\phi \in \Delta_{w^*}(\mathcal{A})$. Then $\mathcal{A}$ is called left (right) ϕ-Connes biprojective if there exists a bounded linear map $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A} \otimes_p \mathcal{A})^*)^*$ such that

$$\rho(ab) = \phi(b)\rho(a) = a \cdot \rho(b), \quad (\rho(ab) = \phi(a)\rho(b) = \rho(a) \cdot b)$$

and

$$\phi \circ \pi_{\sigma wc} \circ \rho(a) = \phi(a)$$

for all $a, b \in \mathcal{A}$, respectively.

In this paper, we study the relationship between this new notion with ϕ-Connes amenability and Connes biprojectivity. In section 2, we investigate that under certain conditions like approximate identity for the dual Banach algebras or the existence of an element in $Z(\mathcal{A})$, left ϕ-Connes biprojectivity implies left ϕ-Connes amenability. In section 3, as an application we characterize the left ϕ-Connes biprojectivity of the dual Banach algebras $\ell^1(S)$ with finiteness of the semigroup S, where S is a unital weakly cancellative semigroup. We show that for a given dual Banach algebra $\mathcal{A}$ with a right identity, the $I \times I$-upper triangular matrices Banach algebra with the usual matrix operations and the finite ℓ^1-norm, under the right ψ_ϕ-Connes biprojectivity is forced to have a singleton index. Also, we show in the case that if I is a finite set and $Z(\mathcal{A}) \cap (\mathcal{A} - \ker\phi) \neq \emptyset$, then the dual Banach algebra $UP(I, \mathcal{A})$ is left ψ_ϕ-Connes biprojective if and only if $|I| = 1$ and $\mathcal{A}$ is left ϕ-Connes biprojective.

2 Left ϕ-Connes biprojectivity

We recall some basic notions. For a Banach algebra $\mathcal{A}$, the projective tensor product $\mathcal{A}\otimes_p\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule via the following actions

$$a\cdot(b\otimes c)=ab\otimes c,\qquad (a\otimes b)\cdot c=a\otimes bc\quad (a,b,c\in\mathcal{A}).$$

If X is a Banach $\mathcal{A}$-bimodule, then X^* is also a Banach $\mathcal{A}$-bimodule via the following actions

$$(a\cdot f)(x)=f(x\cdot a),\qquad (f\cdot a)(x)=f(a\cdot x)\quad (a\in\mathcal{A},x\in X,f\in X^*).$$

By inspiration of methods that used in [12, Theorem 2.6], we state the following Proposition.

Proposition 2.1. Let $\mathcal{A}$ be a Connes biprojective dual Banach algebra and let $\phi\in\Delta_{w^*}(\mathcal{A})$. Then $\mathcal{A}$ is left ϕ-Connes biprojective.

Proof . Suppose that $\mathcal{A}$ is Connes biprojective. Then there exists a bounded $\mathcal{A}$-bimodule morphism $\rho:\mathcal{A}\to(\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*)^*$ such that $\pi_{\sigma wc}\circ\rho=id_{\mathcal{A}}$. We complete the proof in three steps:

Step one: There exist a bounded linear map $\Gamma:(\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*)^*\to(\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*)^*$ such that $a\cdot\Gamma(m)=\Gamma(a\cdot m)$ and $\phi(a)\Gamma(m)=\Gamma(m\cdot a)$, for every $m\in(\sigma\omega c(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ and $a\in\mathcal{A}$. To see this, choose a_0 in $\mathcal{A}$ such that $\phi(a_0)=1$. Define $\theta:\mathcal{A}\otimes_p\mathcal{A}\to\mathcal{A}\otimes_p\mathcal{A}$ by $\theta(a\otimes b)=\phi(b)(a\otimes a_0)$ for every $a,b\in\mathcal{A}$. It is clear that θ is a bounded linear map. One can see that

$$\theta(c\cdot(a\otimes b))=\phi(b)ca\otimes a_0=c\cdot\theta(a\otimes b)\quad (a,b,c\in\mathcal{A})$$

and

$$\theta((a\otimes b)\cdot c)=\theta(a\otimes bc)=\phi(bc)a\otimes a_0=\phi(c)\phi(b)a\otimes a_0=\phi(c)\theta(a\otimes b)\quad (a,b,c\in\mathcal{A}).$$

So we have

$$a\cdot\theta(u)=\theta(a\cdot u)\quad\text{and}\quad\phi(a)\theta(u)=\theta(u\cdot a)\quad (a\in\mathcal{A},u\in\mathcal{A}\otimes_p\mathcal{A})\tag{2.1}$$

and also

$$a\cdot\theta^*(f)=\phi(a)\theta^*(f)\quad\text{and}\quad\theta^*(f)\cdot a=\theta^*(f\cdot a)\quad (a\in\mathcal{A},f\in(\mathcal{A}\otimes_p\mathcal{A})^*).$$

It follows that $\theta^*(\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*)\subseteq\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*$. Then define

$$\Gamma:=(\theta^*|_{\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*})^*:(\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*)^*\to(\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*)^*.$$

Then Γ is a bounded linear map. We know that there exists a continuous $\mathcal{A}$-bimodule map $i:\mathcal{A}\hat{\otimes}\mathcal{A}\longrightarrow(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ which has a wk^*-dense range. We write $\bar{u}$ instead of $i(u)=\hat{u}|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}$ for every $u\in\mathcal{A}\hat{\otimes}\mathcal{A}$. For every $m\in(\sigma\omega c(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, there exists a net (u_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $m=wk^*\text{-}\lim_\alpha\bar{u}_\alpha$. Since $\Gamma(\bar{u})=\widehat{\theta(u)}|_{\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*}$ for every $u\in\mathcal{A}\hat{\otimes}\mathcal{A}$ and Γ is wk^*-continuous, (2.1) implies that

$$a\cdot\Gamma(m)=\Gamma(a\cdot m)\quad\text{and}\quad\phi(a)\Gamma(m)=\Gamma(m\cdot a),\tag{2.2}$$

for every $m\in(\sigma\omega c(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ and $a\in\mathcal{A}$.

Step two: We show that $\phi\circ\pi_{\sigma wc}\circ\Gamma=\phi\circ\pi_{\sigma wc}$. For every $a,b\in\mathcal{A}$

$$\phi\circ\pi_{\mathcal{A}}\circ\theta(a\otimes b)=\phi\circ\pi_{\mathcal{A}}(\phi(b)a\otimes a_0)=\phi(b)\phi(a)=\phi\circ\pi_{\mathcal{A}}(a\otimes b).$$

So $\phi\circ\pi_{\mathcal{A}}\circ\theta=\phi\circ\pi_{\mathcal{A}}$. For every $m\in(\sigma\omega c(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, there exists a net (u_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $m=wk^*\text{-}\lim_\alpha\bar{u}_\alpha$. Since $\pi_{\sigma wc}$ and Γ are wk^*-continuous map,

$$\begin{aligned}\phi\circ\pi_{\sigma wc}\circ\Gamma(m)&=\lim_\alpha\phi\circ\pi_{\sigma wc}\circ\Gamma(\bar{u}_\alpha)=\lim_\alpha\phi\circ\pi_{\sigma wc}\circ\widehat{\theta(u_\alpha)}|_{\sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*}\\&=\lim_\alpha\phi\circ\pi_{\mathcal{A}}\circ\theta(u_\alpha)=\lim_\alpha\phi\circ\pi_{\mathcal{A}}(u_\alpha)=\lim_\alpha\phi\circ\pi_{\sigma wc}(\bar{u}_\alpha)\\&=\phi\circ\pi_{\sigma wc}(wk^*\text{-}\lim_\alpha\bar{u}_\alpha)=\phi\circ\pi_{\sigma wc}(m).\end{aligned}$$

Step three: We show that $\mathcal{A}$ is left ϕ-Connes biprojective. To see this, define $\eta:=\Gamma\circ\rho:\mathcal{A}\to(\sigma\omega c(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. Clearly η is a bounded linear map. (2.2) implies that

$$a\cdot\eta(b)=\eta(ab)\quad\text{and}\quad\phi(b)\eta(a)=\eta(ab)\quad (a,b\in\mathcal{A}).$$

According to step two,

$$\phi \circ \pi_{\sigma wc} \circ \eta(a) = \phi \circ \pi_{\sigma wc} \circ \Gamma \circ \rho(a) = \phi \circ \pi_{\sigma wc} \circ \rho(a) = \phi(a) \quad (a \in \mathcal{A}).$$

□

Example 2.2. Let $\mathcal{A}$ be a dual Banach space and $\dim \mathcal{A} \geq 2$. Suppose that ϕ is a non-zero wk^*-continuous functional on $\mathcal{A}$ with $\|\phi\| \leq 1$. Define a product on $\mathcal{A}$ with $ab = \phi(b)a$, for all $a, b \in \mathcal{A}$. Since ϕ is wk^*-continuous, the multiplication in $\mathcal{A}$ is separately wk^*-continuous. Then $\mathcal{A}$ be a dual Banach algebra [9, Exercise 4.4.1]. Pick x_0 in $\mathcal{A}$ such that $\phi(x_0) = 1$. Define $\rho : \mathcal{A} \to (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ by $\rho(a) = \widehat{a \otimes x_0}|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}$ for every $a \in \mathcal{A}$. It is clear that ρ is a bounded $\mathcal{A}$-bimodule morphism such that $\pi_{\sigma wc} \circ \rho = id_{\mathcal{A}}$. So $\mathcal{A}$ is Connes biprojective. Apply Proposition 2.1, $\mathcal{A}$ is left ϕ-Connes biprojective.

Proposition 2.3. Suppose that $\mathcal{A}$ is a dual Banach algebra and $\phi \in \Delta_{w^*}(\mathcal{A})$. Then the left ϕ-Connes amenability of $\mathcal{A}$ implies that $\mathcal{A}$ is left ϕ-Connes biprojective.

Proof . Let $\mathcal{A}$ be left ϕ-Connes amenable. Then using [7, Proposition 2.3], there exists $m_0 \in \mathcal{A}$ such that $am_0 = \phi(a)m_0$ and $\phi(m_0) = 1$ for all $a \in \mathcal{A}$. Define $M := m_0 \otimes m_0$ and we have $a \cdot M = \phi(a)M$ and $\phi \circ \pi_{\mathcal{A}}(M) = \phi(m_0^2) = \phi(m_0)^2 = 1$. Since $\mathcal{A} \otimes_p \mathcal{A}$ embeds in $(\sigma wc(\mathcal{A} \otimes_p \mathcal{A})^*)^*$, consider M as an element in $(\sigma wc(\mathcal{A} \otimes_p \mathcal{A})^*)^*$. The mapping $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A} \otimes_p \mathcal{A})^*)^*$ defined by $\rho(a) = a \cdot M$. It is clear that $\rho(ab) = a \cdot \rho(b) = \phi(b)\rho(a)$ for all $a, b \in \mathcal{A}$. Also, we have

$$\begin{aligned} \phi \circ \pi_{\sigma wc} \circ \rho(a) &= \phi \circ \pi_{\sigma wc}(a \cdot m_0 \otimes m_0) \\ &= \phi(am_0^2) \\ &= \phi(a) \end{aligned}$$

□ In the following example, we show that there exists a dual Banach algebra which is left ϕ-Connes biprojective but it is not left ϕ-Connes amenable.

Example 2.4. Let $\mathcal{A} = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ be an upper triangular Banach algebra on $\mathbb{C}$. Then it is clear that the map $\phi : \mathcal{A} \longrightarrow \mathbb{C}$ defined by $\phi(\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}) = a_{22}$ is a wk^*-continuous character. Put $a_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. So $\phi(a_0) = 1$. Define $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A} \otimes_p \mathcal{A})^*)^*$ by $\rho(a) = a \otimes a_0$ for each $a \in \mathcal{A}$. It is easy to see that

$$\rho(ab) = \phi(b)\rho(a) = a \cdot \rho(b), \quad \phi \circ \pi_{\sigma wc} \circ \rho(a) = \phi(a), \qquad (a, b \in \mathcal{A}).$$

So $\mathcal{A}$ is left ϕ-Connes biprojective. Now we claim that $\mathcal{A}$ is not left ϕ-Connes amenable. We assume in contradiction that $\mathcal{A}$ is left ϕ-Connes amenable. So by [7, Proposition 2.3] which result, there exists $m = \begin{pmatrix} 0 & m_{12} \\ 0 & m_{22} \end{pmatrix} \in \mathcal{A}$ such that

$$am = \phi(a)m, \quad \phi(m) = m_{22} = 1, \qquad (a \in \mathcal{A}).$$

Let $a = \begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}$ be an arbitrary element of $\mathcal{A}$. Put a at above equation, we have $a_{12}m_{22} = a_{22}m_{12}$ for all $a_{12}, a_{22} \in \mathcal{A}$. Suppose that $a_{12} = 1$ and $a_{22} = 0$. It implies that $m_{22} = 0$ and it is a contradiction.

We show that under certain conditions the left ϕ-Connes biprojectivity of $\mathcal{A}$ implies that $\mathcal{A}$ is left ϕ-Connes amenable:

Theorem 2.5. Let $\mathcal{A}$ be a dual Banach algebra and let $\phi \in \Delta_{w^*}(\mathcal{A})$ such that $\mathcal{A}$ has left approximate identity. If $\mathcal{A}$ is left ϕ-Connes biprojective, then $\mathcal{A}$ is left ϕ-Connes amenable.

Proof . Let $\mathcal{A}$ be left ϕ-Connes biprojective. Then there exists a bounded linear map $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A} \otimes_p \mathcal{A})^*)^*$ such that $\rho(ab) = \phi(b)\rho(a) = a \cdot \rho(b)$ and $\phi \circ \pi_{\sigma wc} \circ \rho(a) = \phi(a)$ for every $a, b \in \mathcal{A}$. By using the main ideas in [12, Proposition 2.4], we complete the proof in three steps:

Step one: There exists a bounded left $\mathcal{A}$-bimodule morphism $\zeta : \mathcal{A} \to (\sigma wc(\mathcal{A}\otimes_p \frac{\mathcal{A}}{\ker \phi})^*)^*$ which $\zeta(l) = 0$, for every $l \in \ker \phi$. To see this, put $\tau := id_{\mathcal{A}} \otimes q : \mathcal{A}\otimes_p\mathcal{A} \to \mathcal{A}\otimes_p \frac{\mathcal{A}}{\ker \phi}$, where $q : \mathcal{A} \to \frac{\mathcal{A}}{\ker \phi}$ is a quotient map. Since τ is a bounded $\mathcal{A}$-bimodule morphism, $\tau^*(\sigma wc(\mathcal{A}\otimes_p \frac{\mathcal{A}}{\ker\phi})^*) \subseteq \sigma wc(\mathcal{A}\otimes_p\mathcal{A})^*$. Now set $\zeta := (\tau^*|_{\sigma wc(\mathcal{A}\otimes_p \frac{\mathcal{A}}{\ker\phi})^*})^* \circ \rho$. So $\zeta : \mathcal{A} \to (\sigma wc(\mathcal{A}\otimes_p \frac{\mathcal{A}}{\ker\phi})^*)^*$ is a bounded left $\mathcal{A}$-module morphism such that $\zeta(ab) = \phi(b)a$ for every $a, b \in \mathcal{A}$. We have

$$\zeta(l) = \lim_\alpha \zeta(e_\alpha l) = \phi(l)\lim_\alpha \zeta(e_\alpha) = 0 \qquad (l \in \ker\phi),$$

where (e_α) is a left approximate identity for $\mathcal{A}$.

Step two: There exists a bounded left $\mathcal{A}$-module morphism $\theta : \frac{\mathcal{A}}{\ker\phi} \to \mathcal{A}$ such that $\phi\circ\theta(a + \ker\phi) = \phi(a)$ for every $a \in \mathcal{A}$. To see this, in the previous step we showed that $\zeta|_{\ker\phi} = 0$. Thus there exists a well-defined map $\bar{\zeta} : \frac{\mathcal{A}}{\ker\phi} \to (\sigma wc(\mathcal{A}\otimes_p \frac{\mathcal{A}}{\ker\phi})^*)^*$ which is defined by $\bar{\zeta}(a + \ker\phi) = \zeta(a)$ for every $a \in \mathcal{A}$. Consider a left $\mathcal{A}$-module morphism $id_\mathcal{A} \otimes \tilde{\phi} : \mathcal{A}\otimes_p \frac{\mathcal{A}}{\ker\phi} \to \mathcal{A}$ defined by $id_\mathcal{A} \otimes \tilde{\phi}(a \otimes (b + \ker\phi)) = \phi(b)a$, where $\tilde{\phi}$ is a character on $\frac{\mathcal{A}}{\ker\phi}$ given by $\tilde{\phi}(a + \ker\phi) = \phi(a)$ for every $a \in \mathcal{A}$. Respect with dual module action we have

$$(id_\mathcal{A} \otimes \tilde{\phi})^*(f)\cdot a = (id_\mathcal{A} \otimes \tilde{\phi})^*(f\cdot a), \quad a\cdot(id_\mathcal{A} \otimes \tilde{\phi})^*(f) = \phi(a)(id_\mathcal{A} \otimes \tilde{\phi})^*(f) \quad (a \in \mathcal{A}, f \in \mathcal{A}^*). \tag{2.3}$$

Since ϕ is wk^*-continuous, (2.3) implies that $(id_\mathcal{A} \otimes \tilde{\phi})^*(\mathcal{A}_*) = (id_\mathcal{A} \otimes \tilde{\phi})^*(\sigma wc(\mathcal{A}_*)) \subseteq \sigma wc(\mathcal{A}\otimes_p \frac{\mathcal{A}}{\ker\phi})^*$. Set $\theta = ((id_\mathcal{A} \otimes \tilde{\phi})^*|_{\mathcal{A}_*})^* \circ \bar{\zeta}$. Then $\theta : \frac{\mathcal{A}}{\ker\phi} \to \mathcal{A}$ is a bounded left $\mathcal{A}$-module morphism. We have

$$\begin{aligned}\phi\circ(id_\mathcal{A} \otimes \tilde{\phi})\circ(id_\mathcal{A}\otimes q)(b\otimes c) &= \phi\circ(id_\mathcal{A} \otimes \tilde{\phi})(b\otimes c + \ker\phi)\\ &= \varphi(b\phi(c)) = \phi(b)\phi(c) = \phi(bc) = \phi\circ\pi_\mathcal{A}(b\otimes c) \quad (b, c \in \mathcal{A}).\end{aligned}$$

Therefore $\phi\circ(id_\mathcal{A} \otimes \tilde{\phi})\circ(id_\mathcal{A}\otimes q) = \phi\circ\pi_\mathcal{A}$.

Now for every $a \in \mathcal{A}$, there exists a net (u_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $\rho(a) = wk^*\text{-}\lim_\alpha \hat{u}_\alpha|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}$. Since ϕ is wk^*-continuous,

$$\begin{aligned}\phi\circ\theta(a + \ker\phi) &= \phi\circ((id_\mathcal{A} \otimes \tilde{\phi})^*|_{\mathcal{A}_*})^* \circ \bar{\zeta}(a + \ker\phi)\\ &= \phi\circ((id_\mathcal{A} \otimes \tilde{\phi})^*|_{\mathcal{A}_*})^* \circ (\tau^*|_{\sigma wc(\mathcal{A}\otimes_p \frac{\mathcal{A}}{\ker\phi})^*})^* \circ \rho(a)\\ &= \phi(wk^*\text{-}\lim_\alpha((id_\mathcal{A} \otimes \tilde{\phi})^*|_{\mathcal{A}_*})^* \circ (\tau^*|_{\sigma wc(\mathcal{A}\otimes_p \frac{\mathcal{A}}{\ker\phi})^*})^*(\hat{u}_\alpha|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}))\\ &= \lim_\alpha \phi\circ id_\mathcal{A}\otimes\tilde{\phi}\circ id_\mathcal{A}\otimes q(u_\alpha) = \lim_\alpha \phi\circ\pi_\mathcal{A}(u_\alpha) \quad (\text{in } (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*).\end{aligned} \tag{2.4}$$

Since for every $u \in \mathcal{A}\hat{\otimes}\mathcal{A}$, $\pi_{\sigma wc}(\hat{u}_\alpha|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}) = \pi_\mathcal{A}(u)$, by (2.4)

$$\begin{aligned}\phi\circ\theta(a + \ker\phi) &= \lim_\alpha \phi\circ\pi_{\sigma wc}(\hat{u}_\alpha|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}) = \phi(wk^*\text{-}\lim_\alpha \pi_{\sigma wc}(\hat{u}_\alpha|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}))\\ &= \phi\circ\pi_{\sigma wc}(wk^*\text{-}\lim_\alpha u_\alpha|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}) = \phi\circ\pi_{\sigma wc}\circ\rho(a) = \phi(a) \quad (a \in \mathcal{A}).\end{aligned}$$

Step three: We prove that $\mathcal{A}$ is left ϕ-Connes amenable. Choose $x_0 \in \mathcal{A}$ such that $\phi(x_0) = 1$. Let $m = \theta(x_0 + \ker\phi)$. Then

$$am = a\theta(x_0 + \ker\phi) = \theta(ax_0 + \ker\phi) = \theta(\phi(a)x_0 + \ker\phi) = \phi(a)m \qquad (a \in \mathcal{A}).$$

According to step two,

$$\phi(m) = \phi\circ\theta(x_0 + \ker\phi) = \phi(x_0) = 1.$$

Using [7, Proposition 2.3], $\mathcal{A}$ is left ϕ-Connes amenable. □

Example 2.6. Let $\mathcal{A}$ be a dual Banach space. Suppose that ϕ is a non-zero wk^*-continuous functional on $\mathcal{A}$ with $\|\phi\| \leq 1$. Define a product on $\mathcal{A}$ with $ab = \phi(b)a$, for all $a, b \in \mathcal{A}$ as the same in Example 2.2. Then $\mathcal{A}$ be a dual Banach algebra. It is known that its unitization $\mathcal{A}^\# = \mathcal{A}\oplus\mathbb{C}$ is a dual Banach algebra with respect to $\mathcal{A}_*\oplus\mathbb{C}$. Consider the character $\psi : \mathcal{A}^\# \to \mathbb{C}$ defined by $\psi(a, \lambda) = \phi(a) + \lambda$. Since ϕ is wk^*-continuous, ψ is wk^*-continuous. We claim that $\mathcal{A}^\#$ is left ψ-Connes biprojective if and only if $\dim\mathcal{A} = 1$. Let $\mathcal{A}^\#$ be a left ψ-Connes biprojective dual Banach algebra. Since $\mathcal{A}^\#$ is unital, the argument as in Theorem 2.5 implies that $\mathcal{A}^\#$ is left ψ-contractible. Then there exists an element $m = (a, \lambda)$ in $\mathcal{A}^\#$ such that $xm = \psi(x)m$ and $\psi(m) = 1$ for every $x \in \mathcal{A}^\#$. So $x = \psi(m)x = \psi(x)m$. For every $b \in \mathcal{A}$, let $x = (b, 0)$. It is easy to see that $b = \phi(b)a$. Thus $\dim(\mathcal{A}) = 1$.
Conversely if $\dim(\mathcal{A}) = 1$, then $\mathcal{A} \cong \mathbb{C}$ as a Banach algebra. It is obvious that $\mathcal{A}^\#$ is left ψ-Connes biprojective.

We denote $Z(\mathcal{A}) = \{a_0 \in A | aa_0 = a_0a, \forall a \in A\}$.

Lemma 2.7. Let $\mathcal{A}$ be a dual Banach algebra with $\phi \in \Delta_{w^*}(\mathcal{A})$ and also let a_0 be an element of $Z(\mathcal{A})$ such that $\phi(a_0) \neq 0$. If $\mathcal{A}$ is left ϕ-Connes biprojective, then $\mathcal{A}$ is left ϕ-Connes amenable.

Proof . Suppose that $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A} \otimes_p \mathcal{A})^*)^*$ is a bounded linear map, $\rho(ab) = \phi(b)\rho(a) = a\rho(b)$ and $\phi \circ \pi_{\sigma wc} \circ \rho(a) = \phi(a)$ for all $a, b \in \mathcal{A}$. Without loss of generality we may assume that $\phi(a_0) = 1$. Since $a_0 \in Z(\mathcal{A})$, we have

$$a \cdot \rho(a_0) = \rho(aa_0) = \rho(a_0 a) = \phi(a)\rho(a_0).$$

Define $M := \pi_{\sigma wc}(\rho(a_0)) \in \mathcal{A}$. The map $\pi_{\sigma wc}$ is an $\mathcal{A}$-bimodule morphism, so we have

$$aM = a\pi_{\sigma wc}(\rho(a_0)) = \pi_{\sigma wc}(a \cdot \rho(a_0)) = \phi(a)M,$$

and

$$\phi(M) = \phi \circ \pi_{\sigma wc} \circ \rho(a_0) = \phi(a_0) = 1.$$

Apply [7, Proposition 2.3] follows that $\mathcal{A}$ is left ϕ-Connes amenable. □

3 Applications

Recall that a discrete semigroup S is weakly left (respectively, right) cancellative if $s^{-1}F = \{x \in S : sx \in F\}$ (respectively, $Fs^{-1} = \{x \in S : xs \in F\}$) is finite for each $s \in S$ and each finite subset F of S. Also S is called weakly cancellative if it is both weakly left cancellative and weakly right cancellative [1, Definition 3.14]. Note that $\ell^1(S)$ is a dual Banach algebra if and only if S is weakly cancelative [1, Theorem 4.6].

For the semigroup algebra $\ell^1(S)$ related to any discrete semigroup S, there exists a multiplicative linear functional (character) on $\ell^1(S)$. In fact the augmentation character on $\ell^1(S)$ defined by

$$\sum_{s \in S} \alpha_s \delta_s \mapsto \sum_{s \in S} \alpha_s,$$

and it is denoted by ϕ_∞ or ϕ_S.

Proposition 3.1. Let S be a unital weakly cancellative semigroup and let $\phi_S \in \Delta_{w^*}(\ell^1(S))$. Then $\ell^1(S)$ is left ϕ_S-Connes biprojective if and only if S is a finite group.

Proof . Suppose that $e \in S$ is unit. Then δ_e is a unit for $\ell^1(S)$. If $\ell^1(S)$ is left ϕ_S-Connes biprojective, then Theorem 2.5 implies that $\ell^1(S)$ is left ϕ_S-contractible. Thus there exists $m \in \ell^1(S)$ such that $am = \phi_S(a)m$ and $\phi_S(m) = 1$ for every $a \in \mathcal{A}$. By [2, Theorem 4.3], S is a group. Also [6, Theorem 6.1] implies that S is compact. Since S is discrete, S must be finite.
Conversely is clear. □

Remark 3.2. Consider $S = \mathbb{N}$ with the binary operation $(m, n) \longmapsto \max\{m, n\}$, where m and n are in $\mathbb{N}$ and it is denoted by $\mathbb{N}_\vee$. Then $\mathbb{N}_\vee$ is a weakly cancellative semigroup [1, Example 3.36]. The augmentation character on $\ell^1(\mathbb{N}_\vee)$ is not wk^*-continuous; to see this, by contrary suppose that the augmentation character is wk^*-continuous. Using [14, Example 5.5; Proposition 2.2], follows that $\ell^1(\mathbb{N}_\vee)$ is left ϕ_∞-amenable. So $\ell^1(\mathbb{N}_\vee)$ is left ϕ_∞-Connes amenable [7, Proposition 2.3]. Hence similar to the proof of previous Proposition $\mathbb{N}_\vee$ is finite, which is a contradiction.

Let $\mathcal{A}$ be a dual Banach algebra and let I be a totally ordered set. Then the set of all $I \times I$-upper triangular matrices with the usual matrix operations and the finite ℓ^1-norm $\| [a_{i,j}]_{i,j \in I} \| = \sum_{i,j \in I} \| a_{i,j} \| < \infty$ becomes a dual Banach algebra [15] and it is denoted by

$$UP(I, \mathcal{A}) = \left\{ \left[\ a_{i,j} \ \right]_{i,j \in I} ; a_{i,j} \in A \text{ and } \ a_{i,j} = 0 \ \text{ for every } i > j \right\}.$$

Let $\mathcal{A}$ be a dual Banach algebra and let $\phi \in \Delta_{wk^*}(\mathcal{A})$. Then define $\psi_\phi : UP(I, \mathcal{A}) \longrightarrow \mathbb{C}$ by $\psi_\phi((a_{ij})) = \phi(a_{i_0 i_0})$, where I be a totally ordered set which has a smallest element. One can see that ψ_ϕ is a wk^*-continuous character on $UP(I, \mathcal{A})$. Suppose that (a_α) is a net in $UP(I, \mathcal{A})$ such that $a_\alpha \xrightarrow{wk^*} b$. Consider $F \in UP(I, \mathcal{A})_*$ which all of entries are zero except (i_0, i_0)-th. It follows that $a^\alpha_{i_0,i_0}(F_{i_0,i_0}) \to b_{i_0,i_0}(F_{i_0,i_0})$, where F_{i_0,i_0} is an arbitrary element in $\mathcal{A}_*$. So $a^\alpha_{i_0,i_0} \xrightarrow{wk^*} b_{i_0,i_0}$. Since ϕ is wk^*-continuous, $\psi_\phi(a_\alpha) \longrightarrow \psi_\phi(b)$.

Theorem 3.3. Let $\mathcal{A}$ be a dual Banach algebra with a right identity and $\phi \in \Delta_{wk^*}(\mathcal{A})$ and also let I be a totally ordered set which has a smallest element. Then $UP(I,\mathcal{A})$ is right ψ_ϕ-Connes biprojective if and only if $|I| = 1$ and $\mathcal{A}$ is right ϕ-Connes biprojective.

Proof . Suppose that $UP(I,\mathcal{A})$ is right ψ_ϕ-Connes biprojective. Since $\mathcal{A}$ has a right identity, $UP(I,\mathcal{A})$ has a right approximate identity [13, Lemma 5.1]. Similar with Theorem 2.5, $UP(I,\mathcal{A})$ is right ψ_ϕ-contractible. So there exist m in $UP(I,\mathcal{A})$ such that $ma = \psi_\phi(a)m$ and $\psi_\phi(m) = \phi(m_{i_0i_0}) = 1$ for every $a \in UP(I,\mathcal{A})$. Suppose that I has at least two elements. Let a be a matrix in $UP(I,\mathcal{A})$ which (i_0, j)-th entry is e and others are zero, where $j \neq i_0$ and e is a right identity for $\mathcal{A}$. So $ma = \psi_\phi(a)m = 0$. It follows that $m_{i_0i_0} = 0$, which is a contradiction with $\phi(m_{i_0i_0}) = 1$. Converse is clear. □ Let $\mathcal{A}$ be a dual Banach algebra and let $\phi \in \Delta_{wk^*}(\mathcal{A})$. Consider I as a finite set $\{i_1, i_2, \dots, i_n\}$. Define $\psi_\phi : UP(I,\mathcal{A}) \longrightarrow \mathbb{C}$ by $\psi_\phi((a_{ij})) = \phi(a_{i_ni_n})$. One can see that ψ_ϕ is a wk^*-continuous character on $UP(I,\mathcal{A})$.

Theorem 3.4. Let $\mathcal{A}$ be a dual Banach algebra and $\phi \in \Delta_{wk^*}(\mathcal{A})$ and I be a finite set such that $Z(\mathcal{A})\cap(\mathcal{A}-\ker\phi) \neq \emptyset$. Then $UP(I,\mathcal{A})$ is left ψ_ϕ-Connes biprojective if and only if $|I| = 1$ and $\mathcal{A}$ is left ϕ-Connes biprojective.

Proof . Let $UP(I,\mathcal{A})$ be left ψ_ϕ-Connes biprojective. Suppose that there exists $a_0 \in \mathcal{A}$ such that $aa_0 = a_0a$ and $\phi(a_0) = 1$. Put $\mathcal{A}_0 = \begin{pmatrix} a_0 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix}_{i_n\times i_n}$. By an easy calculation $\mathcal{A}_0B = B\mathcal{A}_0$ for every $B \in UP(I,\mathcal{A})$ and $\psi_\phi(\mathcal{A}_0) = \phi(a_0) = 1$. Therefore, it conclude that $\mathcal{A}_0 \in Z(UP(I,\mathcal{A})) \cap (UP(I,\mathcal{A}) - \ker\psi_\phi)$. Proposition 2.3 implies that $UP(I,\mathcal{A})$ is left ψ_ϕ-Connes amenable. Clearly, ψ_ϕ is a wk^*-continuous map. By [7, Proposition 2.3], $UP(I,\mathcal{A})$ is left ψ_ϕ-contractible. We assume in contradiction $|I| \geq 2$. Define $J = \{[a_{ij}] \in UP(I,\mathcal{A}) \mid a_{ij} = 0 \quad \forall j \neq i_n\}$. It is clear that J is a closed ideal of $UP(I,\mathcal{A})$ and $\psi_{\phi|_J} \neq 0$. By [6, Proposition 3.8], J is left $\psi_{\phi|_J}$-contractible. It means that there exists $m \in J$ such that $xm = \psi_{\phi|_J}(x)m$ for all $x \in J$ and $\psi_{\phi|_J}(m) = 1$.

Now if $x = \begin{pmatrix} 0 & 0 & \cdots & a_{i_1i_n} \\ 0 & 0 & \cdots & a_{i_2i_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{i_ni_n} \end{pmatrix}$ and $m = \begin{pmatrix} 0 & 0 & \cdots & m_{1n} \\ 0 & 0 & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{nn} \end{pmatrix}$, then we have that $a_{kn} = \phi(a_{nn})m_{kn}$ for all $k = 1, \cdots, n$ and $\psi_{\phi|_J}(m) = \phi(m_{nn}) = 1$. Put $a_{1n} = m_{nn}$ and $a_{nn} = 0$ or ($a_{nn} \in \ker\phi$). Hence $m_{nn}^2 = 0$ and $\phi(m_{nn}^2) = \phi(m_{nn}) = 0$, which is a contradiction □

Example 3.5. Consider the semigroup $S = \mathbb{N}_\vee$ as in Remark 3.2. It is clear that 1 is a unit for $\mathbb{N}_\vee$. Therefore δ_1 is a unit for the dual Banach algebra $\ell^1(\mathbb{N}_\vee)$. Define $\phi_{n_0} : \ell^1(S) \to \mathbb{C}$ by $\phi_{n_0}(\sum_{i=1}^{\infty} a_i\delta_i) = \sum_{i=1}^{n_0} a_i$. One can see that ϕ_{n_0} is a wk^*-continuous character on $\ell^1(\mathbb{N}_\vee)$. We have the following statement:

Let I be a totally ordered set which has a smallest element. Define $\psi : UP(I, \ell^1(\mathbb{N}_\vee)) \longrightarrow \mathbb{C}$ by $\psi_{n_0}((a_{ij})) = \phi_{n_0}(a_{i_0i_0})$. Apply Theorem 3.3 the dual Banach algebra $UP(I, \ell^1(\mathbb{N}_\vee))$ is not right ψ_{n_0}-Connes biflat, unless I is a singleton set.

Example 3.6. Consider the Banach algebra ℓ^1 of all sequences $a = (a(n))$ of complex numbers with

$$\|a\| := \sum_{n=1}^{\infty} |a(n)| < \infty,$$

and the following product

$$(a * b)(n) = \begin{cases} a(1)b(1) & \text{if} \quad n = 1 \\ a(1)b(n) + b(1)a(n) + a(n)b(n) & \text{if} \quad n > 1 \end{cases}$$

for every $a, b \in \ell^1$. Define $\varphi_1 : \ell^1 \longrightarrow \mathbb{C}$ by $\varphi_1(a) = a(1)$ for every $a \in \ell^1$. It is easy to see that φ_1 is a w^*-continuous character on $(\ell^1, *)$. Following [16, Example 3.4], $(\ell^1, *)$ is a dual Banach algebra with respect to c_0, which is not left ϕ_1-Connes amenable. Since $(\ell^1, *)$ has a unit δ_1, where δ_1 is equal to 1 at $n = 1$ and 0 elsewhere, Theorem 2.5 implies that $(\ell^1, *)$ is not left ϕ_1-Connes biprojective.

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