Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1627–1638
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.27884.3747

On the extension of SD sets by using generalized open sets in bitopological spaces

Necati Can Açıkgöz*, Ceren Sultan Elmalı

Department of Mathematics, Erzurum Technical University, Yakutiye, 25050, Turkey

(Communicated by Abasalt Bodaghi)

Abstract

In this paper, we introduce and study ij-somewhere dense sets in bitopological spaces. We present ij-somewhere density properties by using some generalized open sets which are used for generalizing the operators of interior, closure and boundary. In this manner, we also investigate several properties of ij-somewhere dense sets and present several examples for the main results. We also introduce ω-hyperconnectedness as an extension of hyperconnectedness by using ij-somewhere dense sets. We later present some separation axioms based on these types of sets.

Keywords: bitopological spaces, ij-somewhere dense sets, generalized open sets, hyperconnected space
2020 MSC: 54A05, 54E55

1 Introduction

Many of topological properties are introduced and defined in bitopological spaces. A bitopological space (or shortly, bispace) is a set X with two topological structures like τ_i and τ_j and referred to as (X, τ_i, τ_j). The concept of bitopological space has firstly been defined by J. Kelly [11]. In the meantime many of ones defined in topological spaces are defined and currently being kept to be studied and investigated in which ways or conditions those ones hold and preserve themselves with two different topological structures.

In this sense, some of generalized open sets such as *semi-open, pre-open, semi-pre-open, α-open* sets which give an important ways to investigate generalizations of compactness, connectedness, continuity, separation axioms, selection principles, etc., based upon interior and closure operators defined in topological spaces (see [1, 2, 16, 18, 19]) have been introduced in bispaces. This types of open sets such as *ij-pre-open, ij-semi-open, ij-semi-pre-open, ij-α-open* sets have been characterized, respectively in [9, 13, 17, 14] and by means of these types of sets, new closure operators like *pre-closure, semi-closure, semi-pre closure and α-closure* are systematically defined and studied. And based upon these types of open sets being formed by using two different topological structures, many of topological constructions' weak and strong forms have been extended to bispaces and characterized (see [6, 8, 12, 10]).

Main purpose of this paper is to extend somewhere density being a topological notion to the bispaces. First recall the definition of somewhere dense set from [20], a somewhere dense set is a set whose closure contains a non-empty open set in a topological space (X, τ). In [3] properties of somewhere dense sets were studied in topological spaces, also in [4], more notions and mappings via these type of sets were introduced. So these notions provided a way to theoretical area. As this concept is in very study area, and in the light of this definition, we extend this to ij-somewhere density

*Corresponding author
Email addresses: necatican.acikgoz@erzurum.edu.tr (Necati Can Açıkgöz), ceren.elmali@erzurum.edu.tr (Ceren Sultan Elmalı)

Received: July 2022 *Accepted:* October 2022

properties taking these types of ij-open sets into consideration and we scrutinize these properties in ω-hyperconnected spaces and give some separation properties based upon this open sets and density. We investigate that under what conditions these properties closed and if we can form a filter or a proper maximal ideal on a bispace related to this ij-somewhere-dense sets.

2 Preliminaries

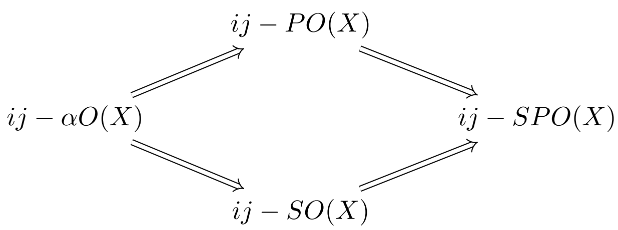
In this section, we will give some elementary definitions, properties and concepts related to topological and bitopological spaces which will be used in the following sections of the paper.
During this paper, when any property of any bitolopological space (X, τ_i, τ_j) is mentioned, it is always written the ij-property even if some well-known topologies like $\tau_D = \tau_i$ the discrete topology or the cofinite topology $\tau_j = \tau_{cof}$ are used. For any A subset of X, the interior and closure of A with respect to the topology τ_i is denoted with $int_{\tau_i}(A)$ and $cl_{\tau_i}(A)$, respectively. If any subset A of X is a dense set with respect to τ_i (or satisfies a property), the set itself is said to be i-dense (i-property). We refer reader to [7] for undefined topological notions and for undefined bitopological notions we refer to [5]. Here are the some definitions and being essential for the paper.

Definition 2.1. Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. Then;

1. [9] *ij-pre-open set* if $A \subseteq int_{\tau_i}(cl_{\tau_j}(A))$
2. [13] *ij-semi-open set* if $A \subseteq cl_{\tau_j}(int_{\tau_i}(A))$
3. [17] *ij-semi-pre-open or ij-β-open set* if $A \subseteq cl_{\tau_j}(int_{\tau_i}(cl_{\tau_j}(A)))$
4. [14] *ij-α-open set* if $A \subseteq int_{\tau_i}(cl_{\tau_j}(int_{\tau_i}(A)))$

The family of all ij-pre-open subsets (respectively ij-semi-open, ij-semi-pre-open, ij-α-open set) of X is denoted with $ij-PO(X)$ (respectively ij-$SO(X)$, ij-$SPO(X)$, ij-$\alpha O(X)$).
The complement of an *ij-preopen set* (respectively *ij-semi-open*, *ij-semi-preopen*, *ij-α-open*) is called *ij-pre-closed set* (respectively *ij-semi-closed*, *ij-semi-pre-closed*, *ij-α-closed*) and the family of all *ij-pre-closed* (respectively *ij-semi-closed*, *ij-semi-pre-closed*, *ij-α-closed*) subsets of (X, τ_i, τ_j) is shown by $ij-PC(X)$ (respectively ij-$SC(X)$, ij-$SPC(X)$, ij-$\alpha C(X)$).

By considering the definitions of these types of ij-open sets, following diagram is obtained.



Denote by P, S, SP, α the abbreviations of *pre*, *semi*, *semi-pre*, α, respectively. We will use this abbreviations during this paper.

Definition 2.2. [22] Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. Then *ij-ω-closure* and *ij-ω-interior* of A for all $\omega \in \{P, S, SP, \alpha\}$ is defined as follows;

1. $ij-\omega cl(A) = \bigcap\{K \subseteq X : K \in ij-\omega C(X), A \subseteq K\}$
2. $ij-\omega int(A) = \bigcup\{G \subseteq X : G \in ij-\omega O(X), A \supseteq G\}$

Definition 2.3. Let (X, τ_i, τ_j) be a bitopological space, $x \in X$ and $U \subseteq X$. If there is a nonempty $G \in ij-\omega O(X)$ such that $x \in G \subseteq U$, then U is called *$ij-\omega$-neighbourhood* of x.

Proposition 2.4. Denote by $\mathcal{U}_\omega(x)$ the family of all ij-ω-neighbourhoods of x in (X, τ_i, τ_j). Then for a subset A of X

1. $x \in ij - \omega int(A)$ if and only if there is a $U \in \mathscr{U}_\omega(x)$ such that $x \in U \subseteq A$
2. $x \in ij - \omega cl(A)$ if and only if for all $U \in \mathscr{U}_\omega(x)$, $U \cap A \neq \emptyset$

Definition 2.5. [5] Let (X, τ_i, τ_j) and (Y, σ_i, σ_j) be bitopological spaces and $f : X \to Y$ be a mapping. If the induced mappings $f_i : (X, \tau_i) \to (Y, \sigma_i)$ and $f_j : (X, \tau_j) \to (Y, \sigma_j)$ are continuous, then f is called *pairwise continuous (d-continuous).*

If a property is preserved under bi-homeomorphism (a mapping which is d-continuous, bijective and whose inverse is d-continuous), then it is called *bi-topological property.*

Definition 2.6. [15] A $\mathscr{I}$ family of subsets of X in topological space (X, τ) is called a *proper ideal* whenever satisfying following three conditions ;

1. $X \notin \mathscr{I}$
2. If $I \in \mathscr{I}$ and $J \subseteq I$ then $J \in \mathscr{I}$
3. For any $I_1, I_2 \in \mathscr{I}$ then $I_1 \cup I_2 \in \mathscr{I}$

Definition 2.7. Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. A is called *ij-ω-dense set* if $ij - \omega cl(A) = X$.

3 ij-ω-somewhere density

In this section, we give some definitions, results related to *ij-ω*-somewhere density and investigate the relations between the generalized open sets in bitopological spaces.

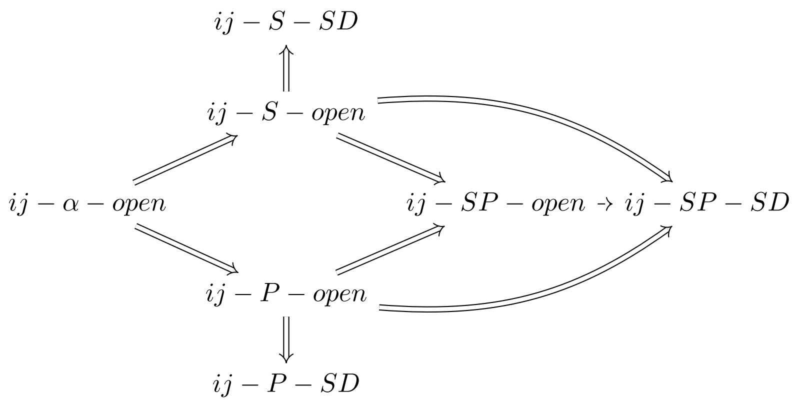
Definition 3.1. Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. If there is a $G \in ij - \omega O(X)$ such that $G \subseteq ij - \omega cl(A)$, then A is called *ij-ω-somewhere dense set (ij-ω-SD, for short).* $\omega \in \{P, S, SP, \alpha\}$

According to the Definition 3.1, we can give the following quick results;

Proposition 3.2. Let (X, τ_i, τ_j) be a bitopological space. Then;

1. Every ij-ω-open set is ij-ω-somewhere dense.
2. Every ij-ω-open set is ij-semi-pre-somewhere dense.
3. Every member of $\mathscr{U}_\omega(x)$ is ij-ω-somewhere dense for all $x \in X$.
4. If $A \subseteq X$ is ij-ω-somewhere dense, then for all $U \subseteq X$ such that $U \supseteq A$ is ij-ω-somewhere dense.

From Proposition 3.2, we also obtain the following diagram.



Denote by $\zeta_\omega(\tau_i, \tau_j)$ the family of all *ij-ω*-somewhere dense subsets of bitopological space (X, τ_i, τ_j). As stated in the diagram above, every $ij - \omega$-open set is $ij - \omega$-somewhere dense, but converse of the statement does not always hold. For example, consider any set X such that $|X| > 2$ with indiscrete topology τ_t and discrete topology τ_D. Observe that for all non-empty $A \subseteq X$, $A \in \zeta_\omega(\tau_t, \tau_D)$ holds. But none of nonempty subsets of X is $ij - \omega$-open for all $\omega \in \{P, S, SP, \alpha\}$.

Definition 3.3. Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. If $X \backslash A \in \zeta_\omega(\tau_i, \tau_j)$, then A is called *ij-ω-co-dense set.*

Theorem 3.4. A D subset of a bitopological space (X, τ_i, τ_j) is ij-ω-co-dense if and only if there is $F \in ij-\omega C(X)$ such that $F \neq X$ and ij-$\omega int(D) \subseteq F$.

Proof . Assume that $X \setminus D \in \zeta_\omega(\tau_i, \tau_j)$, then there is a non-empty $G \in ij-\omega O(X)$ such that $G \subseteq ij-\omega cl(X \setminus D)$ and $X \setminus ij-\omega cl(X \setminus D) \subseteq X \setminus G$. If $x \in ij-\omega int(D)$, then $\exists U_x \in \mathscr{U}_\omega(x)$ such that $U \subseteq D$ and $U_x \cap (X \setminus D) = \emptyset$ implying that $x \in X \setminus ij-\omega cl(X \setminus D) \subseteq X \setminus G$. Now choose $F = X \setminus G \in ij-\omega C(X)$, then $ij-\omega int(D) \subseteq F$. Conversely, if there is a $F \in ij-\omega C(X)$ such that $F \neq X$ and $ij-\omega int(D) \subseteq F$, since $X \setminus F \subseteq X \setminus ij-\omega int(D) \subseteq ij-\omega cl(X \setminus D)$, thus $X \setminus D \in \zeta_\omega(\tau_i, \tau_j)$. □

Proposition 3.5. Let X any set with τ_t and any topology τ_j. Then $\zeta_\omega(\tau_t, \tau_j) = \tau_D$ for $\omega \in \{\alpha, S\}$.

Proof . One can observe that for all non-empty $A \subseteq X$ is ij-ω-dense in X. □

Theorem 3.6. Every subset of a bitopological space is ij-ω-somewhere dense or ij-ω-co-dense.

Proof . If $A \notin \zeta_\omega(\tau_i, \tau_j)$ in the bitopological space (X, τ_i, τ_j), then $ij-\omega int(ij-\omega cl(A)) = \emptyset$ which implies that $X \setminus ij\text{-}\omega\text{-}cl(A) \neq \emptyset$. Since $X \setminus ij-\omega-cl(A) \in ij\text{-}\omega O(X)$ and $X \setminus ij-\omega cl(A) \subseteq X \setminus A$, then $X \setminus A \in \zeta_\omega(\tau_i, \tau_j)$. □

Theorem 3.7. Let (X, τ_i, τ_j) be a bitopological space and $\mathscr{A} = \{A_s : s \in S\} \subseteq \mathscr{P}(X)$ where $\mathscr{P}(X)$ is power set of X. Then $\bigcup_{s\in S} A_s \in \zeta_\omega(\tau_i, \tau_j)$ if and only if $\bigcup_{s\in S} ij\text{-}\omega cl(A_s) \in \zeta_\omega(\tau_i, \tau_j)$.

Proof . It is straightforward from left to right. Now take a nonempty $G \in ij-\omega O(X)$ such that $G \subseteq ij-\omega cl(\bigcup_{s\in S} ij-\omega cl(A_s))$. If $x \in ij-\omega cl(\bigcup_{s\in S} ij-\omega cl(A_s))$, then $U \cap \Big(\bigcup_{s\in S} ij-\omega cl(A_s)\Big) \neq \emptyset$ for all $U \in \mathscr{U}_\omega(x)$. Observe that $U \cap ij-\omega cl\Big(\bigcup_{s\in S} A_s\Big) \neq \emptyset$ which implies that $x \in ij-\omega cl\Big(\bigcup_{s\in S} A_s\Big)$ and $\bigcup_{s\in S} A_s \in \zeta_\omega(\tau_i, \tau_j)$.□

Corollary 3.8. Arbitrary union (intersection) of ij-ω-somewhere dense sets (ij-ω-$co-dense$ sets) is ij-ω-somewhere dense (ij-ω-$co-dense$).

Corollary 3.9. The family $\zeta_\omega(\tau_i, \tau_j)$ is a supra topology on X for all $\omega \in \{P, S, SP, \alpha\}$.

Remark 3.10. $ij-\omega$-somewhere density ($ij-\omega$-co-density) is not closed under finite intersection (arbitrary union). One of clear counterexample is an uncountable set X with the topology τ_D and any topology τ_j. It is clear that $\{x\} \in \zeta_\omega(\tau_D, \tau_j)$ for all $x \in X$ and $\omega \in \{P, S, SP, \alpha\}$ and $\bigcup_{x\in X} X \setminus \{x\} = X$ but $\emptyset \notin \zeta_\omega(\tau_D, \tau_j)$.

Theorem 3.11. Let $\{(X, \tau_i, \sigma_i) : i \in I\}$ be any family of bitopological spaces with $X_i \neq \emptyset$ for all $i \in I$ and $A_i \subseteq X_i$. Then $A_i \in \zeta_\omega(\tau_i, \sigma_i)$ if and only if $\prod_{i\in I} A_i$ is ij-ω-somewhere dense in the box bitopological space $\Big(\prod_{i\in I} X_i, \bigotimes_{i\in I} \tau_i, \bigotimes_{i\in I} \sigma_i\Big)$.

Proof . We give the proof for $\omega = pre$. Others can be obtained likewise.
Let $A_i \in \zeta_\omega(\tau_i, \sigma_i)$ for all $i \in I$. Take a non-empty $G_i \in ij-PO(X)$ such that $G_i \subseteq ij-Pcl(A_i)$. Then $\prod_{i\in I} G_i \subseteq \prod_{i\in I} ij-Pcl(A_i)$.

Fact 1.: $\prod_{i\in I} G_i \in ij-PO\Big(\prod_{i\in I} X_i\Big)$.

If $G_i \in ij-PO(X_i)$ for all $i \in I$, then $\prod_{i\in I} G_i \subseteq \prod_{i\in I} int_{\tau_i}(cl_{\sigma_i}(G_i))$. Now if $x = (x_i)_{i\in I} \in \prod_{i\in I} int_{\tau_i}(cl_{\sigma_i}(G_i))$, then there is a $U_i \in \tau_i$ such that $x_i \in U_i \subseteq cl_{\sigma_i}(G_i)$ and since $\prod_{i\in I} U_i \in \bigotimes_{i\in I}\tau_i$ then $x \in int_{\bigotimes_{i\in I}\tau_i}\Big(\prod_{i\in I} cl_{\sigma_j}(G_i)\Big)$. Also see that $\prod_{i\in I} cl_{\sigma_i}(G_i) = cl_{\bigotimes_{i\in I}\sigma_i}\Big(\prod_{i\in I} G_i\Big)$ and $\prod_{i\in I} G_i \subseteq int_{\bigotimes_{i\in I}\tau_i}(cl_{\bigotimes_{i\in I}\sigma_i}\Big(\prod_{i\in I} G_i\Big)$.

Fact 2.: $\prod_{i\in I} ij - Pcl(A_i) = ij - Pcl\Big(\prod_{i\in I} A_i\Big)$.

Let p_i be projection maps for all $i \in I$. Since $\prod_{i\in I} ij - Pcl(A_i) = \bigcap_{i\in I} p_i^{\leftarrow}(ij - Pcl(A_i))$ and $ij - Pcl(A_i) \in ij - PO(X_i)$ for all $i \in I$, then

$$p_i^{\leftarrow}(X_i \setminus ij - Pcl(A_i)) = \prod_{i\in I} X_i \setminus \Big(p_i^{\leftarrow}(ij - Pcl(A_i))\Big)$$

is satisfied. Now that p_i is both d-open and d-continuous, then $\prod_{i\in I} X_i \setminus \Big(p_i^{\leftarrow}(ij - Pcl(A_i))\Big) \in ij - PO\Big(\prod_{i\in I} X_i\Big)$. Therefore $\prod_{i\in I} ij - Pcl(A_i) \in ij - PC\Big(\prod_{i\in I} X_i\Big)$ as the arbitrary intersection of ij-pre-closed sets which concludes that $ij - Pcl\Big(\prod_{i\in I} A_i\Big) \subseteq \prod_{i\in I} ij - Pcl(A_i)$. Now take $x = (x_i)_{i\in I} \in \prod_{i\in I} ij - Pcl(A_i)$ and $H \in ij - PO\Big(\prod_{i\in I} X_i\Big)$ which contains x. Observe that $p_i(H) \in ij - PO(X_i)$ for all $i \in I$ and thus $H \cap \prod_{i\in I} A_i \neq \emptyset$ concluding that $ij - Pcl\Big(\prod_{i\in I} A_i\Big) = \prod_{i\in I} ij - Pcl(A_i)$ and $\prod_{i\in I} A_i \in \zeta_P(\bigotimes_{i\in I}\tau_i, \bigotimes_{i\in I}\sigma_i)$.

Now if $\prod_{i\in I} A_i \in \zeta_P(\bigotimes_{i\in I}\tau_i, \bigotimes_{i\in I}\sigma_i)$, take $O \in ij - PO\Big(\prod_{i\in I} X_i\Big)$ such that $O \subseteq ij - Pcl\Big(\prod_{i\in I} A_i\Big)$ and conclude that $p_i(O) \subseteq ij - Pcl(A_i) \in \zeta_P(\tau_i, \sigma_i)$. $\square$

Remark 3.12. When the product set is considered with the product topologies, product of ij-ω-somewhere dense sets need not be ij-ω-somewhere dense as the following example illustrates for $\omega = pre$.

Example 3.13. There are ij-pre-somewhere dense sets whose product is not ij-pre-somewhere dense set in the product bitopological space.

Consider the real numbers set $\mathbb{R}$ with its usual topology τ_s and τ_D. Take the $\mathbb{N}$ many copies of the bispace $(\mathbb{R}, \tau_s, \tau_D)$ and denote by τ_p^s and τ_P^D the product topologies of the corresponding topologies, respectively. Observe that $(0, 2) \in \zeta_P(\tau_s, \tau_D)$ and $ij - PO(\mathbb{R}^{\mathbb{N}}) = \tau_p^s$. Now that if $(0, 2)^{\mathbb{N}} \in \zeta_P(\tau_p^s, \tau_p^D)$, then it's ij-pre-closure should contain a non-empty $U \in \tau_p^s$ which is in the form $U = \prod_{i\in N} U_i$ such that for a finite subset $F \subseteq \mathbb{N}$

$$U_i = \begin{cases} U \in \tau_s \setminus \{\mathbb{R}\} & , i \in F \\ \mathbb{R} & , i \in \mathbb{N} \setminus F \end{cases}$$

Now choose $x \in U$ and fix $n \in \mathbb{N} \setminus F$ and build a $y = (y_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ as

$$y_i = \begin{cases} x_i & , i \neq n \\ \sqrt{5} & , i = n \end{cases}$$

and conlude that $ij - Pint(ij - Pcl((0, 2)^{\mathbb{N}})) = \emptyset$.

Corollary 3.14. Since box and product topologies coincides for finite product, finite products of ij-ω-somewhere dense sets is ij-ω-somewhere dense in finite product bitopological space.

Theorem 3.15. Image of an ij-ω-somewhere dense set under d-open and d-continuous mapping is ij-ω-somewhere dense.

Proof . We prove it for $\omega = pre$.
Let (X, τ_i, τ_j) and (Y, σ_i, σ_j) be bitopological spaces, $f : X \longrightarrow Y$ be a d-continuous and d-open mapping and $A \subseteq X$. If $A \in \zeta_\omega(\tau_i, \tau_j)$, then take $U \in ij - PO(X)$ such that $U \subseteq ij - Pcl(A)$. Therefore $f(U) \subseteq f(ij - Pcl(A))$. By using f is $\tau_i - \sigma_i$ open and $\tau_j - \sigma_j$ continuous, observe that $f(U) \in ij - PO(Y)$. Now take any $K \in ij - PC(X)$ such that $f(A) \subseteq K$. On the other hand

$$\begin{aligned} f^{\leftarrow}(Y \setminus K) &\subseteq f^{\leftarrow}(int_{\sigma_i}(cl_{\sigma_j}(Y \setminus K))) \\ &\subseteq int_{\tau_i}(f^{\leftarrow}(cl_{\sigma_j}(Y \setminus K))) \\ &\subseteq int_{\tau_i}(cl_{\tau_j}(f^{\leftarrow}(Y \setminus K))) \end{aligned}$$

holds which implies that $f^{\leftarrow}(K) \in ij - PC(X)$. Now that $A \subseteq f^{\leftarrow}(K)$, then $f(U) \subseteq ij - Pcl(f(A))$ concluding that $f(A) \in \zeta_P(\sigma_i, \sigma_j)$. □

Remark 3.16. ij-ω-somewhere density need not be preserved under d-continuity. For this, consider bispaces $(\mathbb{R}, \tau_{coc}, \tau_D)$ in which τ_{coc} denotes the cocountable topology, and $(\mathbb{R}, \tau_t, \tau_s)$. Let $c \in \mathbb{R}$ and $f(x) = c$ be constant mapping between these bispaces. If we take $A = \{\frac{1}{n} : n = 1, 2, 3, ...\}$ from the first space, then $\mathbb{R} \setminus A \in \zeta_\omega(\tau_{coc}, \tau_D)$ but $\{c\} \notin \zeta_\omega(\tau_t, \tau_s)$.

4 Set operators and ω-hyperconnectedness

In this section, we introduce some set operators based upon $ij - \omega$-somewhere dense sets, ω-hyperconnectedness, and give some results related to this definitions.

Definition 4.1. Let (X, τ_i, τ_j) be bitopological space and $A \subseteq X$. Then;

1. The $SD(\omega)$- interior of A is $ij - int_{SD(\omega)}(A) = \bigcup\{U : U \subseteq A, U \in \zeta_\omega(\tau_i, \tau_j)\}$.
2. The $SD(\omega)$-closure of A is $ij - cl_{SD(\omega)}(A) = \bigcap\{G : G \supseteq A, X \setminus G \in \zeta_\omega(\tau_i, \tau_j)\}$.
3. The $SD(\omega)$-boundary of A is $ij - b_{SD(\omega)}(A) = X \setminus (ij - int_{SD(\omega)}(A) \cup ij - int_{SD(\omega)}(X \setminus A))$

Proposition 4.2. Followings hold for any bitopological space (X, τ_i, τ_j). Let $A \subseteq X$;

1. $ij - int_{SD(\omega)}(A) \subseteq A$. $A \in \zeta_\omega(\tau_i, \tau_j)$ if and only if $ij - int_{SD(\omega)}(A) = A$
2. $A \subseteq ij - cl_{SD(\omega)}(A)$. $X \setminus A \in \zeta_\omega(\tau_i, \tau_j)$ if and only if $A = ij - cl_{SD(\omega)}(A)$
3. $X \setminus ij - int_{SD(\omega)}(A) = ij - cl_{SD(\omega)}(X \setminus A)$.

Proof . Clear from the corresponding definitions. □ Also followings can be easily concluded for any bispace (X, τ_i, τ_j);

Proposition 4.3. For any $A, B \subseteq X$;

1. $ij - int_{SD(\omega)}(A) \cup ij - int_{SD(\omega)}(B) \subseteq ij - int_{SD(\omega)}(A \cup B)$
2. $ij - int_{SD(\omega)}(A \cap B) \subseteq ij - int_{SD(\omega)}(A) \cap ij - int_{SD(\omega)}(B)$
3. $ij - cl_{SD(\omega)}(A \cap B) \subseteq ij - cl_{SD(\omega)}(A) \cap ij - cl_{SD(\omega)}(B)$
4. $ij - cl_{SD(\omega)}(A) \cup ij - cl_{SD(\omega)}(B) \subseteq ij - cl_{SD(\omega)}(A \cup B)$
5. $ij - b_{SD(\omega)}(ij - int_{SD(\omega)}(A)) \subseteq ij - b_{SD(\omega)}(A)$
6. $ij - b_{SD(\omega)}(A) = ij - cl_{SD(\omega)}(A) \cap ij - cl_{SD(\omega)}(X \setminus A)$
7. $ij - b_{SD(\omega)}(A) = ij - cl_{SD(\omega)}(A) \setminus ij - int_{SD(\omega)}(A)$

Theorem 4.4. Let (X, τ_i, τ_j) be a bispace and $A \subseteq X$. Then;

1. $A \in \zeta_\omega(\tau_i, \tau_j)$ if and only if $ij - b_{SD(\omega)}(A) \cap A = \emptyset$.
2. A is ij-ω-co-dense if and only if $ij - b_{SD(\omega)}(A) \subseteq A$.

Proof .

1. Let $A \in \zeta_\omega(\tau_i, \tau_j)$ then $ij - int_{SD(\omega)} = A$ and since $ij - b_{SD(\omega)} \cap ij - int_{SD(\omega)} = \emptyset$ from Definition 4.1(3), conclude what is it to be. Now if $A \notin \zeta_\omega(\tau_i, \tau_j)$ then take $x_0 \in ij - cl_{SD(\omega)}(X \setminus A)$ such that $x_0 \notin X \setminus A$ which clearly implies that $x_0 \in ij - b_{SD(\omega)}(A)$ from Definition 4.1(3) and Proposition 4.3(6) and contradicts that $ij - b_{SD(\omega)}(A) \cap A = \emptyset$.

2. Let $X \setminus A \in \zeta_\omega(\tau_i, \tau_j)$. Then $(X \setminus A) \cap ij - int_{SD(\omega)}(X \setminus A) = \emptyset$. Observe that $ij - b_{SD(\omega)}(A) = ij - b_{SD(\omega)}(X \setminus A)$ and conclude that $(X \setminus A) \cap ij - b_{SD(\omega)}(A) = \emptyset$ which implies the one as wanted. From right to left is obtained likewise.

□

Corollary 4.5. $A, X \setminus A \in \zeta_\omega(\tau_i, \tau_j)$ if and if $ij - b_{SD(\omega)} = \emptyset$.

Theorem 4.6. Let (X, τ_i, τ_j) be a bispace and $A \subseteq X$. If $ij - cl_{SD(\omega)}(A) = X$, then A is $ij - \omega$-dense.

Proof . Suppose that A is not $ij - \omega$-dense subset of X. Then there is a non-empty $G \in ij - \omega O(X)$ such that $A \subseteq X \setminus G$. Since $G \in ij - \omega O(X)$, $G \in \zeta_\omega(\tau_i, \tau_j)$ then $X \setminus G$ is $ij - \omega$-co-dense subset and $ij - cl_{SD(\omega)}(A) \subseteq X \setminus G$, this contradicts that $G \neq \emptyset$. □

Example 4.7. There are sets whose $ij - cl_{SD(\omega)}$ is not the whole space even if it is $ij - \omega$-dense.

When the bispace $(\mathbb{R}, \tau_s, \tau_t)$ is considered, despite the fact that $\mathbb{Q} \subseteq \mathbb{R}$ is $ij - \omega$-dense subset for all $\omega \in \{P, S, SP, \alpha\}$, $ij - cl_{SD(\omega)}(\mathbb{Q}) = \mathbb{Q}$ due to $\mathbb{P} \in \zeta_\omega(\tau_s, \tau_t)$.

Theorem 4.8. Let (X, τ_i, τ_j) be a bitopological space and $G, A \subseteq X$. If $G \in \tau_i \cap \tau_j$ then $G \cap ij - \omega cl(A) \subseteq ij - \omega cl(G \cap A)$.

Proof . For $\omega = pre$, the proof follows;
Take $x \in G \cap ij - Pcl(A)$. Now that $x \in ij - Pcl(A)$, then $U \cap A \neq \emptyset$ for all $x \in U \in ij - PO(X)$. On the other hand,

$$\begin{aligned} G \cap U &\subseteq G \cap (cl_{\tau_j}(int_{\tau_i}(U)) \\ &\subseteq int_{\tau_i}(G \cap cl_{\tau_j}(U))....(G \in \tau_i) \\ &\subseteq int_{\tau_i}(cl_{\tau_j}(G \cap U))....(G \in \tau_j) \end{aligned}$$

holds for all $x \in U \in ij - PO(X)$ which implies that $G \cap U \in ij - PO(X)$ containing x. Then $(G \cap U) \cap (A \cap G) \neq \emptyset$ and $U \cap (A \cap G) \neq \emptyset$ which concludes that $x \in ij - Pcl(A \cap G)$ □ Although finite intersection of ij-ω-somewhere dense sets need not be ij-ω-somewhere dense set in bispaces as Remark 3.10 indicates, ω-hyperconnected spaces allow us construct filters and proper maximal ideals by using $ij - \omega$-somewhere dense sets. We now give some definitions and results.

Definition 4.9. [21] Let (X, τ) be a topological space. If every non-empty open subset is dense in X, then (X, τ) is called *hyperconnected space.*

Theorem 4.10. Let (X, τ_i, τ_j) be τ_i-hyperconnected bitopological space, $A, B \subseteq X$. If $A \in \tau_i \cap \tau_j$ and $B \in \zeta_\omega(\tau_i, \tau_j)$, then $A \cap B \in \zeta_\omega(\tau_i, \tau_j)$.

Proof . Take $\omega = pre$ and obtain the same result for the others. Now that $B \in \zeta_P(\tau_i, \tau_j)$, take a non-empty $G \in ij - PO(X)$ such that $G \subseteq ij - Pcl(B)$ which means $A \cap G \subseteq A \cap ij - Pcl(B)$. Since $G \in ij - PO(X)$, $A \in \tau_i \cap \tau_j$ and the space itself is τ_i-hyperconnected, then $A \cap int_{\tau_i}(cl_{\tau_j}(G) \neq \emptyset \Rightarrow A \cap cl_{\tau_j}(G) = cl_{\tau_j}(A \cap G) \neq \emptyset \Rightarrow A \cap G \neq \emptyset$ and with the same process in Theorem 4.8 it is clear that $A \cap G \subseteq ij - Pcl(A \cap B)$ with $A \cap G \in ij - PO(X)$ which directly leads us to one what we expect.□

Definition 4.11. A bitopological space is called ω-hyperconnected providing that *a subset of the space is ij-ω-dense if and only if it is ij-ω-open.*

One can observe that ω-hyperconnected spaces is τ_i-hyperconnected is a direct consequence.

Theorem 4.12. Let (X, τ_i, τ_j) ω-hyperconnected space and $\emptyset \neq A, B \subseteq X$. If $ij - \omega int(A) = \emptyset = ij - \omega int(B)$, then $ij - \omega int(A \cup B) = \emptyset$.

Proof . Assume that $ij - \omega int(A \cup B) \neq \emptyset$ and take a non-empty $U \in ij - \omega O(X)$ such that $U \subseteq A \cup B$. Since $ij - \omega cl(U) = X$, then $X \setminus ij - \omega cl(A \cup B) = \emptyset$. Then $X \setminus ij - \omega cl(A) \neq \emptyset \neq X \setminus ij - \omega cl(B)$ holds otherwise it would clearly contradict that $A, B \notin ij - \omega O(X)$. But this time it contradicts that $A \cup B$ is $ij - \omega$-dense. Consequently, $ij - \omega int(A \cup B) = \emptyset$. □

Corollary 4.13. If $A \subseteq X$ is $ij-\omega$-co-dense, then $ij-\omega int(A)=\emptyset$.

Theorem 4.14. If (X,τ_i,τ_j) is a ω-hyperconnected bispace, then $\zeta_\omega(\tau_i,\tau_j)$ is a filter on X for all $\omega \in \{P,S,SP,\alpha\}$.

Proof .

1. That $\emptyset \notin \zeta_\omega(\tau_i,\tau_j)$ appears from the Definition 3.1.
2. Take $F_1,F_2 \in \zeta_\omega(\tau_i,\tau_j)$. Then $X\setminus F_1$ and $X\setminus F_2$ are $ij-\omega$-co-dense subsets of X. Then $ij-\omega int(X\setminus F_1)=\emptyset = ij-\omega int(X\setminus F_2)$ and $ij-\omega int(X\setminus (F_1\cap F_2))=\emptyset$ which implies with the Theorem 3.4 that $F_1\cap F_2 \in \zeta_\omega(\tau_i,\tau_j)$.
3. It's the direct consequence from the Proposition 3.2(4).

□

Definition 4.15. [15] Let X be a set and $\mathscr{I}$ be a proper ideal on X. If there is no proper ideal containing $\mathscr{I}$ (equivalently either $A\in\mathscr{I}$ or $X\setminus A\in\mathscr{I}$ for all $A\subseteq X$), then $\mathscr{I}$ is called *proper maximal ideal on X*.

Corollary 4.16. The family of all $ij-\omega$-co-dense subsets of ω-hyperconnected bispace (X,τ_i,τ_j) ;

$$\mathscr{C}=\{U\subseteq X : X\setminus U\in\zeta_\omega(\tau_i,\tau_j)\}$$

is a proper maximal ideal on X with the Theorem 3.6.

5 SD(ω) $-$ T_n spaces

In this last main section of the paper, we introduce some separation axioms taking $ij-\omega$-somewhere density into consideration.

Definition 5.1. Let $\mathscr{X}=(X,\tau_i,\tau_j)$ be a bitopological space. Then;

1. $\mathscr{X}$ is $SD(\omega)-T_0$ if for all $x,y\in X$ with $x\neq y$, there is a $G\in\zeta_\omega(\tau_i,\tau_j)$ such that $x\in G$, $y\notin G$ or $x\notin G$, $y\in G$.
2. $\mathscr{X}$ is $SD(\omega)-T_1$ if for all $x,y\in X$ with $x\neq y$, there are $G,H\in\zeta_\omega(\tau_i,\tau_j)$ such that $x\in G$, $y\notin G$ and $x\notin H$, $y\in H$.
3. $\mathscr{X}$ is $SD(\omega)-T_2$ if for all $x,y\in X$ with $x\neq y$, there are $G,H\in\zeta_\omega(\tau_i,\tau_j)$ such that $x\in G$, $y\in H$ and $G\cap H=\emptyset$.

We now can give the relations as follows;

$$\begin{array}{ccccc}
\tau_i - T_2 & \Longrightarrow & \tau_i - T_1 & \Longrightarrow & \tau_i - T_0 \\
\Downarrow & & \Downarrow & & \Downarrow \\
SD(\omega)-T_2 & \Longrightarrow & SD(\omega)-T_1 & \Longrightarrow & SD(\omega)-T_0
\end{array}$$

Without an extra condition, none of the arrows in the diagram can be reversable as the following examples witness.

Example 5.2.

1. Consider the set of real numbers $\mathbb{R}$ with $\tau^*=\{U\subseteq\mathbb{R} : 0\in U \text{ xor } 1\notin U\}$ and τ_s. The bispace $(\mathbb{R},\tau^*,\tau_s)$ is $SD(\omega)-T_0$ but not $\tau_i-T_n(n=0,1,2)$. To see this, observe that $(0,1]\in\zeta_\omega(\tau^*,\tau_s)$ and for all $x,y\in\mathbb{R}\setminus\{0,1\}$ with $x\neq y$, choose $\{x\}\in\zeta_\omega(\tau^*,\tau_s)$. But no $G\in\tau^*$ such that $0\in G$, $1\notin G$ or vice versa can be found.
2. Let $X=\{a,b\}$ and $\tau_i=\{\emptyset,X,\{a\}\}$ is the Sierpiński topology. Consider the bispace (X,τ_i,τ_{cof}). Since (X,τ_i,τ_{cof}) is τ_i-T_0 then $SD(\omega)-T_0$ for all $\omega\in\{P,S,SP,\alpha\}$. It can be easily seen that $\{b\}\notin\zeta_\omega(\tau_i,\tau_{cof})$. Then only ij-ω-somewhere dense set containing b is X. This shows that (X,τ_i,τ_{cof}) is not $SD(\omega)-T_1$.

3. Let $\mathbb{R}$ be equipped with the topologies τ_{cof} and τ_D. Then the bispace $(\mathbb{R}, \tau_{cof}, \tau_D)$ is $SD(\omega) - T_1$ since ij-$\omega O(X) = \tau_{cof}$ for all $\omega \in \{P, S, SP, \alpha\}$. Then it can not be $SD(\omega) - T_2$.
4. Let $\omega \in \{P, SP\}$. Consider $\mathbb{R}$ with the topologies $\tau_i = \{U \subset \mathbb{R} : a \notin U\} \cup \{\mathbb{R}\}$ and $\tau_j = \{U \subset \mathbb{R} : a \in U\} \cup \{\emptyset\}$ where $a \in \mathbb{R}$ is fixed. (See [21]). Since only τ_i-open set containing a is $\mathbb{R}$, the bispace $(\mathbb{R}, \tau_i, \tau_j)$ is not $\tau_i - T_1$ and so not $\tau_i - T_2$. For all $x, y \in \mathbb{R}$ such that $x \neq y$ and $a \notin \{x, y\}$, choose $\{x\}, \{y\} \in \zeta_\omega(\tau_i, \tau_j)$ and for $a \neq x$, choose $\mathbb{R} \setminus \{x\}, \{x\} \in \zeta_\omega(\tau_i, \tau_j)$. Then $(\mathbb{R}, \tau_i, \tau_j)$ is both $SD(\omega) - T_1$ and $SD(\omega) - T_2$.
5. Let $\omega \in \{\alpha, S\}$. Let τ_i be as in (4). With the same processes as in (4), it can be easily obtained that the bispace $(\mathbb{R}, \tau_i, \tau_t)$ is not $\tau_i - T_1$ and $\tau_i - T_2$. But it is $SD(\omega) - T_1$ and $SD(\omega) - T_2$.

Theorem 5.3. Let $\mathscr{X} = (X, \tau_i, \tau_j)$ be a bitopological space. Then the followings are equivalent.

1. $\mathscr{X}$ is $ij - SD(\omega) - T_1$.
2. $X \setminus \{x\} \in \zeta_\omega(\tau_i, \tau_j)$ for every $x \in X$.
3. $A = \bigcap\{U : U \in \zeta_\omega(\tau_i, \tau_j), U \supseteq A\}$ for all $A \subseteq X$.

Theorem 5.4. Followings are also equivalent for $\mathscr{X} = (X, \tau_i, \sigma_j)$.

1. $\mathscr{X}$ is $ij - SD(\omega) - T_2$.
2. $\Delta(X) = \{(x, x) : x \in X\}$ is $ij - \omega$-co dense in $(X \times X, \bigotimes_{i=1}^{2} \tau_i, \bigotimes_{j=1}^{2} \sigma_j)$.
3. $\{x\} = \bigcap\{ij - cl_{SD(\omega)}(A) : x \in A, A \in \zeta_\omega(\tau_i, \sigma_j)\}$ for all $x \in X$.

Proof . (1) $\Rightarrow$ (2). Let $x, y \in X$ and $x \neq y$. Take $G, H \in \zeta_\omega(\tau_i, \sigma_j)$ such that $x \in G$, $y \in H$ and $G \cap H = \emptyset$. Then from Corollary 3.14, $G \times H \in \zeta_\omega(\bigotimes_{i=1}^{2} \tau_i, \bigotimes_{j=1}^{2} \sigma_j)$ and since $G \cap H = \emptyset$, then $G \times H \subseteq X \times X \setminus \Delta(X)$. Now that $G, H \in \zeta_\omega(\tau_i, \sigma_j)$, choose $O_1, O_2 \in ij - PO(X)$ such that $O_1 \times O_2 \subseteq ij - \omega cl(G \times H) \subseteq ij - \omega cl(X \times X \setminus \Delta(X))$ implying that $\Delta(X)$ is $ij - \omega$-co-dense.
(2) $\Rightarrow$ (1). *Straightforward.*
(1) $\Rightarrow$ (3). If there is $y \notin \{x\}$ such that $y \in \bigcap\{ij - cl_{SD(\omega)}(A) : x \in A, A \in \zeta_\omega(\tau_i, \sigma_j)\}$ for $x \in X$, then take $G_1, G_2 \in \zeta_\omega(\tau_i, \sigma_j)$ such that $x \in G_1, y \in G_2$ and $G_1 \cap G_2 = \emptyset$. Choose $O_1 \in ij - \omega O(X)$ witnesses that $O_1 \subseteq ij - \omega cl(X \setminus H) \in \zeta_\omega(\tau_i, \sigma_j)$. But this clearly is a contradiction due to $H \in \zeta_\omega(\tau_i, \sigma_j)$ and $x \in ij - cl_{SD(\omega)}(X \setminus H) = X \setminus H$.
(3) $\Rightarrow$ (1). *Straightforward.* $\square$

Theorem 5.5. Let $\mathscr{X} = (X, \tau_i, \tau_j)$ be a bispace. If every τ_i-convergent filter on X has a unique limit point, then the space is $ij - SD(\omega) - T_2$.

Proof . If $\mathscr{X}$ is not $ij - SD(\omega) - T_2$, then there exists $x, y \in X$ with $x \neq y$ such that $y \in U$ for all $x \in U \in \zeta_\omega(\tau_i, \tau_j)$. Since $\mathscr{U}_{\tau_i}(x) \subseteq \zeta_\omega(\tau_i, \tau_j)$, then

$$\mathscr{B} = \{G \cap V : G \in \mathscr{U}_{\tau_i}(x), V \in \mathscr{U}_{\tau_i}(y)\}$$

is a base for some filter $\mathscr{F}$ witnesses that both $\mathscr{F} \to x$ and $\mathscr{F} \to y$. $\square$

However, converse of the statement in Theorem 5.5 does not always hold which may be the one of the most important difference separating being $\tau_i - T_2$ and being $SD(\omega) - T_2$. For this, consider the bispace $(\mathbb{R}, \tau^*, \tau_s)$ from the Example 5.2(1). Choose a $V_x \in \zeta_\omega(\tau^*, \tau_s)$ for all $x \in \mathbb{R}$ as;

$$V_x = \begin{cases} \{x\} & , x \notin \{0, 1\} \\ (0, 1] & , x = 1 \\ (-1, 0] & , x = 0 \end{cases}$$

implying that the bispace is $SD(\omega) - T_2$. Now see that $\mathscr{F} = \mathscr{U}_{\tau^*}(1)$ is filter on $\mathbb{R}$ and observe that $\mathscr{F} \supseteq \mathscr{U}_{\tau^*}(0)$ which concludes both $\mathscr{F} \to 1$ and $\mathscr{F} \to 0$

Remark 5.6. The property of being $SD(\omega) - T_n (n = 0, 1, 2)$ is not hereditary. One can see this by taking the same bispace $(\mathbb{R}, \tau^*, \tau_s)$ and consider its $A = \{0, 1\}$ subset with its corresponding topologies ${\tau^*}_A$ and ${\tau_s}_A$, respectively. Since $\zeta_\omega({\tau^*}_A, {\tau_s}_A) = {\tau_t}_A$, then $(A, {\tau^*}_A, {\tau_s}_A)$ is not $SD(\omega) - T_n (n = 0, 1, 2)$ for all $\omega \in \{P, S, SP, \alpha\}$.

Also, we can give the following ;

Theorem 5.7.

1. Arbitrary product of $SD(\omega)-T_n$ spaces is $SD(\omega)-T_n(n=0,1,2)$ in the product topology.
2. Being $SD(\omega)-T_n(n=0,1,2)$ is a bitopological property.

Definition 5.8. Let $\mathscr{X}=(X,\tau_i,\tau_j)$ be a bitopological space, $x\in X$ and $x\notin K\in ij-\omega C(X)$. $\mathscr{X}$ is called *$SD(\omega)$-regular bitopological space* if there are $U,V\in\zeta_\omega(\tau_i,\tau_j)$ such that $x\in U$, $K\subseteq V$ and $U\cap V=\emptyset$.

Clearly, every τ_i-regular space is $SD(\omega)$-regular, but;

Example 5.9. $\mathbb{R}$ with the topologies $\tau=\{\emptyset,\mathbb{Q},\mathbb{R}\}$ and $\tau^{'}=\{\emptyset,\mathbb{P},\mathbb{R}\}$ is $SD(P)-regular$ not being $\tau_i-regular$ saying $SD(\omega)-regularity$ does not always imply $\tau_i-regularity$. For seeing this, one can conclude that $ij-PO(\mathbb{R})=\mathscr{P}(\mathbb{R})=ij-PC(\mathbb{R})$. For $\omega=\{S,SP,\alpha\}$, simple counterexamples can be built.

Definition 5.10. Let (X,τ_i,τ_j) be a bitopological space, $x\in X$ and $A\subseteq X$. Then A is said to be a *$SD(\omega)$-neighbourhood* of x if there is a nonempty $G\in\zeta_\omega(\tau_i,\tau_j)$ such that $x\in G\subseteq A$.

Theorem 5.11. A bispace $\mathscr{X}=(X,\tau_i,\tau_j)$ is $SD(\omega)-regular$ if and only if there is a neighbourhood base consisting of $ij-\omega$-co-dense subsets for all $x\in X$.

Proof . Let $\mathscr{X}$ be $SD(\omega)$-regular and $x\in X$. Take $U\in\mathscr{U}_\omega(x)$ and $G\in ij-\omega O(X)$ such that $x\in G\subseteq U$. Then there are disjoint $H_1,H_2\in\zeta_\omega(\tau_i,\tau_j)$ such that $x\in H_1$, $X\setminus G\subseteq H_2$. Choose $V_U=X\setminus H_2$ for all $U\in\mathscr{U}_\omega(x)$ then $\mathscr{B}(x)=\{V_U:U\in\mathscr{U}_\omega(x)\}$ is what as wanted. Now if there is a neighbourhood base $\mathscr{B}(x)$ as described, take $x\in X$ and $K\in ij-PC(X)$ such that $x\notin K$, then there is a $V_{X\setminus K}$ such that $x\in V_{X\setminus K}\subseteq X\setminus K$. From here, $K\subseteq X\setminus V_{X\setminus K}$ and $X\setminus V_{X\setminus K}\in\zeta_\omega(\tau_i,\tau_j)$. On the other hand, $x\in ij-int_{SD(\omega)}(V_{X\setminus K})$ which conluding that $\mathscr{X}$ is $ij-SD(\omega)$-regular. □

Also we now can give the same for $ij-SD(\omega)$-regularity;

Theorem 5.12.

1. Arbitrary product of $SD(\omega)$-regular spaces is $SD(\omega)$-regular in the product topology.
2. Being $SD(\omega)$-regular is a bitopological property.

Remark 5.13. $SD(\omega)$-regularity is not hereditary. We now give a counterexample for $\omega=pre$.

Consider an uncountable set X, fix $x_0\in X$ and take X with the topologies $\tau=\{U\subseteq X: x_0\in U\}\cup\{\emptyset\}$ and τ_{cof}. Then we claim that $\mathscr{X}=(X,\tau,\tau_{cof})$ is $SD(P)$-regular. For this, we firstly determine $ij-PO(X)$. If a non-empty subset A of X is ij-pre-open, then $A\subseteq int_\tau(cl_{\tau_{cof}}(A))$ is satisfied. Now then, we obtain that;

$$ij-PO(X)=\{A\subseteq X:\ A \text{ is finite and } x_0\in A \text{ or } A \text{ is infinite }\}.$$

For $x_0\in X$ and any $K\in ij-PC(X)$ such that $x_0\notin K$, choose $G_{x_0},G_K\in\zeta_P(\tau,\tau_{cof})$ as $G_{x_0}=\{x_0\}$ and $G_K=X\setminus\{x_0\}$. On the other hand, for all $x\in X\setminus\{x_0\}$ and $x\notin K\in ij-PC(X)$, choose $G_x,G_K\in\zeta_P(\tau,\tau_{cof})$ such that

$$G_x=\begin{cases} X\setminus K & ,x_0\in K\\ \{x_0,x\} & ,x_0\notin K\end{cases}\quad \textit{and } G_K=\begin{cases} K & ,x_0\in K\\ X\setminus\{x_0,x\} & ,x_0\notin K\end{cases}$$

and conclude that $\mathscr{X}$ is $SD(P)$-regular. But, if we consider the set $A_x=\{x_0,x\}$ for any $x\in X\setminus\{x_0\}$ with its corresponding subspace topologies, observe that $\{x\}\in ij-PC(A_x)$ and $x_0\notin\{x\}$ and the only ij-pre-somewhere dense set S such that $\{x\}\subseteq S$ is the space itself.

Following theorem says us hereditability of $SD(\omega)$-regularity is closed under some very special subsets in ω-hyperconnected bispaces.

Theorem 5.14. Let $\mathscr{X}=(X,\tau_i,\tau_j)$ be a ω-hyperconnected $SD(\omega)$-regular bitopological space, denote by τ_j^{CL} the family of all clopen (closed+open) subsets of (X,τ_j). If $F\in\tau_i\cap\tau_j^{CL}$, then $(F,\tau_{i_F},\tau_{j_F})$ is $SD(\omega)$-regular.

Proof . Let give the proof for $\omega = pre$. Take an arbitrary $x \in F$ and $K \in ij - PC(F)$ such that $x \notin K$. Since;

$$\begin{aligned} X \setminus K &\subseteq int_{\tau_{i F}}(cl_{\tau_{j F}}(X \setminus K)) \\ &= int_{\tau_{i F}}(F \cap cl_{\tau_j}(X \setminus K)) \\ &\subseteq F \cap int_{\tau_{i F}}(cl_{\tau_j}(X \setminus K)) \\ &\subseteq int_{\tau_{i F}}(cl_{\tau_j}(X \setminus K)) \\ &= int_{\tau_i}(cl_{\tau_j}(X \setminus K)) \end{aligned}$$

hence $K \in ij - PO(X)$. Then there is $G, H \in \zeta_P(\tau_i, \tau_j)$ such that $x \in G$, $K \subseteq H$ and $G \cap H = \emptyset$. Now we claim that $G \cap F, F \cap H \in \zeta_P(\tau_{i F}, \tau_{j F})$. Since $G \in \zeta_P(\tau_i, \tau_j)$, take a non-empty $U \in ij - PO(X)$ such that $U \subseteq ij - Pcl(G)$. Then $F \cap U \neq \emptyset$ because of that the bispace is P-hyperconnected and $F \cap U \subseteq ij - Pcl(G)$. Observe that $ij - Pcl(G) = ij - Pcl_F(G)$ and since $F \in \tau_i \cap \tau_j$,

$$\begin{aligned} F \cap U &\subseteq F \cap ij - Pcl(G) \\ &\subseteq F \cap ij - Pcl_F(G) \\ &\subseteq ij - Pcl_F(F \cap G) \end{aligned}$$

is obtained. On the other hand, since $U \in ij - PO(X)$, then;

$$\begin{aligned} F \cap U &\subseteq F \cap int_{\tau_i}(cl_{\tau_j}(U)) \\ &\subseteq int_{\tau_i}(F \cap cl_{\tau_j}(U)) \\ &\subseteq int_{\tau_i}(cl_{\tau_j}(F \cap U)) \\ &\subseteq int_{\tau_{i F}}(cl_{\tau_{j F}}(F \cap U)) \end{aligned}$$

which implies that $F \cap U \subseteq ij - Pcl_F(F \cap G)$ and $x \in F \cap G \in \zeta_P(\tau_{i F}, \tau_{j F})$. With all the same process, observe that $K \subseteq F \cap H \in \zeta_P(\tau_{i F}, \tau_{j F})$ and $(F \cap G) \cap (F \cap H) = \emptyset$. □

6 Conclusion

All in all, we define the concepts ij-ω-somewhere density properties for $\omega \in \{P, S, SP, \alpha\}$. We obtained some weaker forms of somewhere density by using generalized open sets in bitopological spaces. We derive many properties and interesting results of this ij-ω-SD sets and we obtain a filter and proper maximal ideal in ω-hyperconnected spaces. These types of sets obtained by using two different topological structures can give new viewpoints to computer science applications and digital topologies. The related filter and ideals obtained in ω-hyperconnected spaces can give a new way on studying different type compactifications of these spaces. In upcoming paper, we plan to study new types of compactness and other covering properties by using ij-ω-SD sets. Also it would be interesting to study the selective versions of this types of density properties. In addition, we are going to define this concept on the contexts of soft and fuzzy bitopological spaces and will provide an approximations and accuracy measures of rough sets by using this types of sets as new framework for the applicable area. We believe in that this work will contribute for the researchers who are interested in bitopological spaces to study this kind of properties.

References

[1] M.E. Abd El Monsef, S.N. El Deeb and R.A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.

[2] D. Andrijevic, *On b-open sets*, Mat. Vesn. **48** (1996), 59–64.

[3] T.M. Al-Shami, *Somewhere dense sets and ST1-spaces*, Punjab Univ. J. Math. **49** (2017), no. 2, 101–111.

[4] T.M. Al-Shami, T. Noiri, *More notions and mappings via somewhere dense sets*, Afr. Mat. **30** (2019), no. 7, 1011–1024.

[5] B.P. Dvalishvili, *Bitopological spaces, Theory, Relations with Generalized Algebraic structures and Applications*, Mathematic Studies, vol. 199, Elsevier Science B.V, Amsterdam, 2005.

[6] O. El-Tantawy and H. Abu-Donia, *Generalized separation axioms in bitopological spaces*, Arab J. Sci. Eng. **30** (2005), no. 1, 117–129.

[7] R. Engelking, *General Topology*, Sigma Series in Pure Mathematics, vol. 6, Heldermann-Verlag, Berlin, 1989.

[8] M. İlkhan, M. Akyiğit and E.E. Kara, *On new types of sets via γ-open sets in bitopological spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **67** (2017), no. 1, 225-234.

[9] M. Jelic, *A decomposition of pairwise continuity*, J. Inst. Math. Comput. Sci. Math. Ser. **3** (1990), no. 1, 25–29.

[10] M. Jelic, *On a decomposition of p-continuity*, J. Inst. Math. Comput. Sci. Math. Ser. **4** (1991), no. 1, 57–60.

[11] J.C. Kelly, Bitopological spaces, Proc. Lond. Math. Soc. **13** (1963), no. 3, 71-89.

[12] A.B. Khalaf, H.M. Darwesh and K. Kannan, *Some types of separation axioms in topological spaces*, Tamsui Oxf. J. Math. Sci. **28** (2012), no. 3, 303–326.

[13] F.H. Khedr, S.M. Al Areefi and T. Noiri, *Precontinuity and semi-precontinuity in bitopological spaces*, Indian J. Pure Appl. Math. **23** (1992), no. 9, 625–633.

[14] S.S. Kumar, *On decomposition of pairwise continuity*, Bull. Cal. Math. Soc. **89** (1997), 441–446.

[15] K. Kuratowski, *Topology: Volume I*, vol. 1, Elsevier, 2014.

[16] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Month. **70** (1963), no. 1, 36–41.

[17] S.N. Maheshwari and R. Prasad, *Semi-open sets and semi-continuous functions in bitopological spaces*, Math. Notae **26** (1977-78), 29–37.

[18] A.S. Mashour, M.E. Abd El Monsef and S.N. El Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.

[19] O. Njastad, *On some classes of nearly open sets*, Pac. J. Math. **15** (1965), no. 3, 961–970.

[20] C.C. Pugh, *Real Mathematical analysis*, Undergraduate Texts in Mathematics, vol. 2011, Springer Science and Business Media, 2003.

[21] L.A. Steen and J.A. SeebachJr, *Counterexamples in Topology*, vol. 18, Springer, New York, 1978.

[22] G. Thamizharasi and P. Thangavelu, *Remarks on closure and interior operators in bitopological spaces*, J. Math. Sci. Comput. Appl. **1** (2010), no. 1, 1–8.